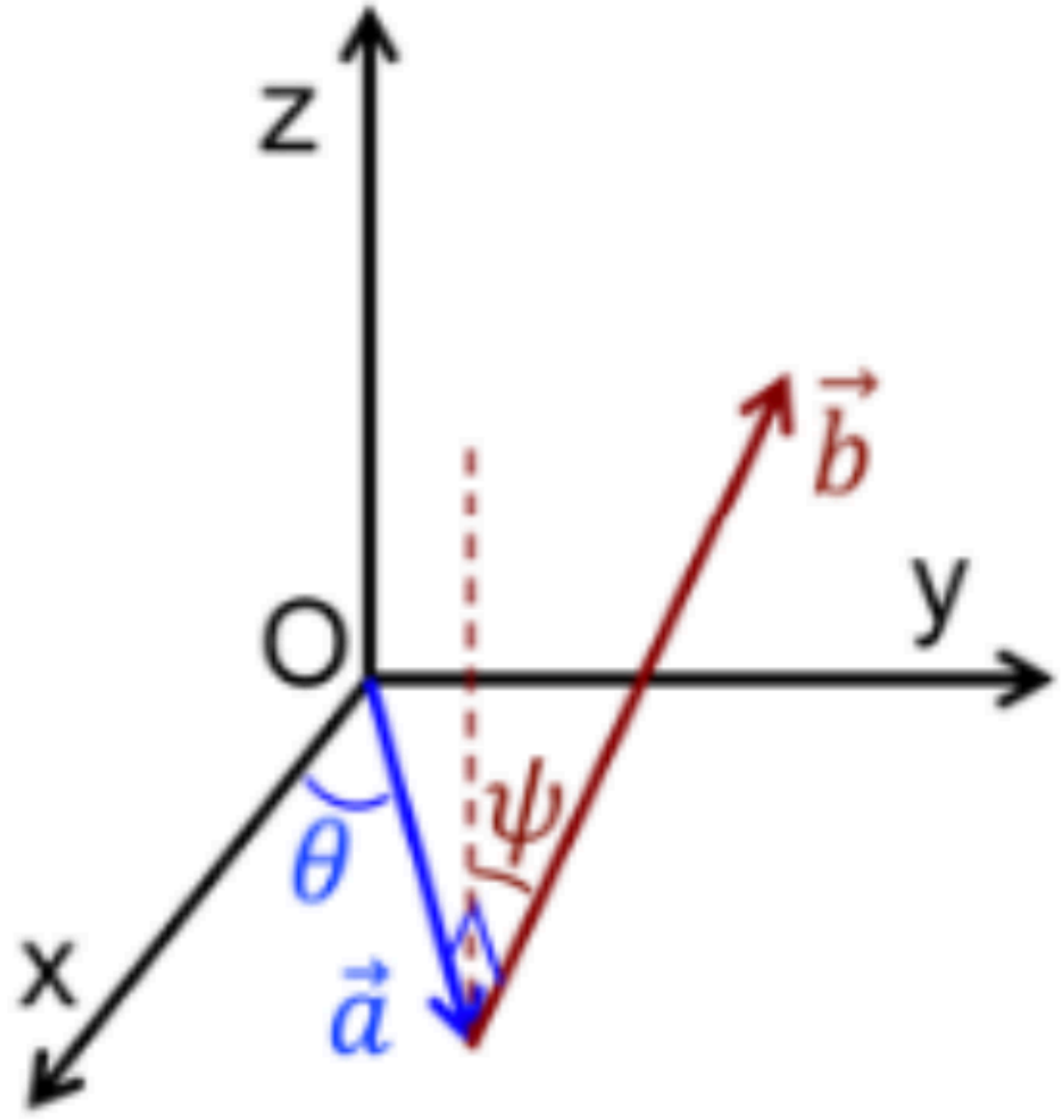
EXERCISE



We consider an orthonormal coordinate system (xyz), with the z-axis as the vertical axis. The vector $\vec{a}$, with magnitude a, lies in the xy plane and makes an angle θ with the x-axis. The vector $\vec{b}$, with magnitude b, is perpendicular to $\vec{a}$ and forms an angle ψ with the vertical.

A. Visualize the vectors $\vec{a}$ and $\vec{b}$ in space and find their Cartesian coordinates in (xyz). Use sketches to represent the projections of these vectors on the axes.

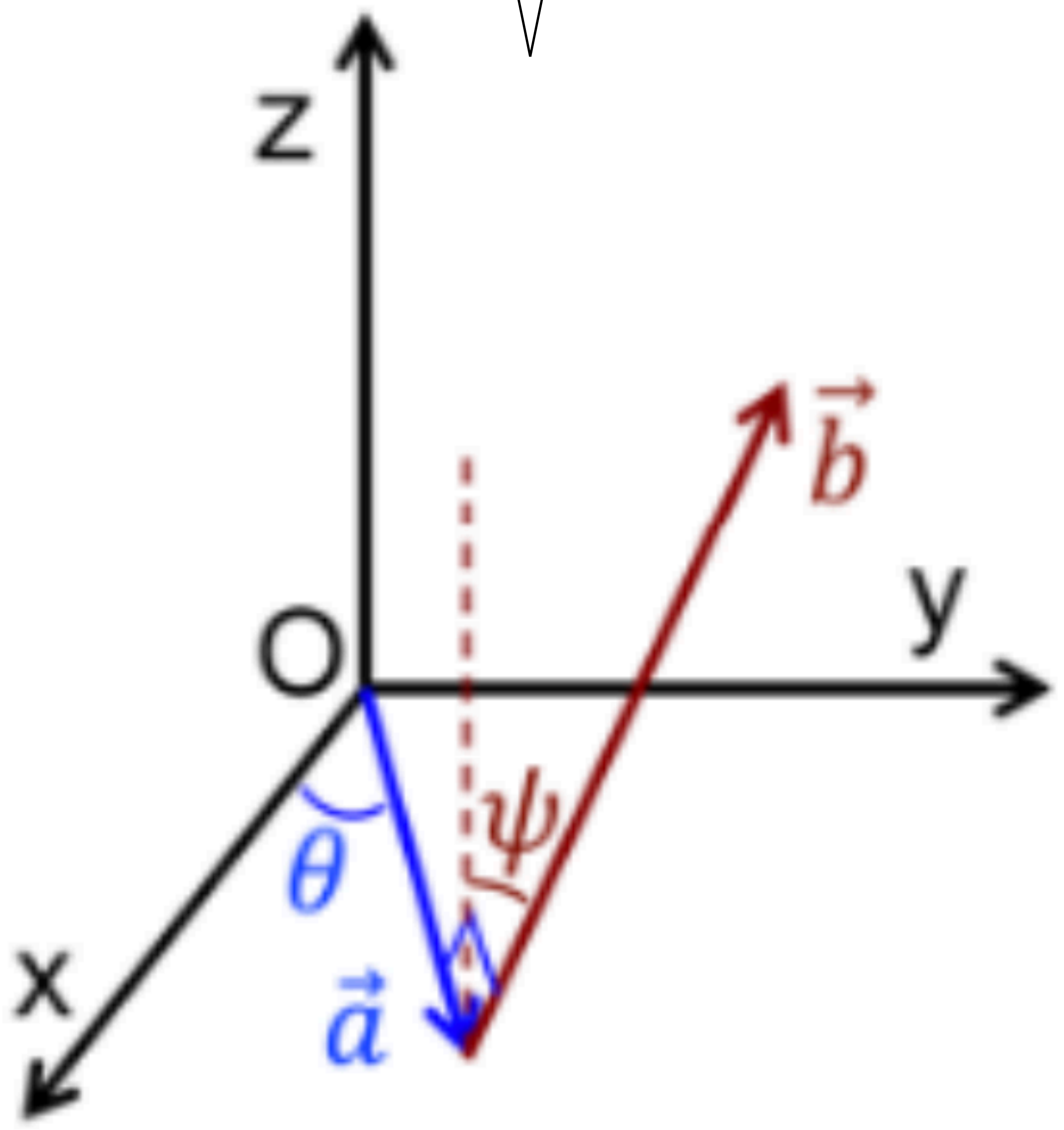
B. We define a vector $\vec{c} = \vec{a} \wedge \vec{b}$. Calculate the coordinates of $\vec{c}$ by formulating the cross product in (xyz). Visualize vector $\vec{c}$ in space.

C. Calculate the values of $\vec{c} \cdot \vec{a}$, $\vec{c} \cdot \vec{b}$, and $\|\vec{c}\|$:

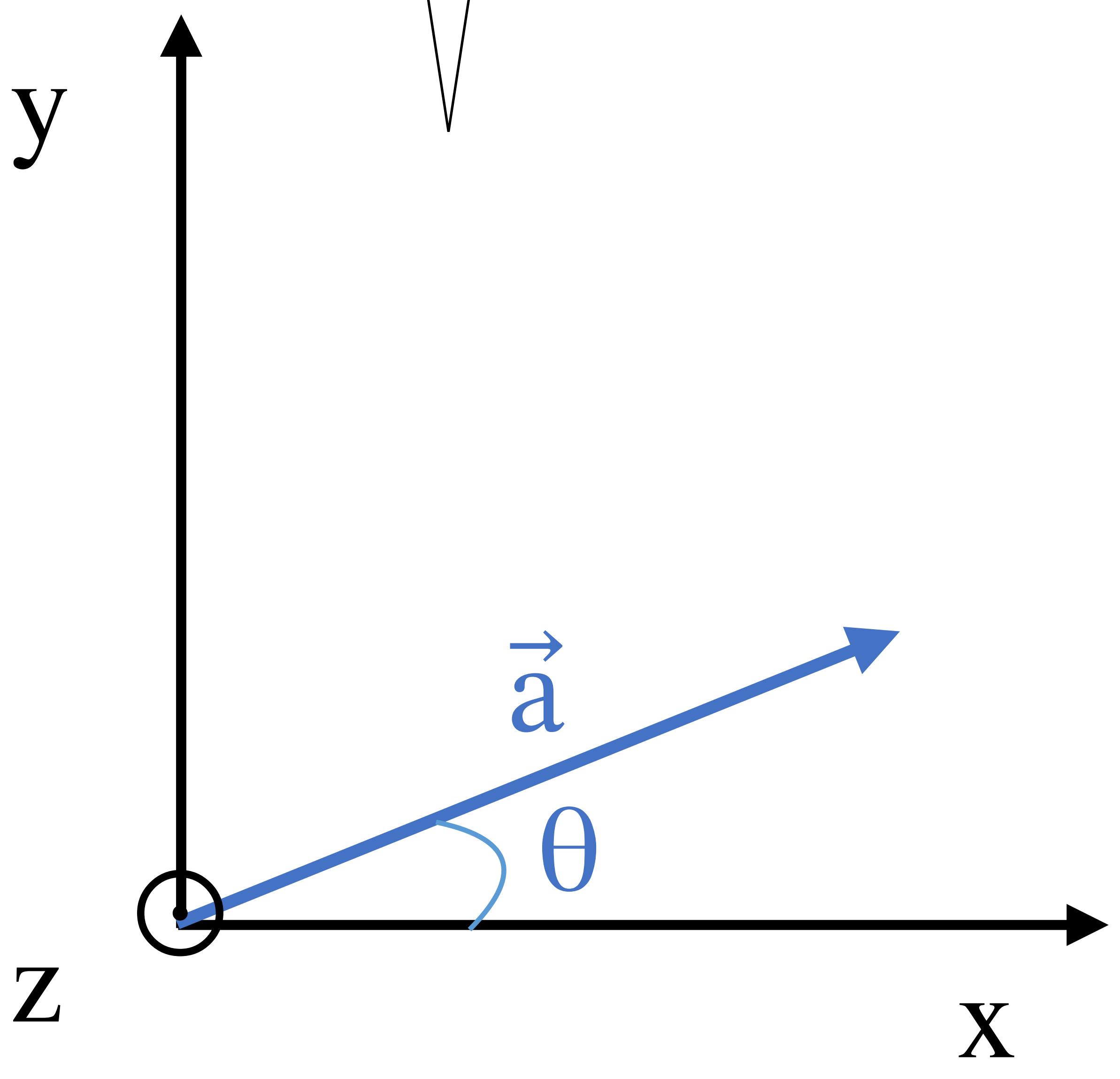
- without calculation, relying on simple geometric considerations
- by performing the calculation using the coordinates of $\vec{a}$, $\vec{b}$, and $\vec{c}$ found previously.

Projection of $\vec{a}$ is straightforward in the xy plane.

Original "3D" view



Top-down view with xy plane in the plane of the screen



$$\vec{a}\left|\begin{array}{l} a\cos\theta \\ a\sin\theta \\ 0 \end{array}\right.$$

Column/vertical notation

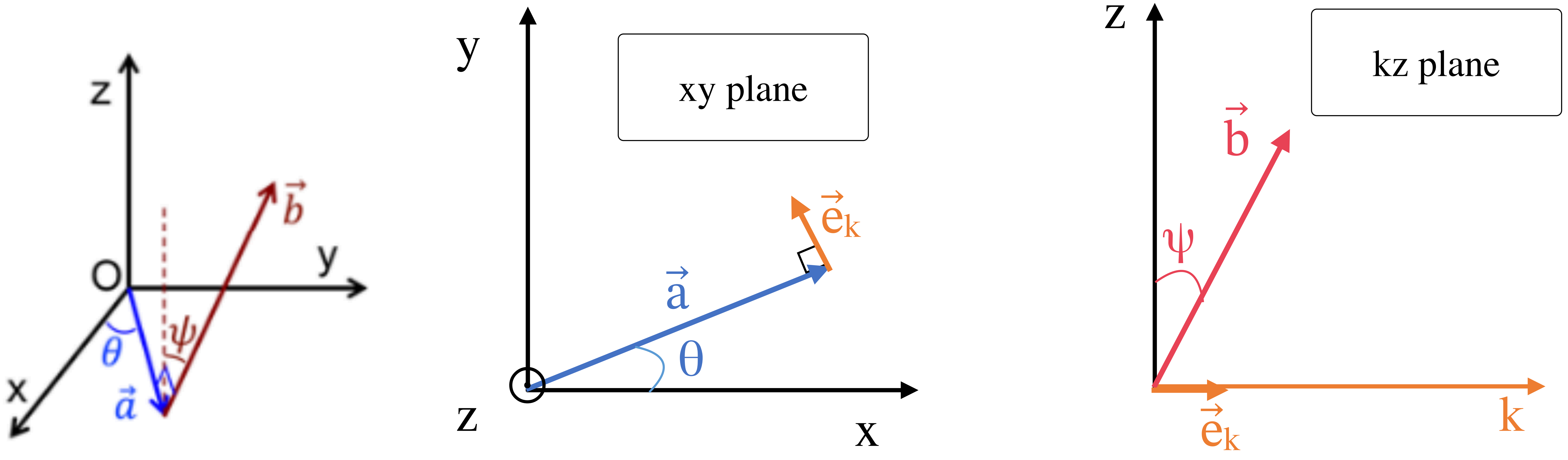
$$\vec{a} = a\cos\theta\,\vec{e}_x + a\sin\theta\,\vec{e}_y$$

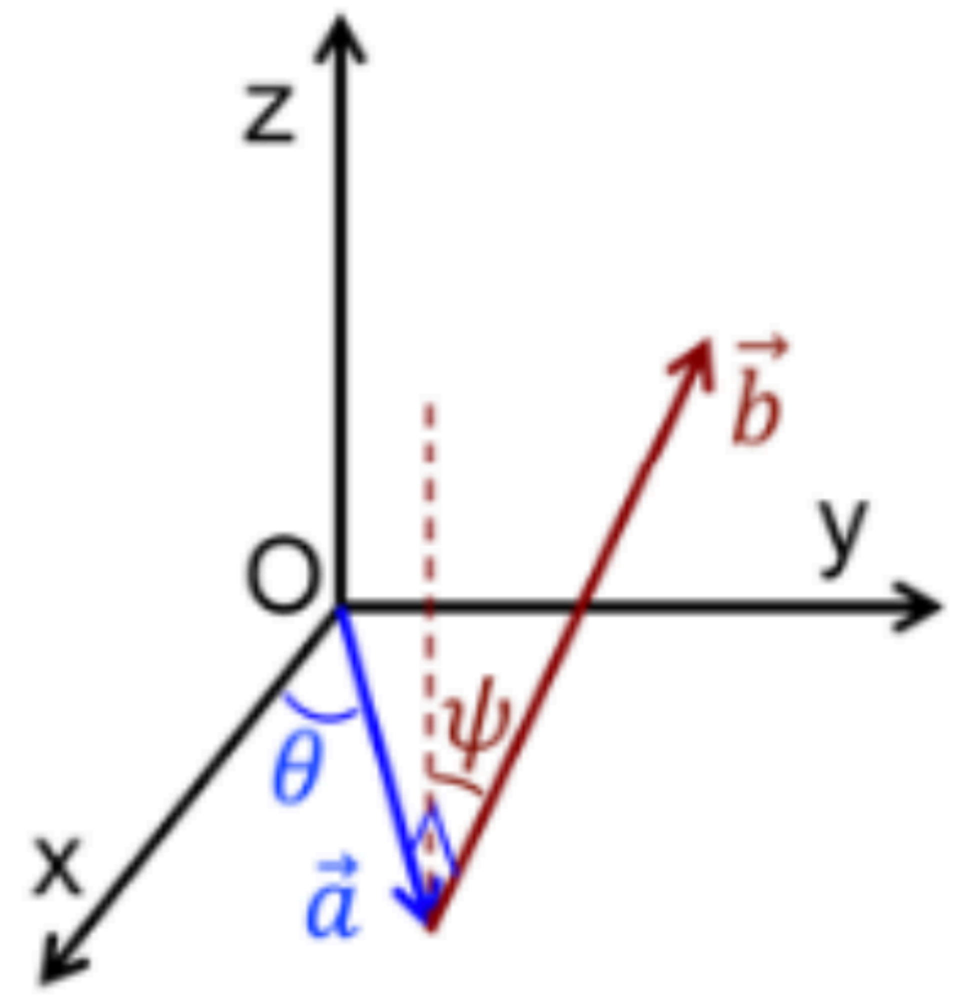
Written in terms of Cartesian basis vectors

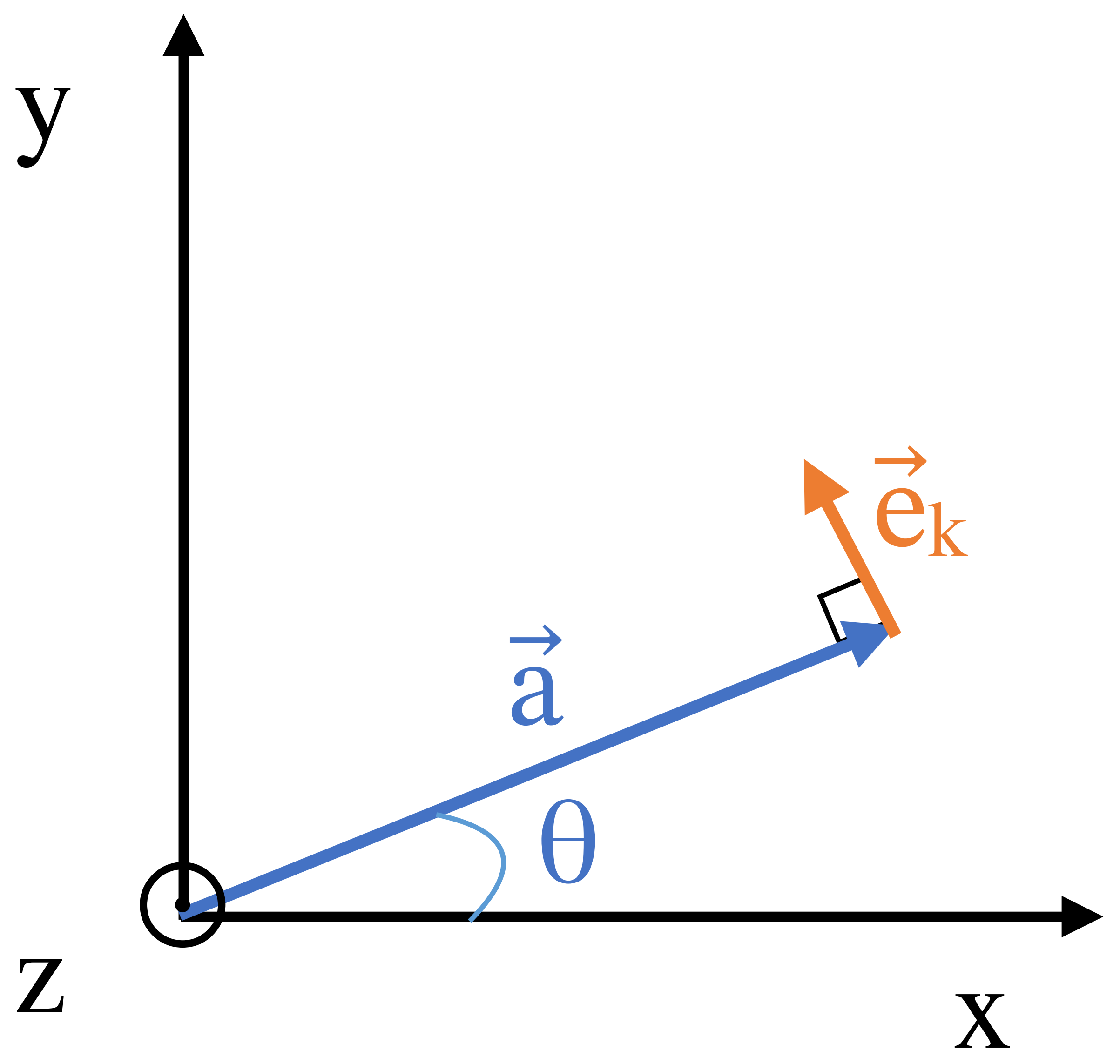
The vector $\vec{b}$ is a bit more complicated. Components of b are determined by both θ and ψ:

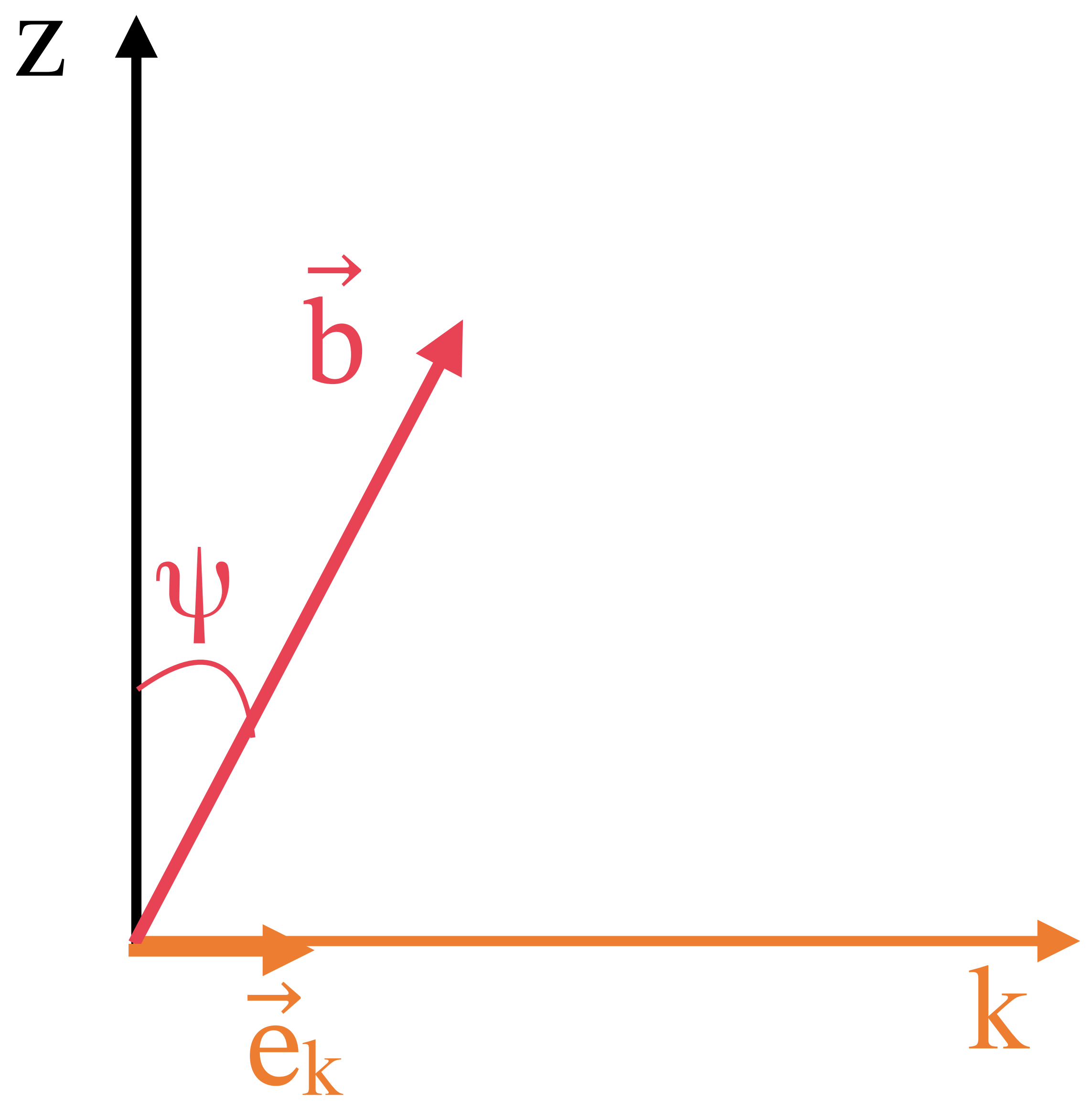
- ψ determines the component of $\vec{b}$ along the z-axis. If $\psi = 0$, $\vec{b}$ is collinear with z. If $\psi = \pi/2$, $\vec{b}$ lies in the xy plane.
- θ determines the components of $\vec{b}$ along the x-axis vs y-axis. If $\theta = 0$, $\vec{b}$ is perpendicular to the x-axis. If $\theta = \pi/2$, $\vec{b}$ is perpendicular to the y-axis. θ does not change the component of $\vec{b}$ along the z-axis.

In this example, because the component of $\vec{b}$ along z only depends on one angle ψ, we begin by finding the component of $\vec{b}$ along z. To do the projection, we need a plane which always contains both the z-axis and $\vec{b}$. To do this, **we define an intermediate axis k** with unit basis vector $\vec{e}_k$, perpendicular to $\vec{e}_z$, therefore lying in the xy-plane. Because $\vec{b} \perp \vec{a}$, we also set $\vec{e}_k \perp \vec{a}$.









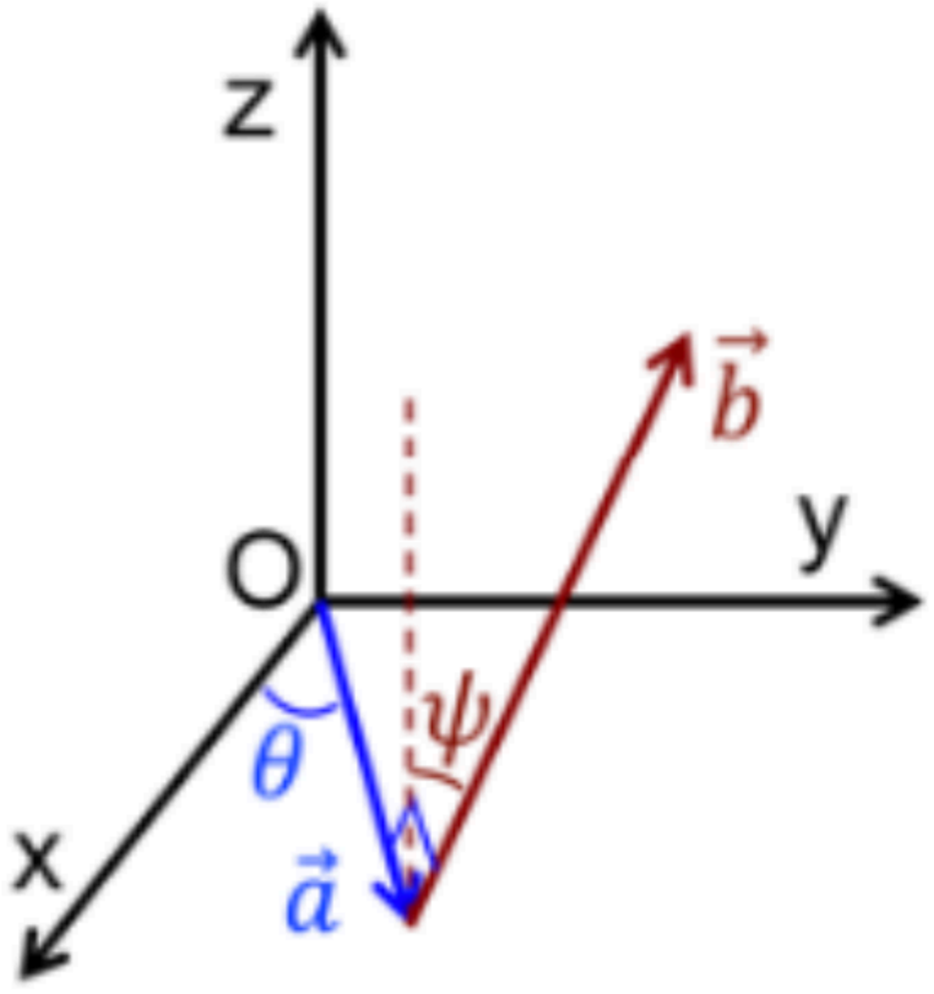
$$\vec{e}_k = -\sin\theta\ \vec{e}_x + \cos\theta\ \vec{e}_y$$

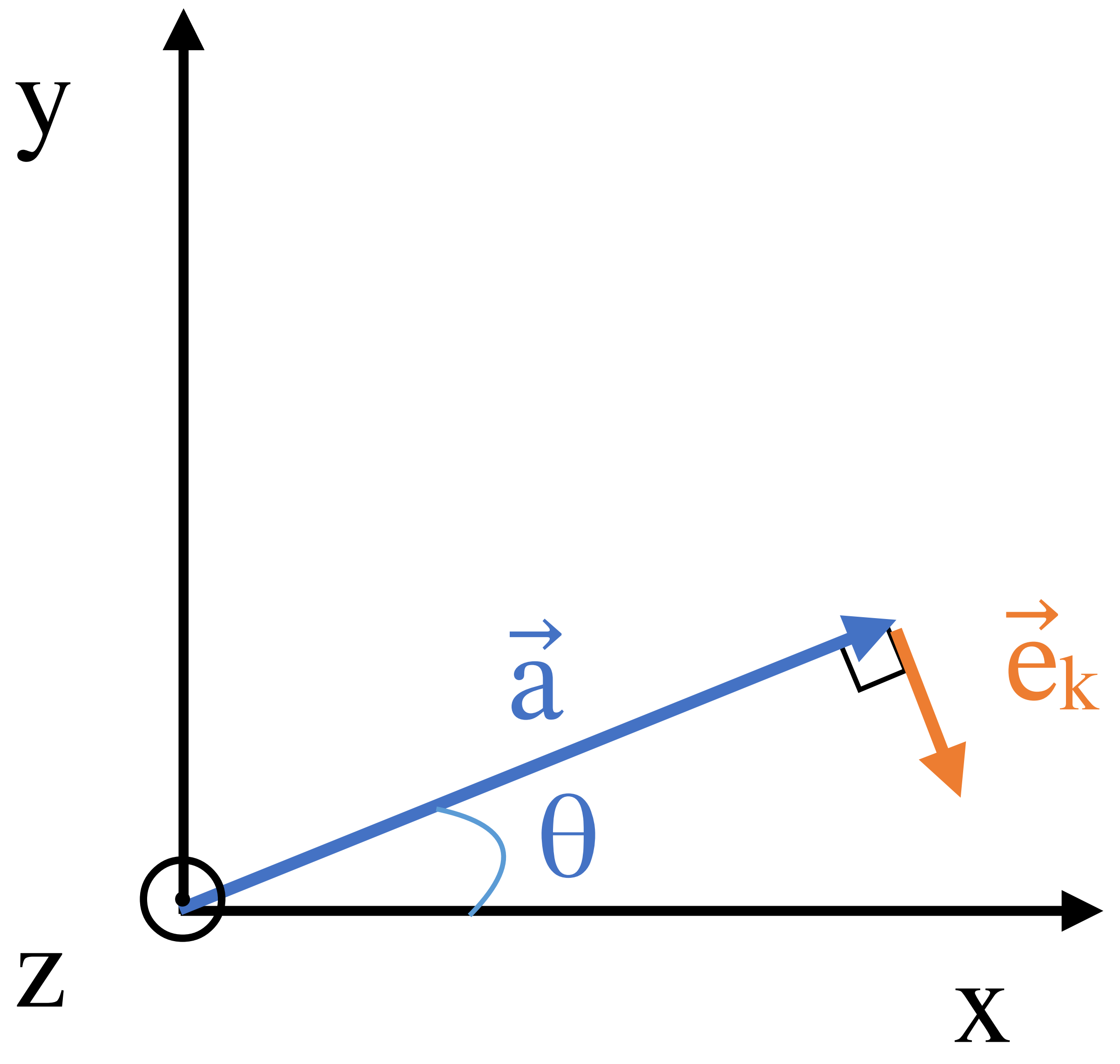
$$\vec{b} = b\cos\psi\ \vec{e}_z + b\sin\psi\ \vec{e}_k$$

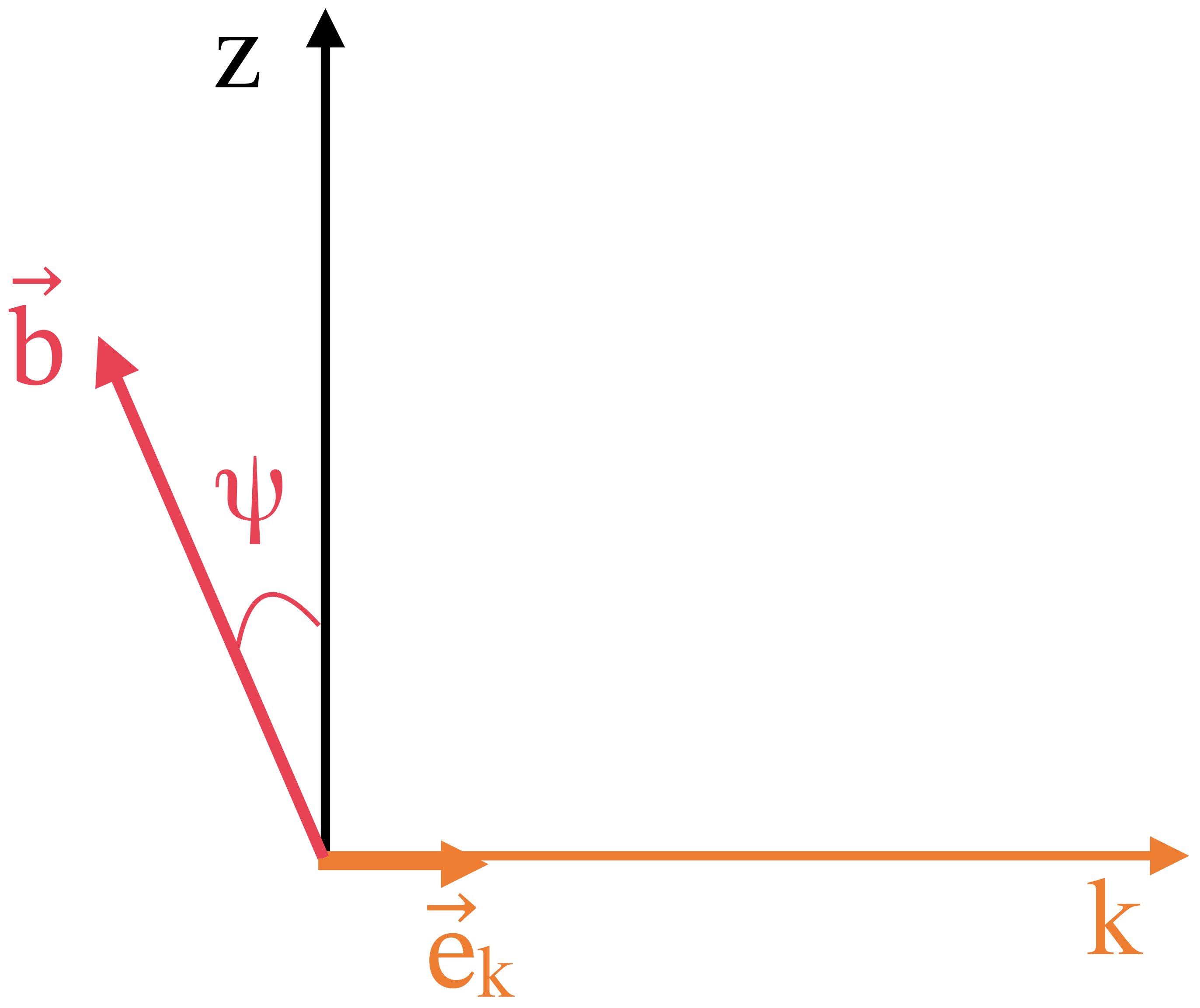
$$\vec{b} = b\cos\psi\ \vec{e}_z + b\sin\psi\ (-\sin\theta\ \vec{e}_x + \cos\theta\ \vec{e}_y)$$

$$\vec{b}\left|\begin{array}{l} -b\sin\psi\sin\theta \\ b\sin\psi\cos\theta \\ b\cos\psi \end{array}\right.$$

Note that we would have gotten the same result if we had defined the axis k pointing in the other direction







$\vec{e}_k = \sin\theta\, \vec{e}_x - \cos\theta\, \vec{e}_y$

$\vec{b} = b \cos\psi\, \vec{e}_z - b \sin\psi\, \vec{e}_k$

$$\vec{b} = b \cos\psi\, \vec{e}_z - b \sin\psi\, (\sin\theta\, \vec{e}_x - \cos\theta\, \vec{e}_y)$$

$$\vec{b} \left| \begin{array}{l} -b \sin\psi \sin\theta \\ b \sin\psi \cos\theta \\ b \cos\psi \end{array} \right.$$

$\vec{c} = \vec{a} \wedge \vec{b}$

$$\vec{a} \begin{vmatrix} a\cos\theta \\ a\sin\theta \\ 0 \end{vmatrix} \wedge \vec{b} \begin{vmatrix} -b\sin\psi\sin\theta \\ b\sin\psi\cos\theta \\ b\cos\psi \end{vmatrix} = \vec{c} \begin{vmatrix} ab\cos\psi\sin\theta \\ ab\cos\psi\cos\theta \\ ab\sin\psi\cos^2\theta + ab\sin\psi\sin^2\theta = ab\sin\psi\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} \end{vmatrix}$$

$$\vec{c} \begin{vmatrix} ab\cos\psi\sin\theta \\ ab\cos\psi\cos\theta \\ ab\sin\psi \end{vmatrix}$$

$\vec{a}.\vec{c} = 0$ car $\vec{c} = \vec{a} \wedge \vec{b} \perp \vec{a}$; $\vec{b}.\vec{c} = 0$

$|\vec{c}| = |\vec{a} \wedge \vec{b}| = |\vec{a}|.|\vec{b}|.\underbrace{\sin(\widehat{\vec{a},\vec{b}})}_{1} = ab$