XI. Application du solide indéformable

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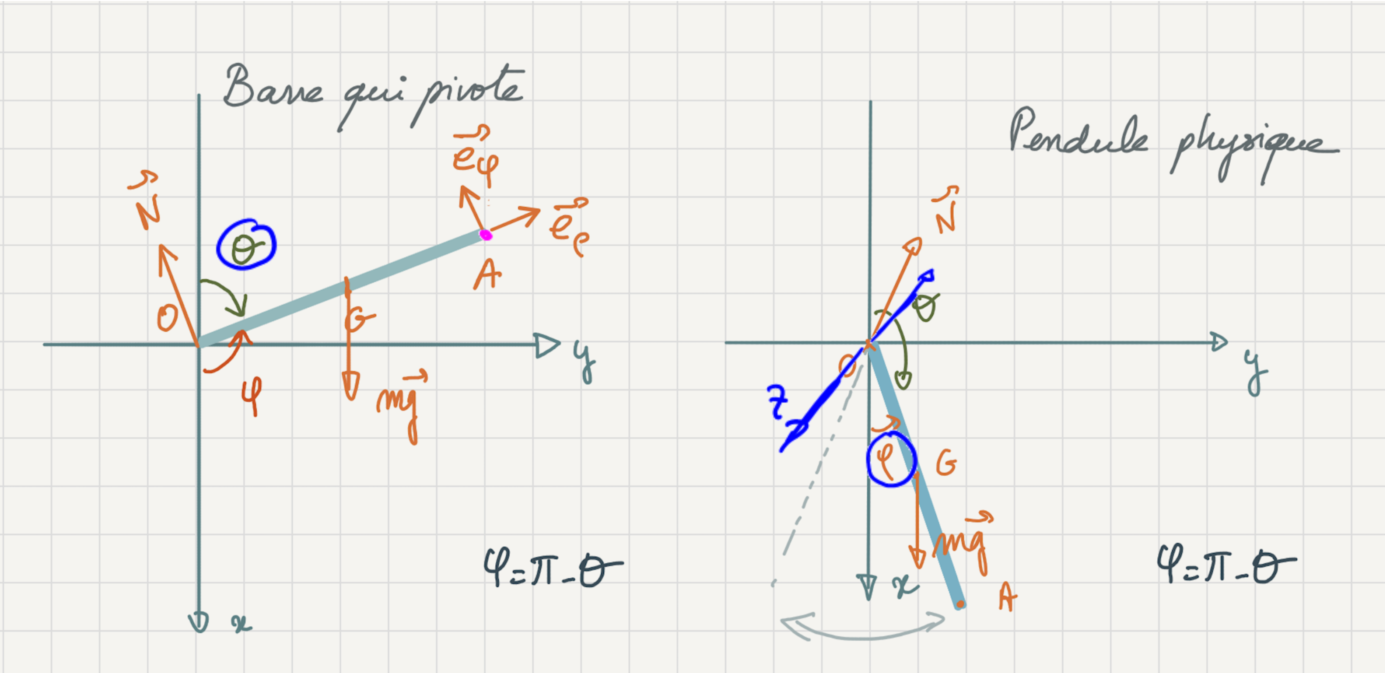
Plan du cours

Table des matières

Barre homogène de masse m et de longueur l pivotant autour de 0, point fixe.



θ [0, 2π]

φ [−π, π]

$\varphi = \pi - \theta$

• $\sum \vec{F}^{ext} = M\vec{a}_G$

• $\sum \vec{M}_x^{ext} = \frac{d}{dt}\vec{L}_x$ "x" = O

$$\sum \vec{M}_O^{ext} = \underbrace{\vec{OO}\times\vec{N}}_{0} + \vec{OG}\times m\vec{g}$$

θ

$$\vec{OG}\times m\vec{g} = -\frac{\ell}{2}mg\sin(\pi-\theta)\,\vec{e}_z$$
$$= -\frac{\ell}{2}mg\sin(\theta)\,\vec{e}_z$$

φ (angle polaire)

$$\vec{OG}\times m\vec{g} = \frac{\ell}{2}\vec{e}_\rho \times mg\,\vec{e}_x$$
$$= \frac{\ell}{2}mg\;\vec{e}_\rho\times\vec{e}_x$$
$$= \frac{\ell}{2}mg\;\sin\varphi\;(-\vec{e}_z)$$
$$= -\frac{\ell}{2}mg\sin\varphi\;\vec{e}_z$$

$$\vec{L}_O = I_{Oz}\,\vec{\omega}$$

th. de Steiner

$$I_{Oz} = I_{Gz} + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{12}m\ell^2 + \frac{1}{4}m\ell^2 = \frac{1}{12}m\ell^2 + \frac{3}{12}m\ell^2$$

$$\underline{I_{Oz} = \frac{1}{3}m\ell^2}$$

$$\vec{L}_O = \frac{1}{3} m\ell^2 \vec{\omega} \qquad \frac{d\vec{L}_O}{dt} = \frac{1}{3} m\ell^2 \dot{\vec{\omega}}$$

θ

$$\vec{\omega} = -\dot{\theta}\,\vec{e}_z$$

$$\dot{\vec{\omega}} = -\ddot{\theta}\,\vec{e}_z$$

$$\frac{d\vec{L}_O}{dt} = -\frac{1}{3} m\ell^2 \ddot{\theta}\,\vec{e}_z$$

$$-\frac{\ell}{2} mg \sin(\theta)\,\vec{e}_z = -\frac{1}{3} m\ell^2 \ddot{\theta}\,\vec{e}_z$$

$$\boxed{\ddot{\theta} - \frac{3}{2}\frac{g}{\ell}\sin(\theta) = 0}$$

$$\theta(t) = \dots$$

φ

$$\vec{\omega} = \dot{\varphi}\,\vec{e}_z \qquad \theta = \pi - \varphi$$

$$\dot{\vec{\omega}} = \ddot{\varphi}\,\vec{e}_z \qquad \dot{\theta} = -\dot{\varphi}$$

$$\frac{d\vec{L}_O}{dt} = \frac{1}{3} m\ell^2 \ddot{\varphi}\,\vec{e}_z$$

$$\frac{\ell}{2} mg \sin\varphi\,\vec{e}_z = \frac{1}{3} m\ell^2 \ddot{\varphi}\,\vec{e}_z$$

$$\boxed{\ddot{\varphi} + \frac{3}{2}\frac{g}{\ell}\sin\varphi = 0}$$

$$\varphi(t) = \dots$$

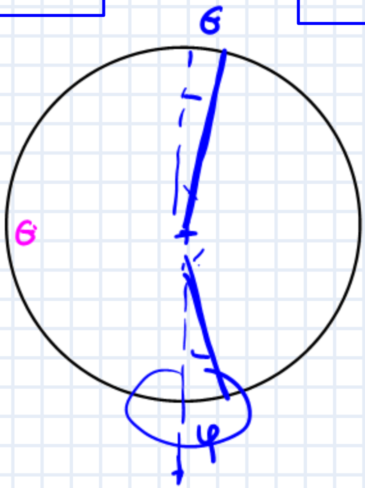
$$\ddot{\theta} - \frac{3}{2}\frac{g}{\ell}\sin(\theta) = 0$$

$\theta \ll 1$

$\sin\theta \cong \theta$

$$\ddot{\theta} - \frac{3}{2}\frac{g}{\ell}\theta = 0$$

$\ddot{\theta} \sim \theta$



$$\ddot{\theta} = \underbrace{\frac{3}{2}\frac{g}{\ell}}_{\Omega^2}\theta$$

$\theta(t) = e^{\Omega t}$ $\quad e^{-\Omega t}$

$\dot{\theta} = \Omega e^{\Omega t}$

$\ddot{\theta} = \Omega^2 e^{\Omega t} = \Omega^2 \theta$

$\theta(t) = A e^{\Omega t} + B e^{-\Omega t}$

si $\dot{\theta}(t=0) = 0$

$\theta(t) = C\left(e^{\Omega t} + e^{-\Omega t}\right)$

$$\ddot{\varphi} + \frac{3}{2}\frac{g}{\ell}\sin\varphi = 0$$

$\varphi \ll 1$

$\sin\varphi \cong \varphi$

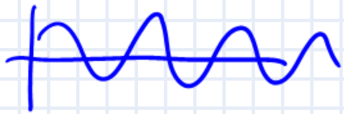
$$\ddot{\varphi} + \frac{3}{2}\frac{g}{\ell}\varphi = 0$$

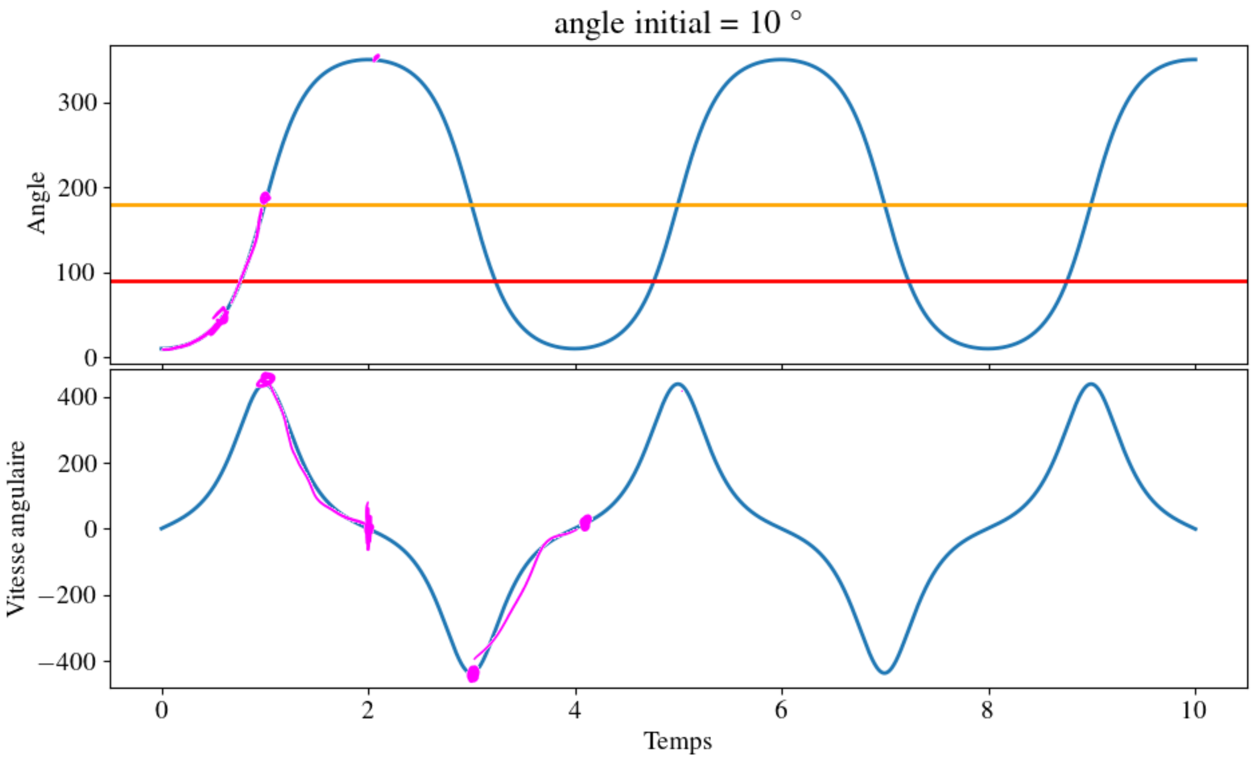
$\ddot{\varphi} \sim -\varphi$

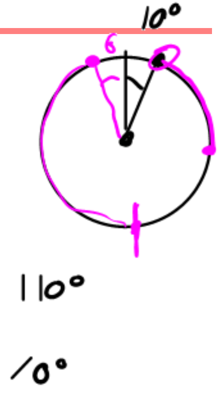
$\Omega^2 = \frac{3}{2}\frac{g}{\ell}$

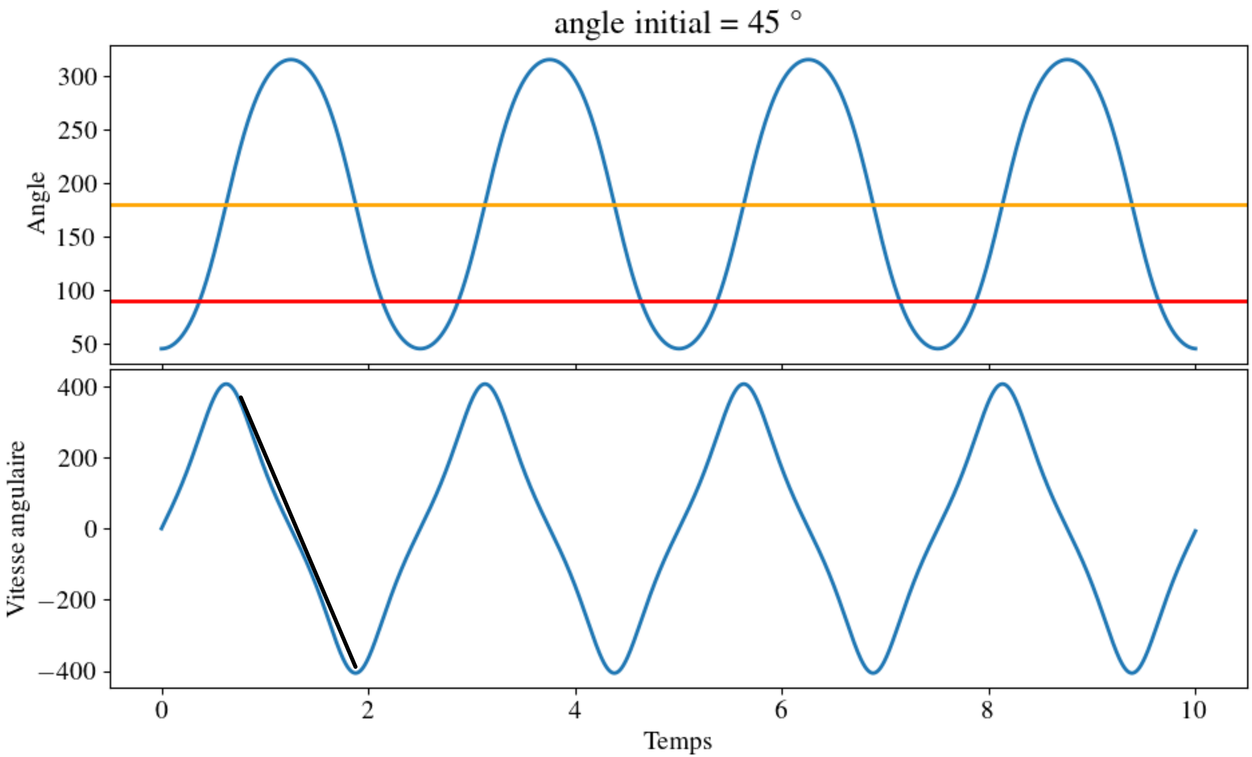
$\varphi(t) = C\cos(\Omega t + \alpha)$

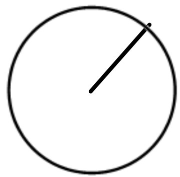
$= A\cos(\Omega t) + B\sin(\Omega t)$

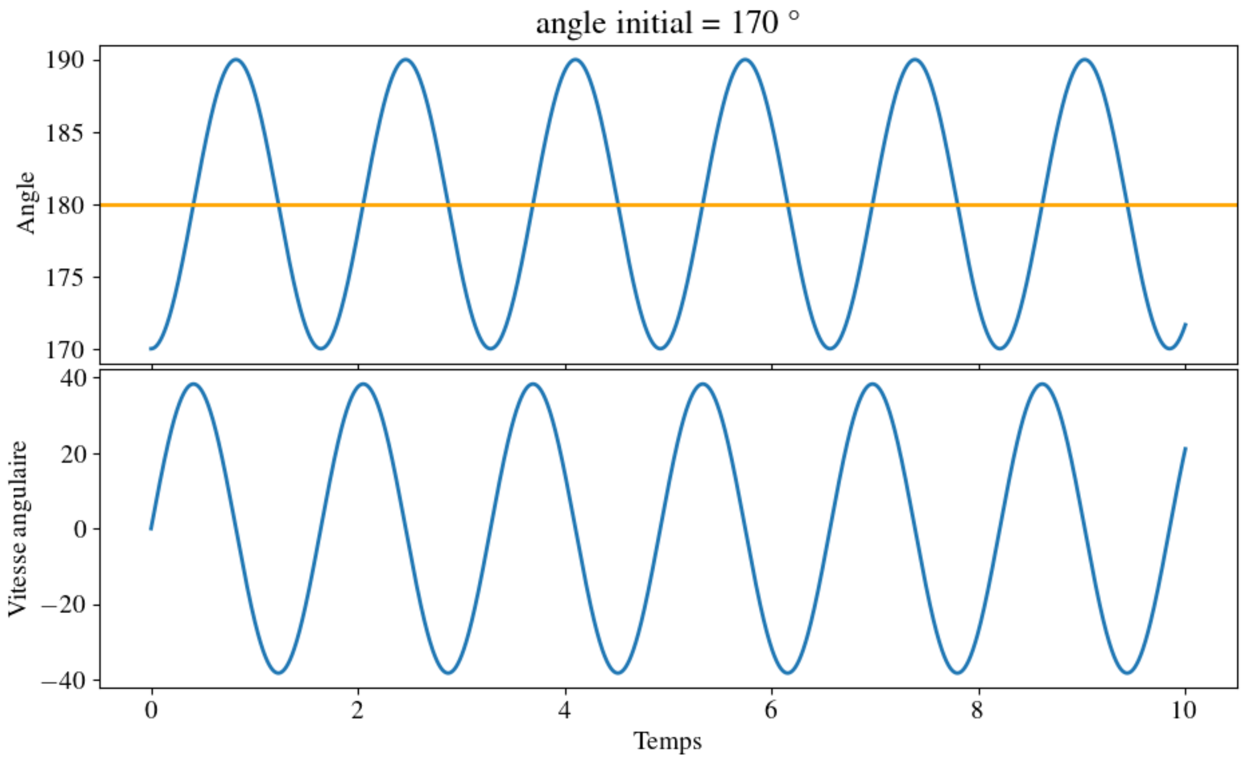


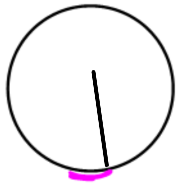

angle initial = 10 °
Angle
300
200
100
0
Vitesse angulaire
400
200
0
−200
−400
0
2
4
6
8
10
Temps

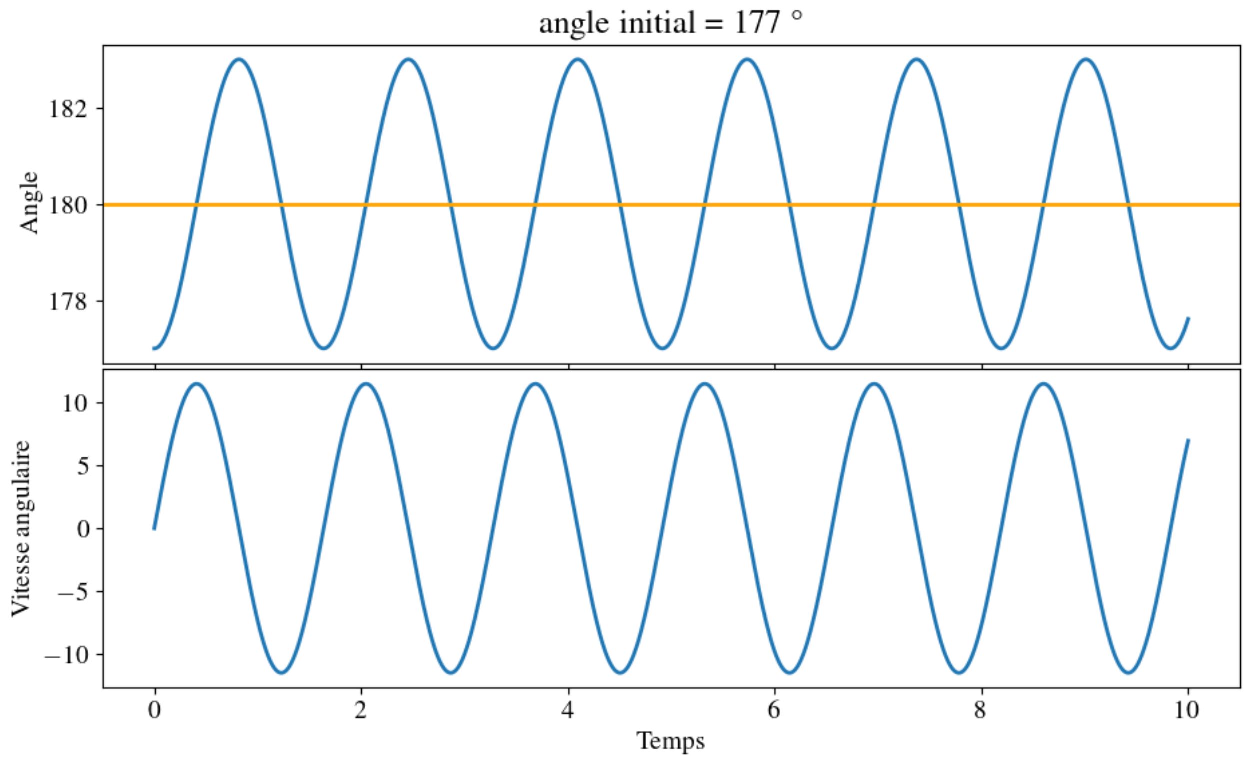



angle initial = 45 °
300
250
200
150
100
50
Angle
400
200
0
−200
−400
Vitesse angulaire
0
2
4
6
8
10
Temps

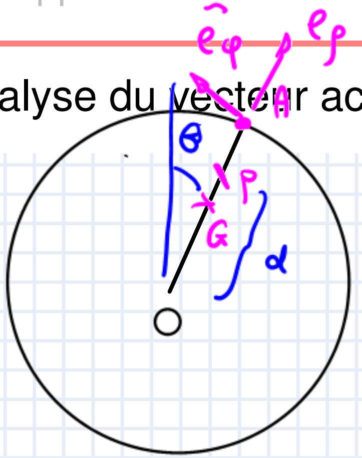



angle initial = 170 °
190
185
180
175
170
Angle
40
20
0
−20
−40
Vitesse angulaire
0
2
4
6
8
10
Temps




angle initial = 177 °
182
180
178
Angle
10
5
0
−5
−10
Vitesse angulaire
0
2
4
6
8
10
Temps

Analyse du vecteur accélération



$\theta(t) = \theta_0 \qquad \dot\theta(t) = 0$

accélération d'un point P ?

$$\ddot\theta - \frac{3}{2}\frac{g}{\ell}\sin\theta = 0$$

$\varphi = \pi - \theta$

$\dot\varphi = -\dot\theta$

$\ddot\varphi = -\ddot\theta$

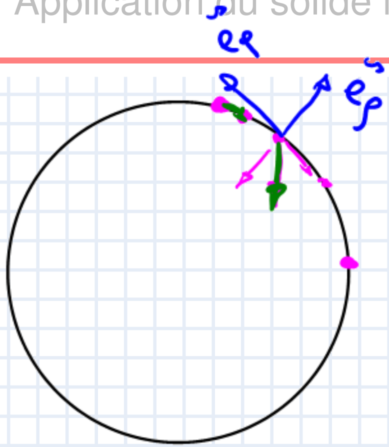
(ρ, φ) polaires $\quad \rho = \rho_0 = d$

$\dot\rho = \ddot\rho = 0$

$$\vec a = -\rho\dot\varphi^2 \vec e_\rho + \rho\ddot\varphi \vec e_\varphi$$

$$= -d\dot\theta^2 \vec e_\rho - d\ddot\theta \vec e_\varphi$$

$$\boxed{\vec a = -d\dot\theta^2 \vec e_\rho - d\frac{3}{2}\frac{g}{\ell}\sin\theta\, \vec e_\varphi}$$



- $\theta(t)$ $\dot\theta(t)$ numériquement
- choix du Δt

$t_0 = 0$

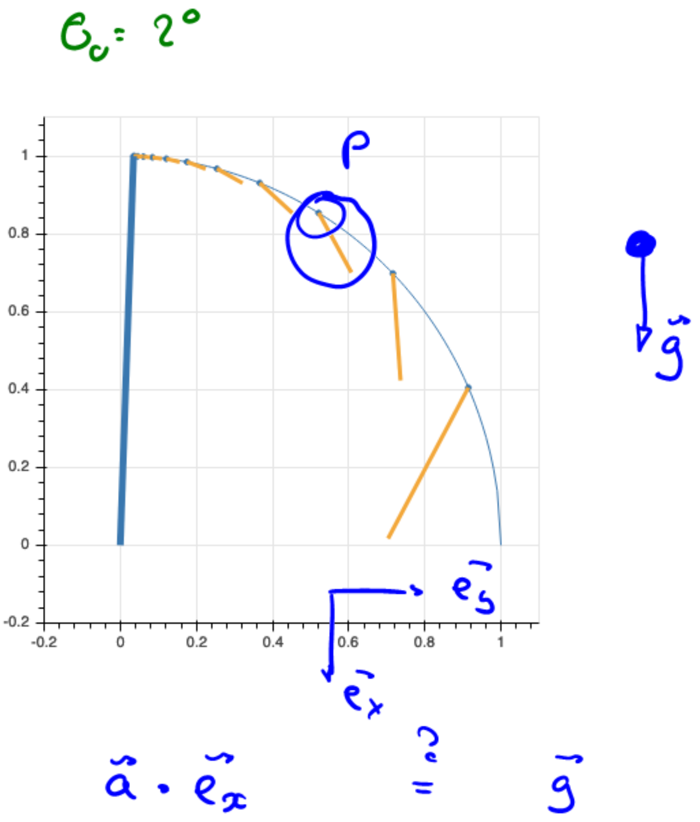
$t_1 = \Delta t$

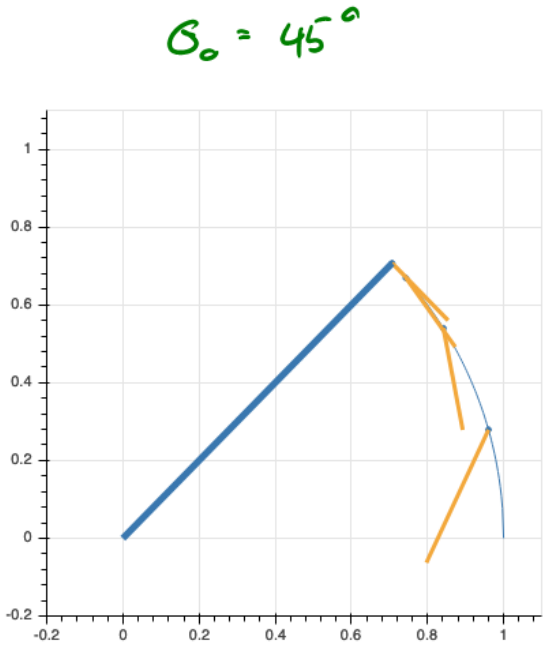
$t_2 = 2\Delta t$

$t_3 = 3\Delta t$

$\vdots$

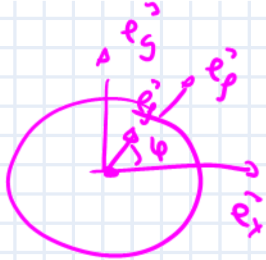
$$\vec a = -d\,\dot\theta^2\,\vec e_\rho - d\,\frac{3}{2}\,\frac{g}{\ell}\,\sin\theta\,\vec e_\varphi$$


$\theta_0 = 2°$
P
$\vec{g}$
$\vec{e}_y$
$\vec{e}_x$
$\vec{a} \cdot \vec{e}_x \overset{?}{=} \vec{g}$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.2
0
0.2
0.4
0.6
0.8
1


$\theta_0 = 45°$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.2
0
0.2
0.4
0.6
0.8
1

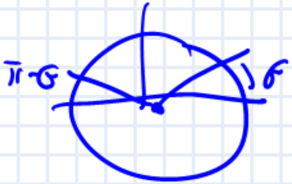
$$\vec{a}\cdot\vec{e}_x = \left(-d\,\dot{\theta}^2\,\vec{e}_\rho - d\,\frac{3}{2}\frac{g}{\ell}\sin\theta\,\vec{e}_\varphi\right)\cdot\vec{e}_x$$

$$= -d\,\dot{\theta}^2\left(\vec{e}_\rho\cdot\vec{e}_x\right) - d\,\frac{3}{2}\frac{g}{\ell}\sin\theta\left(\vec{e}_\varphi\cdot\vec{e}_x\right)$$



$$\left(\vec{e}_\rho\cdot\vec{e}_x\right) = \left(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y\right)\cdot\vec{e}_x = \cos\varphi$$

$$\left(\vec{e}_\varphi\cdot\vec{e}_x\right) = \left(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y\right)\cdot\vec{e}_x = -\sin\varphi$$

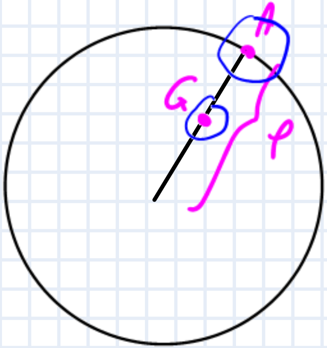


$$\left(\vec{e}_\rho\cdot\vec{e}_x\right) = \cos(\varphi) = \cos(\pi-\theta) = -\cos(\theta)$$

$$\left(\vec{e}_\varphi\cdot\vec{e}_x\right) = -\sin(\varphi) = -\sin(\pi-\theta) = -\sin(\theta)$$

$$a_x = \vec{a}\cdot\vec{e}_x = +d\,\dot{\theta}^2\cos(\theta) + d\,\frac{3}{2}\frac{g}{\ell}\sin^2\theta$$

$$a_x = \vec{a} \cdot \vec{e}_x = + d\,\dot{\theta}^2 \cos(\theta) + d\,\frac{3}{2}\frac{g}{\ell}\sin^2\theta$$



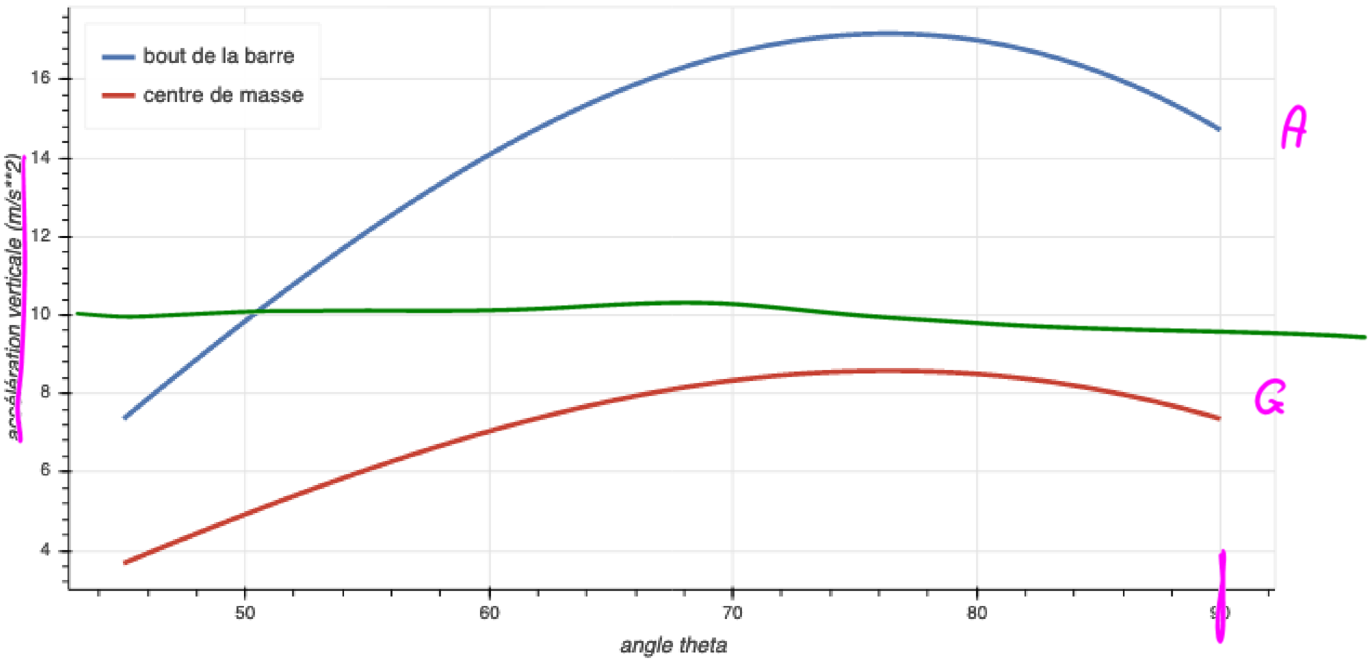
A : ℓ

$$a_{x,A} = \ell\,\dot{\theta}^2 \cos(\theta) + \frac{3}{2} g \sin^2\theta$$

G : ℓ/2

$$a_{x,G} = \frac{\ell}{2}\,\dot{\theta}^2 \cos\theta + \frac{3}{4} g \sin^2\theta$$

$\theta_0 = 45^\circ$



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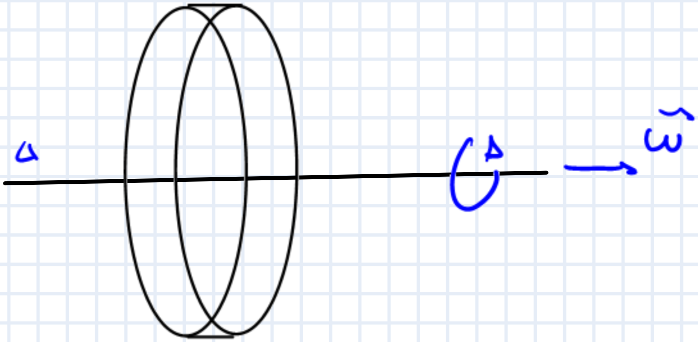
XI-2. Mouvement gyroscopique

$M \,,\, R$



$\vec{L} = I_\Delta \vec{\omega}$

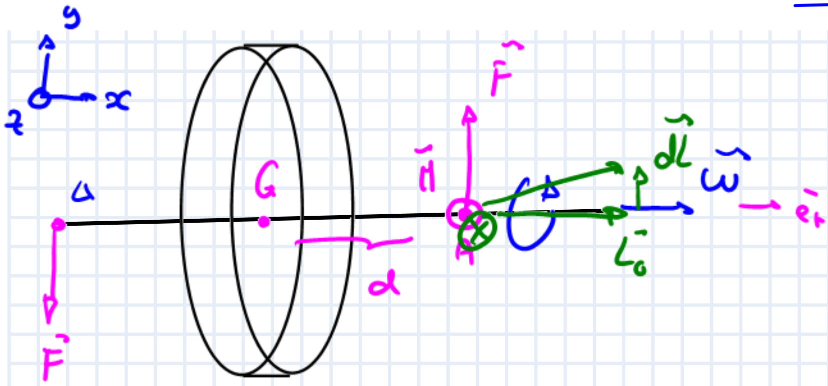
$\hat{=} M R^2 \omega$

① mvt de translation

$\vec{L} = \text{cte} \qquad \frac{d\vec{L}}{dt} = 0 = \underbrace{\vec{M}_O}_{=0}$

XI-2. Mouvement gyroscopique

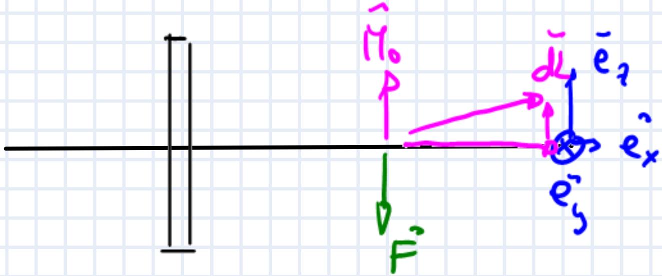
Rotation



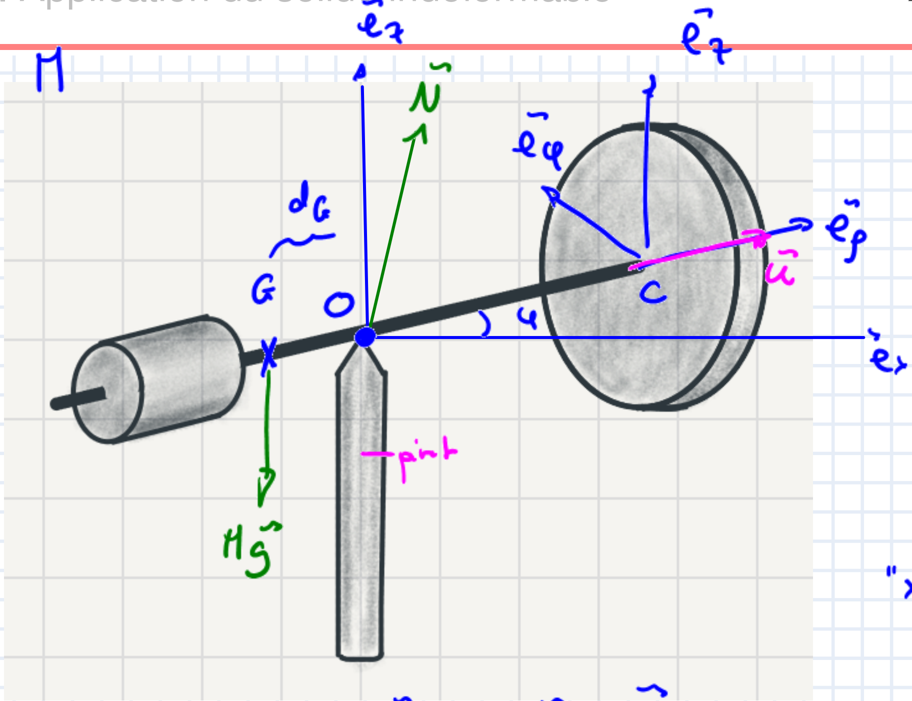
$$\vec{M}_O = 2(d\,\vec{e}_x \times F\,\vec{e}_y) = 2dF\,\vec{e}_z$$

$$\vec{M}_O = \frac{d}{dt}\vec{L}_O \sim d\vec{L}_O$$

$$\vec{L}_O(t+dt) = \vec{L}_O + dt\,\frac{d\vec{L}_O}{dt}$$



$$\vec{M}_O \sim \vec{e}_y$$



coord. cylindrique
(ρ, φ, z)

$\vec{\omega} = \omega \vec{e}_\rho \qquad \dot{\omega} = 0$

$M\vec{a}_G = M\vec{g} + \vec{N}$

$\sum \vec{M}_x = \frac{d}{dt} \vec{L}_x$

"x" $\overset{?}{=}$ 0

$$\sum \vec{M}_O = \underbrace{\vec{OO}}_{=0} \times \vec{N} + \vec{OG} \times M\vec{g}$$

$$\vec{L}_O = I_O \vec{\omega}$$ (on néglige les autres moments cinétiques)

$$\sum \vec{M}_O = \underbrace{\vec{OO} \times \vec{N}}_{=0} + \vec{OG} \times \vec{M}_g = \frac{d}{dt}\left(I_o \vec{\omega} \right) = \frac{d}{dt}\left(I_o \omega \vec{e}_\rho \right)$$

$$\left(-d_G \vec{e}_\rho\right) \times \left(-M g \vec{e}_z\right) = I_o \omega \frac{d}{dt} \vec{e}_\rho = I_o \omega \dot{\vec{e}}_\rho$$

$$-d_G M g \, \vec{e}_\varphi = I_o \omega \, \dot{\varphi} \, \vec{e}_\varphi$$

$$\boxed{\dot{\varphi} = - \frac{d_G M g}{I_o \omega}}$$

$(I_o \sim M a^2)$

vitesse angulaire de précession

$\dot{\varphi}(t) = ct$

$\dot{\varphi} \sim d_G$

$\sim M$

$\sim \frac{1}{I_o}$

$\sim \frac{1}{\omega}$