Solutions to Problem Set 4

Circular motion

PHYS-101(en)

1. Rotating space station

1. The required centripetal acceleration at the rim must equal g. With angular speed ω the centripetal acceleration is $\omega^2 R$ and

$$\omega^2 R = g \quad \Longrightarrow \quad \omega = \sqrt{\frac{g}{R}}$$

2. The period T is related to ω by $T = \dfrac{2\pi}{\omega}$. Using the result from (a):

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}} \quad \Longrightarrow \quad T = 2\pi\sqrt{\frac{R}{g}}$$

3. The tangential speed at the rim is $v = \omega R$. Using $\omega = \sqrt{g/R}$ from (a) gives:

$$v = \omega R = R\sqrt{\frac{g}{R}} = \sqrt{gR} \quad \Longrightarrow \quad v = \sqrt{gR}$$

   If the radius is changed from R to $2R$, the new speed is:

$$v' = \sqrt{g(2R)} = \sqrt{2}\,\sqrt{gR} = \sqrt{2}\,v$$

   so v increases by a factor of $\sqrt{2}$.

2. Banked turn

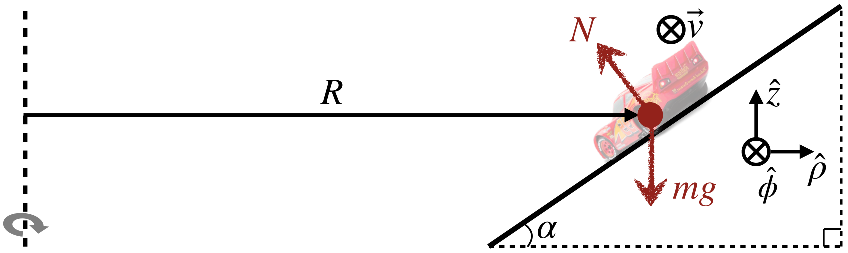
1. In this part, the static friction can be considered to be $F_s \approx 0$ because the coefficient of static friction is so small. As we are analyzing circular motion, we choose a cylindrical coordinate system as shown in the figure below, where the unit vector $\hat{\rho}$ points in the outward radial direction, $\hat{\phi}$ points into the page around the curve, and $\hat{z}$ points upwards. The free body diagram on the car is also shown.

   Given the free body diagram and the form of the centripetal acceleration $\vec{a} = -(v_0^2/R)\hat{\rho}$ for circular motion, Newton's second law $\sum \vec{F} = m\vec{a}$ in the $\hat{\rho}$ direction is

$$-N\sin\alpha = -\frac{mv_0^2}{R}$$

   We can tell that the trigonometric function in this equation is sine rather than cosine by imagining the case that $\alpha = 0$. If $\alpha = 0$, the $\hat{\rho}$ component of $\vec{N}$ would be 0. Since $\sin(0) = 0$ and $\cos(0) = 1$, the $\hat{\rho}$ component of $\vec{N}$ should contain $\sin(\alpha)$. In the $\hat{z}$ direction Newton's second law is

$$N\cos\alpha - mg = ma_z$$



Because the car is traveling in a circle and does not slide up or down, the acceleration in the $\hat{z}$ direction is zero $a_z = 0$. Thus, the components of Newton's second law become

$$N \sin\alpha = \frac{mv_0^2}{R}$$

and

$$N \cos\alpha = mg$$

Dividing these equations to eliminate N yields

$$\tan\alpha = \frac{v_0^2}{Rg}$$

which we can solve for the speed v_0 that is necessary to maintain circular motion. We find

$$v_0 = \sqrt{Rg\tan\alpha}$$

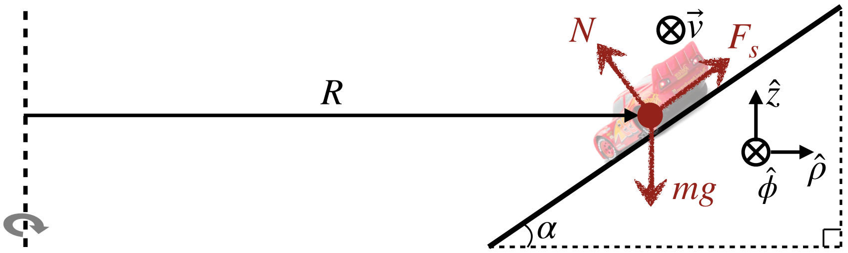
2. We now reconsider the problem with a non-zero coefficient of static friction μ_s. In this part, the speed of the car $v = v_{min}$ is so slow that it just barely doesn't slip down the bank. The static friction force must point up the incline as we know that it is preventing the car from slipping down. The free body diagram on the car is shown in the figure below, from which we see that Newton's second law is

$$-N\sin\alpha + F_s\cos\alpha = -\frac{mv^2}{R}$$

in the $\hat{\rho}$ direction and

$$N\cos\alpha + F_s\sin\alpha - mg = ma_z$$

in the $\hat{z}$ direction.



When $v = v_{min}$, the car still isn't slipping so the acceleration in the $\hat{z}$ direction $a_z = 0$ is still zero, but the static friction has its maximum magnitude of $F_s = \mu_s N$. Thus, Newton's second law becomes

$$-N\sin\alpha + \mu_s N\cos\alpha = -\frac{mv_{min}^2}{R}$$

and

$$N\cos\alpha + \mu_s N \sin\alpha = mg$$

Dividing these equations to eliminate N yields

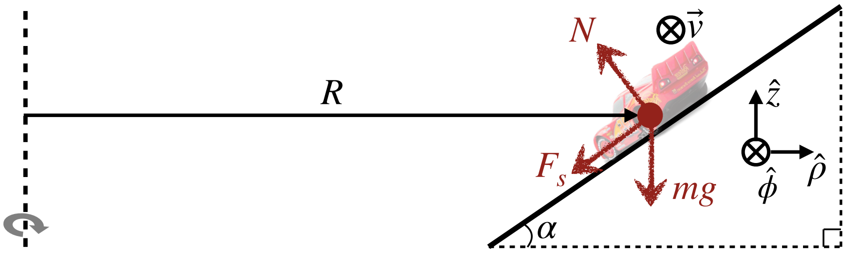
$$\frac{-\sin\alpha + \mu_s\cos\alpha}{\cos\alpha + \mu_s\sin\alpha} = -\frac{v_{min}^2}{Rg}$$

which can then be solved for the minimum speed necessary to avoid sliding down the embanked turn

$$v_{min} = \sqrt{Rg\frac{\sin\alpha - \mu_s\cos\alpha}{\cos\alpha + \mu_s\sin\alpha}} = \sqrt{Rg\frac{\tan\alpha - \mu_s}{1 + \mu_s\tan\alpha}}$$

The limiting cases of this result can be checked. In the limit $\mu_s \to 0$, $v_{min} \to \sqrt{Rg\tan\alpha}$, which is consistent with the result for part 1. In the limit $\mu_s \to \tan\alpha$, $v_{min} \to 0$, which is the solution for the static case of a block sitting on an incline.

3. We now consider the case where the car is at the maximum speed $v = v_{max}$ such that it is almost slipping up the inclined plane. For this case, the direction of static friction now points down the incline plane and the free body diagram is shown below.



The analysis is identical to the previous case, except the static friction force changes sign. Thus Newton's second law become

$$-N\sin\alpha - F_s\cos\alpha = -\frac{mv^2}{R} \tag{1}$$

in the $\hat{\rho}$ direction and

$$N\cos\alpha - F_s\sin\alpha - mg = 0 \tag{2}$$

in the $\hat{z}$ direction. When $v = v_{max}$, the static friction has its maximum value of $F_s = \mu_s N$, so Newton's second law becomes

$$-N\sin\alpha - \mu_s N\cos\alpha = -\frac{mv_{max}^2}{R}$$

and

$$N\cos\alpha - \mu_s N\sin\alpha = mg$$

Dividing these two equations to eliminate N yields

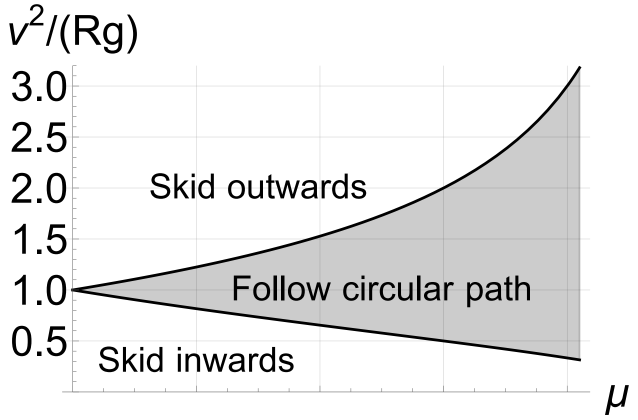
$$\frac{-\sin\alpha - \mu_s\cos\alpha}{\cos\alpha - \mu_s\sin\alpha} = -\frac{v_{max}^2}{Rg},$$

which can then be solved for the maximum speed v_{max} to avoid sliding up the embanked turn

$$v_{max} = \sqrt{Rg\frac{\sin\alpha + \mu_s\cos\alpha}{\cos\alpha - \mu_s\sin\alpha}} = \sqrt{Rg\frac{\tan\alpha + \mu_s}{1 - \mu_s\tan\alpha}}$$

This solution is identical to that of part 2, except the sign in front of μ_s is opposite.

The figure below shows a plot of $v^2/(Rg)$ versus μ_s when $\alpha = 45°$. The shaded area represents the set of $(\mu_s, v^2/(Rg))$ points where the car remains in a circular path. Above the shaded region the car will slid up and out, while below the shaded region the car will slide down and in.



4. The analysis is the same as in part 3, but the magnitude of the static friction is less than its maximum value. However, now the problem statement gives us the velocity v as a known quantity. Hence, we can combine equations (1) and (2) to eliminate N and solve for F_s. To do so, we multiply (1) by $\cos\alpha$ and (2) by $\sin\alpha$ to find

$$-N\sin\alpha\cos\alpha - F_s\cos^2\alpha = -\frac{mv^2}{R}\cos\alpha$$

and

$$N\cos\alpha\sin\alpha - F_s\sin^2\alpha - mg\sin\alpha = 0$$

Adding these two equations yields

$$-F_s\left(\cos^2\alpha + \sin^2\alpha\right) - mg\sin\alpha = -\frac{mv^2}{R}\cos\alpha$$

Using the identity $\cos^2\alpha + \sin^2\alpha = 1$ gives

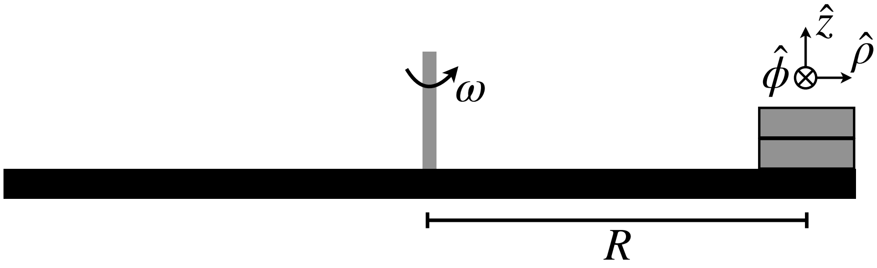
$$F_s = m\left(\frac{v^2}{R}\cos\alpha - g\sin\alpha\right)$$

for the magnitude of the static friction force.

3. Angular speed of coins

1. We choose to use a cylindrical coordinate system (because of the circular motion). For this part of the problem, it would be more efficient to analyze the situation by considering the two coins as a single combined object, which enables us to ignore internal forces (e.g. the normal and friction forces between the two coins). However, it will be helpful for the second part of this problem to treat the two coins separately. Thus, we will start by drawing separate free body diagrams for each coin (shown below). We have the weight of each coin $m\vec{g}$, the forces of static friction between the coins (both action $\vec{f}_{bt}$ and reaction $\vec{f}_{tb} = -\vec{f}_{bt}$), the force of static friction from the turntable on the bottom coin $\vec{f}_{gb}$, the

normal forces between the coins (both action $\vec{N}_{bt}$ and reaction $\vec{N}_{tb} = -\vec{N}_{bt}$), and the normal force of the turntable on the bottom coin $\vec{N}_{gb}$. Here the top coin is denoted by the t subscript, the bottom coin by b, and the turntable by g.



To determine the magnitude of the radial force exerted by the turntable on the bottom coin $\vec{f}_{gb}$, we will apply Newton's second law to each coin. A key point is that the static friction between the coins form an action-reaction pair, so the static friction that makes the top coin accelerate inward also acts to push the bottom coin radially outward.

The radial component of Newton's second law for the bottom coin is

$$f_{tb} - f_{gb} = -mR\omega^2 \tag{3}$$

where we've used that the centripetal acceleration is $\vec{a} = -R\omega^2\hat{\rho}$. The radial component of Newton's second law on the top coin is given by

$$-f_{bt} = -mR\omega^2 \quad \Rightarrow \quad f_{bt} = mR\omega^2 \tag{4}$$

Since the static friction force of the bottom coin on the top coin f_{bt} and the static friction force of the top coin on the bottom coin f_{tb} are an action-reaction pair, Newton's third law requires that their magnitudes respect $f_{bt} = f_{tb}$. Substituting this into equation (4) gives

$$f_{tb} = mR\omega^2 \tag{5}$$

Then substituting equation (5) into equation (3) yields

$$mR\omega^2 - f_{gb} = -mR\omega^2 \tag{6}$$

Hence the turntable exerts a inward radial force on the bottom coin with a magnitude of

$$f_{gb} = 2mR\omega^2 \tag{7}$$

Comparing with equation (4), we see that the static friction force on the bottom coin from the turntable is twice as large as the static friction force on the top coin.

2. When two surfaces slip, it is because the static friction force required to hold them in place has exceeded its maximum possible strength of $f^{max} = \mu N$. To calculate this, we must first find the magnitude of the normal force N between the relevant surfaces. Applying Newton's second law to the top coin in the vertical $\hat{z}$ direction yields

$$N_{bt} - mg = 0 \tag{8}$$

where we note that none of the objects in this problem are accelerating vertically so $a_z = 0$. Thus, we can calculate the normal force of the bottom coin on the top coin to be

$$N_{bt} = mg \tag{9}$$

Since this force forms an action-reaction pair with the normal force of the top coin on the bottom coin, we know that their magnitudes follow $N_{tb} = N_{bt} = mg$.

Substituting equation (9) into the form of the static friction force, we see that the top coin will slip when the static friction between the two coins reaches its maximum value of

$$f_{bt}^{max} = \mu_2 N_{bt} = \mu_2 mg \tag{10}$$

We then substitute this result into equation (4) to find

$$\mu_2 mg = mR\left(\omega_t^{max}\right)^2 \tag{11}$$

where ω_t^{max} is the maximum angular speed for which the top coin does not slip. Rearranging this, we find

$$\omega_t^{max} = \sqrt{\frac{\mu_2 g}{R}} \tag{12}$$

Newton's second law for the bottom coin in the $\hat{z}$ direction is

$$N_{gb} - N_{tb} - mg = 0 \tag{13}$$

Noting again from Newton's third law that $N_{tb} = N_{bt} = mg$, we can rearrange this equation to show that the normal force between the turntable and the bottom coin is

$$N_{gb} = 2mg \tag{14}$$

Using this and the form of the static friction force, we see that the bottom coin will slip when the static friction between it and the turntable exceeds

$$f_{gb}^{max} = \mu N_{gb} = 2\mu_1 mg \tag{15}$$

From equation (7), we can determine that the maximum angular speed after which the bottom coin slips will satisfy

$$2\mu_1 mg = 2mR\left(\omega_b^{max}\right)^2 \tag{16}$$

Rearranging we find that

$$\omega_b^{max} = \sqrt{\frac{\mu_1 g}{R}} \tag{17}$$

Comparing equations (12) and (17) and remembering that $\mu_2 < \mu_1$, we see that

$$\omega_t^{max} < \omega_b^{max} \tag{18}$$

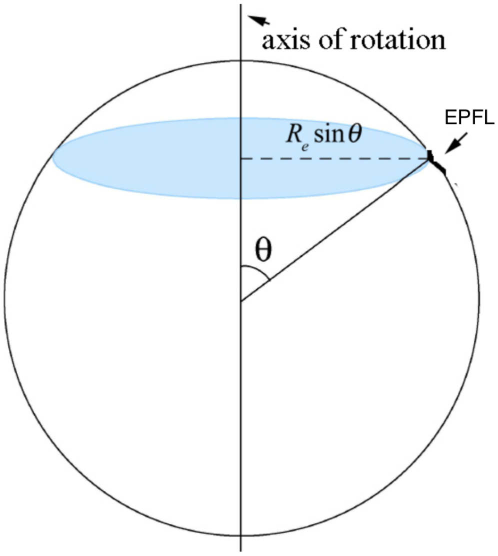
Thus, as we increase the angular velocity of the turntable, the top coin will slip first.

4. Circular motion of the earth

1. The rotational period of the earth is given by

$$T_e = (23\text{ hr})\left(60\,\frac{\text{min}}{\text{hr}}\right)\left(60\,\frac{\text{s}}{\text{min}}\right) + (56\text{ min})\left(60\,\frac{\text{s}}{\text{min}}\right) + 4\text{ s} = 86164\text{ s}$$

which is less than 24 hr. Twenty-four hours is one solar day (i.e. noon to noon), while the above period is one "sidereal" day. "Sidereal" means with respect to the fixed stars and, if you think about things, you should be able to see why the two are different. A person at EPFL undergoes circular motion about the axis of the earth (as shown in the picture). The radius of the orbit is given by



$$R = R_e \sin\theta$$

where θ is the angle between EPFL and the axis of rotation as shown in the figure. Since the latitude $\lambda = 46°31'\text{N} = (46 + 31/60)° = 46.52°$ is measured from the equator,

$$\theta = \frac{\pi}{2} - \lambda$$

and

$$\sin\theta = \sin\left(\frac{\pi}{2} - \lambda\right) = \cos\lambda$$

using trigonometric identities. The angle θ is sometimes called the "colatitude". The radius of the orbit of a person at EPFL is

$$R = R_e \cos\lambda = \left(6.38 \times 10^6\text{ m}\right)\cos(46.52°) = 4.39 \times 10^6\text{ m}$$

Because the circular motion is uniform, during one period of rotation T the person travels a distance

$$d = 2\pi R = vT$$

at a constant speed v, where $d = 2\pi R$ is the circumference. Solving for the speed gives

$$v = \frac{2\pi R}{T}$$

Thus a person at EPFL has a velocity of

$$\vec{v} = \frac{2\pi R}{T}\hat{\phi} = \frac{2\pi\left(4.39 \times 10^6 \text{ m}\right)}{(86164 \text{ s})}\hat{\phi} = \left(320 \ \frac{\text{m}}{\text{s}}\right)\hat{\phi}$$

where $\hat{\phi}$ is the unit vector pointing east.

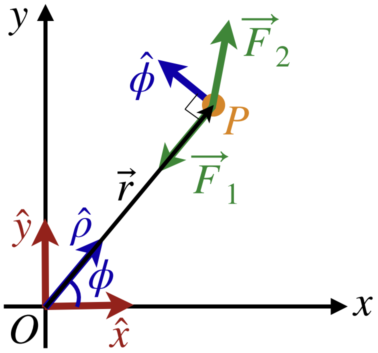
2. The centripetal acceleration is given by

$$\vec{a} = -\frac{v^2}{R}\hat{\rho} = -\frac{(320 \text{ m/s})^2}{\left(4.39 \times 10^6 \text{ m}\right)}\hat{\rho} = -2.33 \times 10^{-2} \ \frac{\text{m}}{\text{s}^2}\hat{\rho}$$

where $-\hat{\rho}$ is the unit vector pointing towards the closest point on the axis of rotation (not towards the center of the earth).

5. Spiral motion of a point mass

1. We start by representing the system in polar coordinates, as shown below. Note that, until we solve the equations of motion, we don't know the direction of $\vec{v}$ and, hence, $\vec{F}_2$, so we draw it with an arbitrary direction (and include both radial and azimuthal components to be as general as possible).



2. We are given that the forces acting on the mass are

$$\vec{F}_1 = -mk^2\vec{r}$$

$$\vec{F}_2 = -2m\lambda\vec{v} \quad \text{where} \quad 0 < \lambda < k$$

To calculate motion from forces, we will apply Newton's second law

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 = m\vec{a}$$

but we must first rewrite $\vec{F}_1$, $\vec{F}_2$, and $\vec{a}$ in polar coordinates. The position vector in polar coordinates is $\vec{r} = \rho\hat{\rho}$ (which does not include a $\hat{\phi}$ component because the direction of the radial unit vector $\hat{\rho}$ changes with time such that it always points at the point mass). Thus, the spring-like force can be written as

$$\vec{F}_1 = -mk^2\left(\rho\hat{\rho}\right)$$

The problem statement gives the form of the velocity in polar coordinates, which we can substitute to write the friction-type force as

$$\vec{F}_2 = -2m\lambda\left(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}\right)$$

Lastly, the problem statement tells us that the acceleration vector is

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi}$$

in polar coordinates.

Substituting these three equations into Newton's second law gives the equations of motion

$$\ddot{\rho} + 2\lambda\dot{\rho} + \left(k^2 - \dot{\phi}^2\right)\rho = 0$$

in the $\hat{\rho}$ direction and

$$\rho\ddot{\phi} + 2\left(\dot{\rho} + \lambda\rho\right)\dot{\phi} = 0$$

in the $\hat{\phi}$ direction. Since $\dot{\phi} \neq 0$ and $\ddot{\phi} = 0$, the equation in the $\hat{\phi}$ direction simplifies to

$$\dot{\rho} = -\lambda\rho$$

3. Given the solution form in the problem statement, we can immediately solve the $\hat{\phi}$ equation and use the initial condition $\rho(0) = \rho_0 = C$ to find

$$\rho\left(t\right) = \rho_0 e^{-\lambda t}$$

We can then rearrange the $\hat{\rho}$ equation to isolate $\dot{\phi}$ according to

$$\dot{\phi}^2 = \frac{\ddot{\rho}}{\rho} + \frac{2\lambda\dot{\rho}}{\rho} + k^2$$

Substituting our solution for $\rho(t)$, we find

$$\dot{\phi} = \pm\sqrt{k^2 - \lambda^2}$$

where we note that this is a real number as the problem statement tells us that $k > \lambda$. Integrating this equation with respect to time and using the initial condition that $\phi\left(0\right) = 0$ to determine the integration constant, we find

$$\phi\left(t\right) = \pm t\sqrt{k^2 - \lambda^2}$$

Solving this for t and substituting it into our expression for $\rho(t)$ gives

$$\rho\left(\phi\right) = \rho_0 e^{\mp\frac{\lambda\phi}{\sqrt{k^2-\lambda^2}}}$$

From the problem statement we know that the velocity in polar coordinates is

$$v = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}$$

Substituting our solutions from above, we find

$$\vec{v}(t) = -\rho(t)\left(\lambda\hat{\rho} \mp \sqrt{k^2 - \lambda^2}\hat{\phi}\right)$$

The speed is just the norm of this, which simplifies to

$$v(t) = k\rho(t)$$