IV - Balistique

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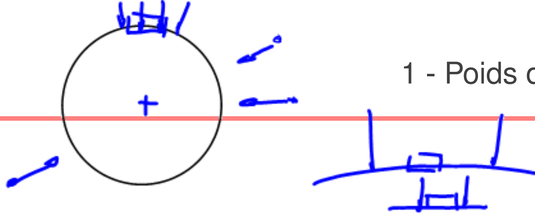
2025

Plan du cours

Table des matières



1 - Poids d'un objet

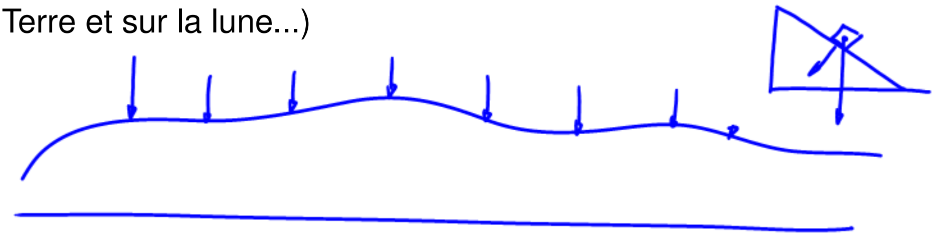
À l'échelle du laboratoire, la Terre est **plate** et l'accélération de la pesanteur $\vec{g}$ dirigée **vers le bas** et est **constante**.

$\vec{g} \triangleq 10 \frac{m}{s^2}$ $\vec{g} = 9.81... \frac{m}{s^2}$

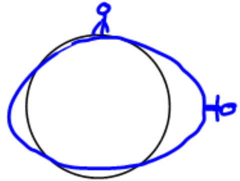
La force qui s'exerce sur une masse m est son poids $\vec{P}$:

$\vec{P} = m\vec{g}$ $g \cdot \frac{m}{s^2}$

La masse est une **propriété intrinsèque** du corps. Le poids dépend du lieu (le poids d'un cosmonaute n'est pas le même sur Terre et sur la lune...)



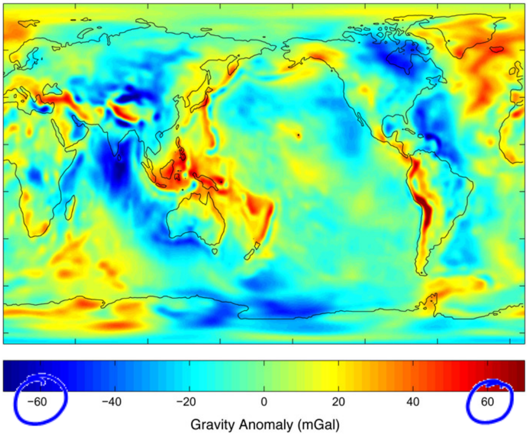
1)



2)

$g = 9.81\ldots$

$g = 10\,\frac{m}{s^2}$



$1\ \text{Gal} = 1\ \text{cm}\cdot\text{s}^{-2} = 10^{-2}\,\frac{m}{s^2}$

$1\ \text{mGal} = 10^{-5}\,\frac{m}{s^2}$

Anomalie de g par rapport à l'ellipsoide applati.

$60\ \text{mGal} = 60\ 10^{-5}\,\frac{m}{s^2} = 6\ 10^{-5}\,g$

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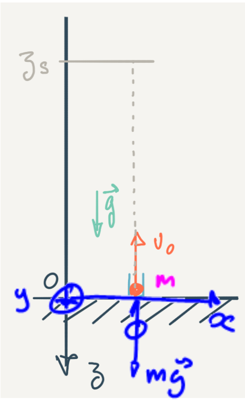
2 - Cas d'un lancer vertical (1 dimension)

$\mathcal{R}$



$\frac{d}{dt}\vec{v} = \vec{a}$

$\mathcal{R}$ inertiel (O, x, y, z)

z vers le bas

$t = 0$ $\quad x = 0 \quad y = 0, \quad z = 0$ $\qquad v_0 > 0$

$v_x = 0 \quad v_y = 0 \quad v_z \neq 0 \qquad \vec{v}(t=0) = -v_0 \vec{e}_z$

$$\boxed{\sum \vec{F}^{ext} = m\vec{a}}$$

$\vec{P} = mg\,\vec{e}_z \qquad m\vec{a} = m\,a\,\vec{e}_z$

$a(t) = g = \text{cte} \qquad \vec{a}(t) = g\,\vec{e}_z$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \int g\,\vec{e}_z \cdot dt = g\,t\,\vec{e}_z + \vec{\text{cte}}$

$\vec{v}(t=0) = \vec{\text{cte}} = -v_0 \vec{e}_z \qquad \boxed{\vec{v}(t) = g\,t\,\vec{e}_z - v_0\,\vec{e}_z}$

$$\vec{v}(t) = g t \vec{e}_z - v_0 \vec{e}_z$$

$$\vec{r}(t) = \int \vec{v}(t)\, dt = \int g t \vec{e}_z - v_0 \vec{e}_z$$

$$= \frac{1}{2} g t^2 \vec{e}_z - v_0 t \vec{e}_z + z_0 \vec{e}_z$$

$$\vec{r}(t=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad z_0 = 0 \qquad \vec{r}(t=0) = z_0 \vec{e}_z \qquad z_0 = 0$$

$$\boxed{\vec{r}(t) = \frac{1}{2} g t^2 \vec{e}_z - v_0 t \vec{e}_z} \qquad \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} g t^2 - v_0 t \end{pmatrix}$$

$$z(t) = \frac{1}{2} g t^2 - v_0 t$$

Temps de chute (chute libre)

$$\boxed{\vec{v}(t) = g t \vec{e}_z \cancel{- v_0 \vec{e}_z}}$$

$$\boxed{\vec{r}(t) = \frac{1}{2} g t^2 \vec{e}_z \cancel{- v_0 t \vec{e}_z} + z_0 \vec{e}_z}$$

$$z(t=0) = -h$$

$$\vec{r}(0) = -h \vec{e}_z$$

$$\boxed{\vec{r}(t) = \frac{1}{2} g t^2 \vec{e}_z - h \vec{e}_z}$$

$$\vec{r}(t) = 0$$

$$\frac{1}{2} g t^2 \vec{e}_z - h \vec{e}_z = \vec{0}$$

$$\frac{1}{2} g t^2 - h = 0$$

$$\boxed{t = \sqrt{2 \frac{h}{g}}}$$

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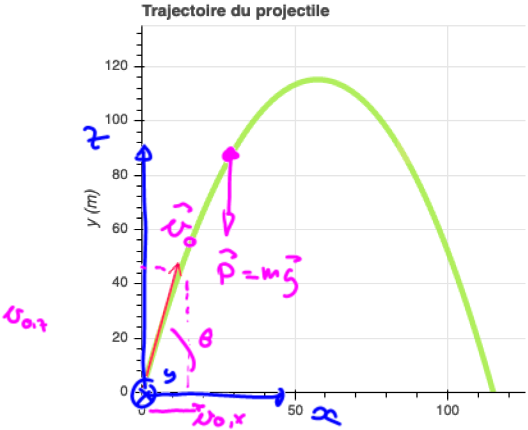
3 - Cas général



$\mathcal{R}$ in. $O(x, y, z)$

$$\vec r(t=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad v_0 = |\vec v_0|$$

$$\vec v(t=0) = \vec v_0 = \begin{pmatrix} v_0 \cos\theta \\ 0 \\ v_0 \sin\theta \end{pmatrix}$$

$$\vec P = m\vec g = -mg\,\vec e_z = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

$$\vec F = m\vec a$$

$$\begin{pmatrix} 0 \\ 0 \\ -\cancel{m}g \end{pmatrix} = \cancel{m} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$$\vec v = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \int \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} dt = \begin{pmatrix} A \\ B \\ -gt + C \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \int \begin{matrix} a_x \\ a_y \\ a_z \end{matrix} \, dt = \begin{pmatrix} A \\ B \\ \underline{-gt} + C \end{pmatrix} \qquad \vec{v}(t=0) = \begin{pmatrix} v_0 \cos\theta \\ 0 \\ v_0 \sin\theta \end{pmatrix}$$

$$A = v_0 \cos\theta \qquad B = 0 \qquad C = v_0 \sin\theta$$

$$\vec{v}(t) = \begin{pmatrix} v_0 \cos\theta \\ 0 \\ -gt + v_0 \sin\theta \end{pmatrix} \qquad \vec{v}(t) = \frac{d}{dt}\vec{r}(t) \qquad \vec{r}(t) = \int v(t)\, dt$$

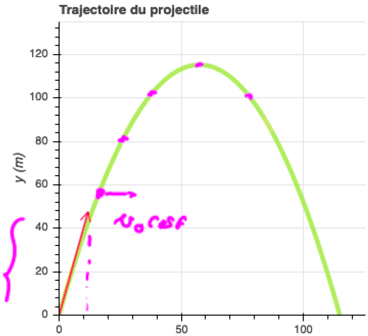
$$\vec{r}(t) = \begin{pmatrix} v_0 \cos\theta \cdot t + D \\ E \\ -\frac{1}{2} g t^2 + v_0 \sin\theta \cdot t + F \end{pmatrix} \qquad \vec{r}(t=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} D \\ E \\ F \end{pmatrix}$$

$$\vec{r}(t) = \begin{pmatrix} v_0 \cos\theta \cdot t \\ 0 \\ -\frac{1}{2} g t^2 + v_0 \sin\theta \cdot t \end{pmatrix}$$

Equ. horaire

$v_{0,z}$

$v_0 \sin\theta$



$v_0 \cos\theta$

$$\vec{a} \left| \begin{array}{l} 0 \\ 0 \\ -g \end{array} \right.$$

$$\vec{v} \left| \begin{array}{l} v_0 \cos\theta \\ 0 \\ -gt + v_0 \sin\theta \end{array} \right.$$

$$\vec{r} \left| \begin{array}{l} (v_0 \cos\theta)\, t \\ 0 \\ -\frac{1}{2} g t^2 + \underbrace{(v_0 \sin\theta)}_{v_{0,z}}\, t \end{array} \right.$$

$\vec{v}_0$

$|\vec{v}_0| = v_0$

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4 - Trajectoire, hauteur maximale, point d'impact

Chercher la trajectoire, c'est chercher z en fonction de x

$z(x)$

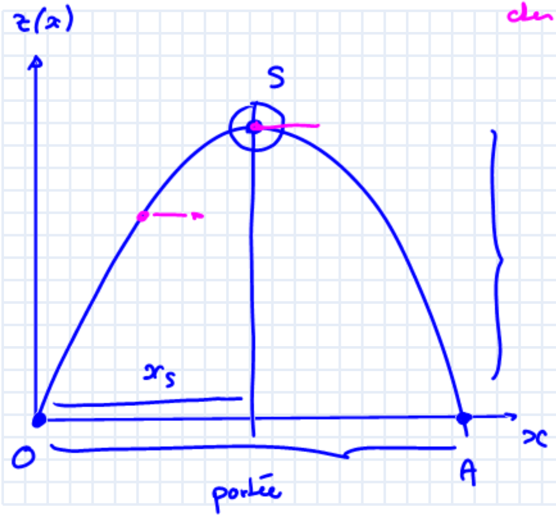
$$\vec{a}\left|\begin{array}{l}0\\0\\-g\end{array}\right. \qquad \vec{v}\left|\begin{array}{l}v_0\cos\theta\\0\\-gt+v_0\sin\theta\end{array}\right. \qquad \vec{r}\left|\begin{array}{ll}(v_0\cos\theta)t & x(t)\\0 & y(t)\\-\frac{1}{2}gt^2+(v_0\sin\theta)t & z(t)\end{array}\right.$$

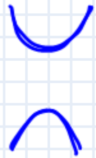
$$x(t) = v_0\cos\theta \cdot t \qquad t(x) = \frac{x}{v_0\cos\theta}$$

$$z(t(x)) = z(x) = -\frac{1}{2}g\left(\frac{x^2}{v_0^2\cos^2\theta}\right) + v_0\sin\theta\,\frac{x}{v_0\cos\theta}$$

$$z(x) = -\frac{g}{2\,v_0^2\cos^2\theta}\,x^2 + \tan\theta\; x$$

$$z = \underbrace{\overbrace{-\frac{1}{2}\frac{g}{v_0^2\cos^2\theta}}^{a}}_{\text{cte}} x^2 + \underbrace{\overbrace{\tan\theta}^{b}}_{\text{cte}} x = a x^2 + b x + \overset{\overset{0}{\shortparallel}}{c}$$





$$S = \begin{pmatrix} x_s \\ y_s = 0 \\ z_s \end{pmatrix} \rightarrow \text{hauteur}$$

$$A = \begin{pmatrix} x_A \\ 0 \\ 0 \end{pmatrix} \qquad x_A = 2 x_s$$

$$\vec{v}_s = \begin{pmatrix} v_0 \cos\theta \\ 0 \\ 0 \end{pmatrix} \qquad \vec{v}(t) = \begin{pmatrix} v_0 \cos\theta \\ 0 \\ -gt + v_0 \sin\theta \end{pmatrix}$$

$$-g t_s + v_0 \sin\theta = 0 \qquad \underline{t_s} = \frac{v_0 \sin\theta}{g}$$

$$\vec{r}(t) = \begin{pmatrix} v_0 \cos\theta \cdot t \\ 0 \\ -\frac{1}{2} g t^2 + v_0 \sin\theta \cdot t \end{pmatrix} = \begin{pmatrix} v_0 \cos\theta \, \frac{v_0 \sin\theta}{g} \\ 0 \\ -\frac{1}{2} g \frac{v_0^2 \sin^2\theta}{g^2} + v_0 \sin\theta \cdot \frac{v_0 \sin\theta}{g} \end{pmatrix}$$

$$x_s = x(t_s) = \frac{v_0^2}{g} \sin\theta \cos\theta$$

$$z_s = z(t_s) = -\frac{1}{2} \frac{v_0^2 \sin^2\theta}{g} + \frac{v_0^2}{g} \sin^2\theta = \frac{1}{2} \frac{v_0^2}{g} \sin^2\theta$$

Sommet S :

$$S \left| \begin{array}{l} \frac{v_0^2}{g} \sin\theta \cos\theta \\ 0 \\ \frac{v_0^2}{2g} \sin^2\theta \end{array} \right.$$

Point d'impact A

$$A \left| \begin{array}{l} 2\frac{v_0^2}{g} \sin\theta \cos\theta \\ 0 \\ 0 \end{array} \right. \quad = 2 \times x_S$$

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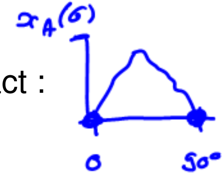
5 - Portée maximale ou atteindre une cible

On veut lancer le plus *loin* possible. A est le point d'impact :

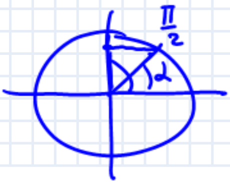


$$A \left| \begin{array}{l} 2\frac{v_0^2}{g}\sin\theta\cos\theta \\ 0 \\ 0 \end{array} \right.$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$x_A = \frac{v_0^2}{g}\sin(\underbrace{2\theta}_{\alpha})$$

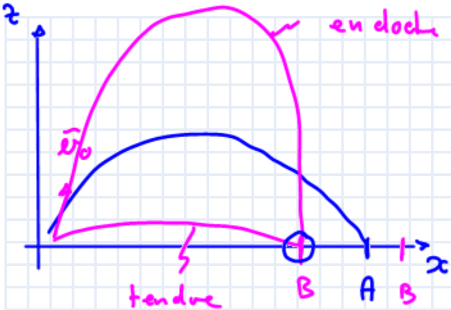
$$\theta = \frac{\alpha}{2}$$

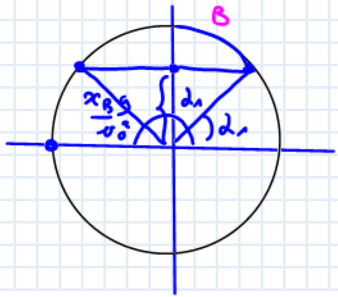


$$\alpha = \frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{4} = 45°}$$

Atteindre une cible en B (x_B)





v_0 = ok

$\theta = 45°$

$$x_B = \frac{v_0^2}{g}\sin(\underbrace{2\theta}_{\alpha}) \qquad \theta = \frac{\alpha}{2}$$

$$\boxed{\sin(\underbrace{2\theta}_{\alpha}) = \frac{x_B g}{v_0^2}}$$

$\alpha_1 \in [0, \frac{\pi}{2}] \rightarrow [0, \frac{\pi}{4}] \ni \theta_1$

$\alpha_2 \in [\pi, \frac{\pi}{2}] \rightarrow [\frac{\pi}{2}, \frac{\pi}{4}] \ni \theta_2$

$$\theta_1 = \frac{1}{2}\arcsin\left(\frac{x_B g}{v_0^2}\right)$$

$$\theta_2 = \frac{1}{2}\left[\pi - \arcsin\left(\frac{x_B g}{v_0^2}\right)\right]$$

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6 - Temps de vol

À quel temps t_A l'objet est-il en A ?

$$\vec{r}_A \left| \begin{array}{l} (v_0\cos\theta)\, t_A \\ 0 \\ -\frac{1}{2} g t_A^{\cancel{2}} + (v_0 \sin\theta)\, \cancel{t_A} \end{array} \right. = \left| \begin{array}{l} \frac{2v_0^2}{g}\sin\theta\cos\theta \\ 0 \\ 0 \end{array} \right.$$

$$t_A = \frac{\frac{2v_0^{\cancel{2}}}{g}\sin\theta\,\cancel{\cos\theta}}{\cancel{v_0}\,\cancel{\cos\theta}} = \frac{2v_0}{g}\sin\theta$$

Analyse conceptuelle

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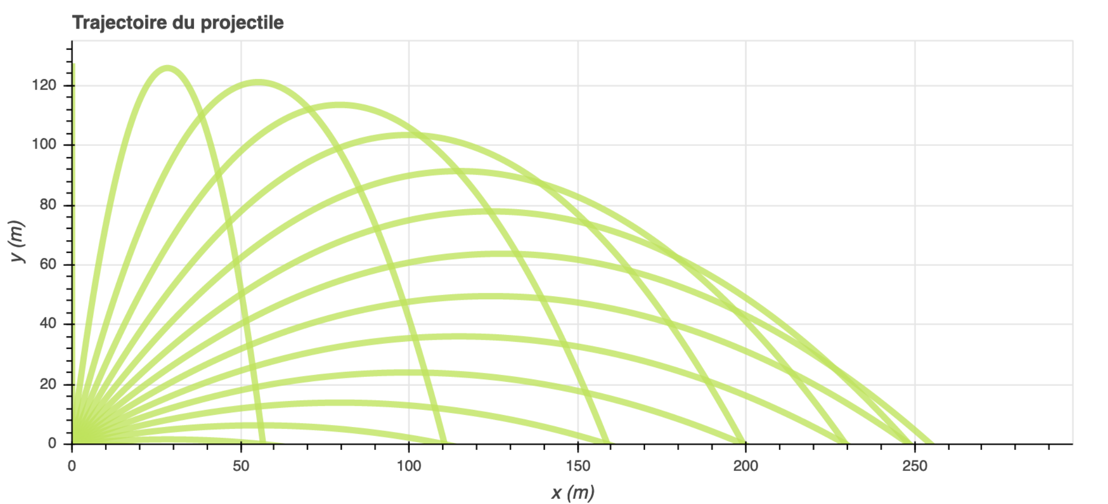
7 - Parabole de sureté

Parabole de sureté. Pour une vitesse initiale v_0 donnée, un projectile ne peut pas atteindre les points en dehors de la parabole de sureté.



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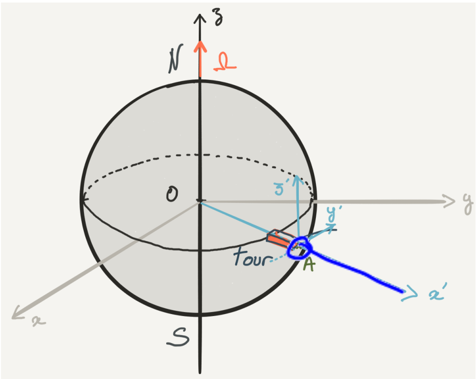
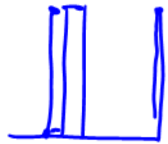
8 - Effet de la rotation de la Terre : pierre qui tombe d’une tour

On considère qu’on lâche une pierre d’une hauteur h depuis une tour située à l’équateur. De quelle distance et dans quelle direction la pierre est-elle déviée ?



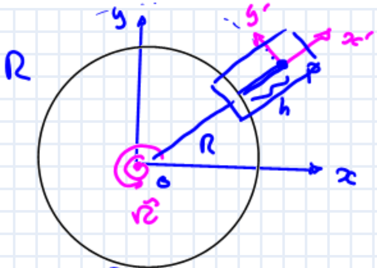


$\mathcal{R}$ inertiel (O, x, y, z)

$\mathcal{R}'$ non inertiel (A, x', y', z')

$z' = z$

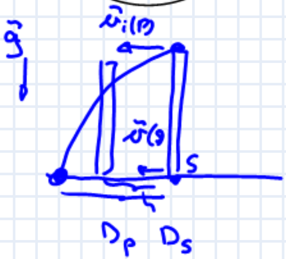
Calcul dans un référentiel innertiel, prenant en compte que les vitesses de rotations :



$$\vec{v}_i(P) = \Omega\cdot(R+h)\ \vec{e}_{y'}$$

$$\vec{v}_i(S) = \Omega\cdot(R)\ \vec{e}_{y'}$$

$$t_{chute} = \sqrt{\frac{2h}{g}}$$



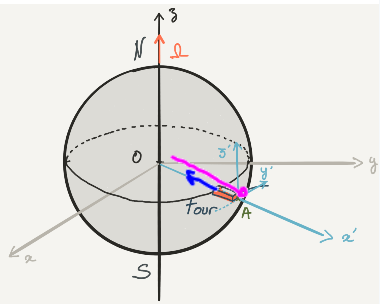
$$D_P = t_{chute}\cdot\Omega(R+h) = \sqrt{\frac{2h}{g}}\,\Omega(R+h)$$

$$D_S = t_{chute}\cdot\Omega R = \sqrt{\frac{2h}{g}}\,\Omega R$$

$$\boxed{D = D_P - D_S = \sqrt{\frac{2h}{g}}\,\Omega h} = \underline{17\ \text{cm}}$$

$h = 300\ \text{m}$

Calcul complet : On prend un repère $\mathcal{R}(O, x, y, z)$ fixe avec O centre de la Terre et un repère lié à la tour $\mathcal{R}'(A, x', y', z')$. A est le sommet de la tour (point d'où on lâche la pierre).



$$\vec{g} = -g\,\vec{e}_{x'}$$

$$\vec{\Omega} = \Omega\,\vec{e}_{z'}$$

$$\underline{\sum \vec{F}^{ext} = m\,\vec{a}_{R}(P)} = \vec{P} = m\vec{g}$$

$$\vec{a}_{R'}(P) \to \vec{v}_{R'}(P) \to \vec{r}_{R'}(P)$$

$$\vec v_R(A) = \vec\Omega \wedge \vec{OA}$$

$$\vec a_R(A) \quad \vec\Omega\times\vec\Omega\times\vec{OA}$$

$$\overbrace{\vec a_R(P)}^{\vec g} = \overbrace{\vec a_{R'}(P)}^{0} + \underbrace{\underbrace{\vec a_R(A)}_{\vec\Omega\times\vec\Omega\times\vec{OA}} + \cancel{\dot{\vec\Omega}\times\vec{AP}} + \vec\Omega\times\vec\Omega\times\vec{AP}}_{\vec\Omega\times\vec\Omega\times\vec{OP}} + 2\vec\Omega\times\vec v_{R'}(P)$$

$$\vec a_{R'}(P) = \vec g - \underbrace{\vec\Omega\times\vec\Omega\times\vec{OP}}_{\Omega\vec e_{y'}\times R\vec e_{x'}} - 2\vec\Omega\times\vec v_{R'}(P) \qquad \vec{OP} = R\vec e_{x'}$$

$$-g\vec e_{x'} \quad -\Omega\vec e_{y'}\times(\Omega R\vec e_{z'})$$

$$+\Omega^2 R\vec e_{x'}$$

$0.003\,g$

$$\vec a_{R'}(P) = -g_{eff}\,\vec e_{x'} - 2\vec\Omega\times\vec v_{R'}(P)$$

$$\vec{a}_{R'}(P) = -g_{eff}\,\vec{e}_{x'} - 2\,\vec{\Omega}\times\vec{v}_{R'}(P)$$

$$\int_0^t \vec{a}_{R'}(P)\,dt = -g_{eff}\,\vec{e}_{x'}\int_0^t dt - 2\,\vec{\Omega}\times\int_0^t \vec{v}_{R'}(P)\,dt$$

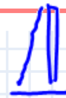
$$\vec{v}_{R'}(t) - \overbrace{\vec{v}_{R'}(t=0)}^{=0} = -g_{eff}\,\vec{e}_{x'}\,t - 2\,\vec{\Omega}\times\left(\vec{r}_{R'}(t) - \overbrace{\vec{r}_{R'}(t=0)}^{=0}\right)$$

$$\begin{pmatrix}\dot{x}'\\ \dot{y}'\\ \dot{z}'\end{pmatrix} = \begin{pmatrix}-g_{eff}\,t\\ 0\\ 0\end{pmatrix} - 2\begin{pmatrix}0\\ 0\\ \Omega\end{pmatrix}\times\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}$$

$$\begin{pmatrix}\dot{x}'\\ \dot{y}'\\ \dot{z}'\end{pmatrix} = \begin{pmatrix}-g_{eff}\,t\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}2\Omega y'\\ -2\Omega x'\\ 0\end{pmatrix}$$

$$\left.\begin{array}{l} g_{eff} = 10\,\frac{m}{s^2} \\ t \sim 1\,s \end{array}\right\} \quad g_{eff}\cdot t \sim 10\,\frac{m}{s}$$

$$y' = 0.1\,m \qquad \Omega = 7\cdot 10^{-5} \quad \sim \quad 7\,10^{-6}\,\frac{m}{s}$$



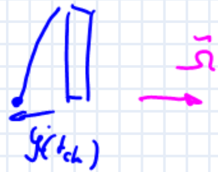
$$\left\{\begin{array}{lcll} \dot{x}' & = & -g_{eff}\,t & + \cancel{2\Omega y'} \\ \dot{y}' & = & 0 & -2\Omega x' \\ \dot{z}' & = & 0 & \end{array}\right.$$

$$z(t) = z_0 \qquad \boxed{z'(t) = 0}$$

$$\boxed{x'(t) = -\frac{1}{2} g_{eff}\, t^2}$$

$$\dot{y}'(t) = +\Omega\, g_{eff}\, t^2$$

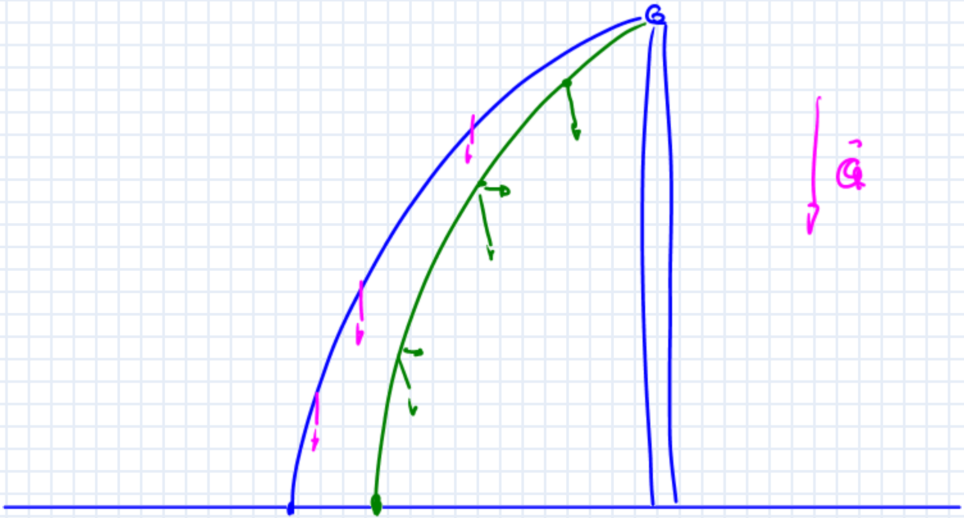
$$\boxed{y'(t) = \frac{1}{3}\Omega\, g_{eff}\, t^3} \qquad t_{chute} = \sqrt{\frac{2h}{g_{eff}}}$$



$$y'(t_{chute}) = \frac{1}{3}\Omega\, g_{eff} \left(\frac{2h}{g_{eff}}\right)^{3/2} = \frac{1}{3}\Omega\, \cancel{g_{eff}}\, \frac{2h}{\cancel{g_{eff}}} \cdot \sqrt{\frac{2h}{g_{eff}}} = \frac{2}{3}\Omega\, h \sqrt{\frac{2h}{g_{eff}}}$$

$$\sqrt{\frac{2h}{5}}\;\Omega\, h$$

8 - Effet de la rotation de la Terre



Ferdinand Reich 1799–1882

$\lambda = 51°$

Expérience en 1833 ; $h = 158$m