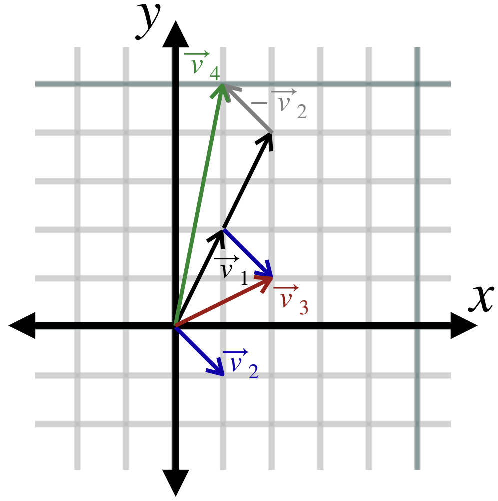
Solutions to Problem Set 2

Ballistics

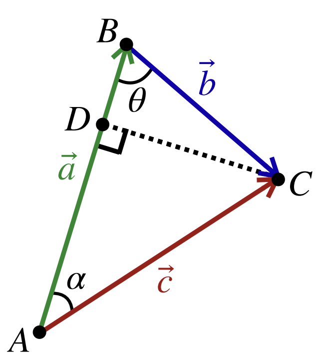
PHYS-101(en)

1. Vectors

1. See figure below.



2. We can divide the main triangle ABC into two parts using a line perpendicular to $\vec{a}$ as shown in the figure.



For the triangle BCD, simple trigonometry gives

$$\sin\theta = \frac{CD}{b}, \tag{1}$$

where $b = |\vec{b}|$ is the magnitude (i.e. length) of $\vec{b}$ and CD is the length of the line connecting points C and D. Similarly, we can see that

$$\cos\theta = \frac{BD}{b}, \tag{2}$$

where BD is the length of the line connecting points B and D.

For the triangle ACD, the Pythagorean theorem gives

$$c^2 = AD^2 + CD^2, \tag{3}$$

where $c = |\vec{c}|$ and AD is the length of the line connecting points A and D.

Lastly, we see from the figure that

$$AD + BD = a, \tag{4}$$

where $a = |\vec{a}|$.

Since $\vec{a}$, $\vec{b}$, and θ are given, we see that equations (1) through (4) are four equations that contain four unknowns (CD, BD, AD, and c). We can solve equations (1) and (4) for CD and AD respectively. Then, substitute the results into equation (3) to find

$$c^2 = (a - BD)^2 + (b\sin\theta)^2. \tag{5}$$

Now we solve equation (2) for BD and substitute the result to find

$$c^2 = (a - b\cos\theta)^2 + (b\sin\theta)^2. \tag{6}$$

Simplifying this expression and using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ gives

$$c = \sqrt{a^2 + b^2 - 2ab\cos\theta}, \tag{7}$$

which is the law of cosines.

3. From the above figure we see that

$$\sin\alpha = \frac{CD}{c}.$$

Then solving equation (1) for CD and substituting, we find

$$\sin\alpha = \frac{b}{c}\sin\theta.$$

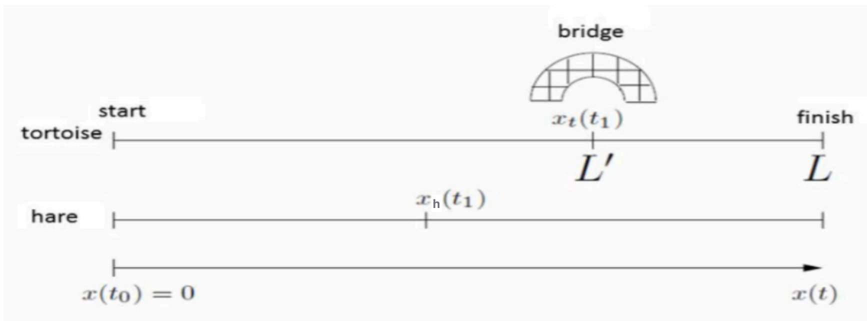
Therefore, the solution is

$$\alpha = \arcsin\left(\frac{b}{c}\sin\theta\right).$$

2. The tortoise and the hare revisited

In this problem we take as given the length of the race L, the distance to the bridge L', the velocity of the tortoise v_t, and the initial velocity of the hare v_h. We must calculate the accelerate a needed for the hare to win the race.

As in problem 3 of problem set 1, we will define $t_0 = 0$ as the time the race starts, t_1 as the time at which the tortoise reaches the bridge (which is when the hare starts to accelerate), and t_2 as the arrival time of the tortoise. Additionally, can also define the time durations $\Delta t = t_1 - t_0$ and $\Delta t' = t_2 - t_1$ of the first and second phases of the race respectively. We draw a schematic of the race in the figure below, where x_t is the position of the tortoise and x_h is the position of the hare.



During both phases of the race, the tortoise travels at the same constant speed v_t. Given that we have defined the origin to be its initial position, the tortoise's equation of motion is

$$x_t(t) = v_t t.$$

Thus, it reaches the bridge when

$$x_t(t_1) = L' = v_t t_1,$$

so

$$t_1 = \frac{L'}{v_t}. \tag{8}$$

Similarly, the tortoise reaches the finish line when

$$x_t(t_2) = L = v_t t_2,$$

so

$$t_2 = \frac{L}{v_t}. \tag{9}$$

During the first phase of the race (from t_0 to t_1), the hare maintains a constant speed v_h. Thus, its equation of motion is

$$x_h(t) = v_h t.$$

However, when the *tortoise* reaches the bridge at time t_1, the *hare's* motion changes to constant acceleration a. At t_1, the hare's velocity is still v_h and we can calculate its position to be

$$x_h(t_1) = v_h t_1.$$

By substituting equation (8), we find

$$x_h(t_1) = L' \frac{v_h}{v_t}.$$

The general equation of motion for constant acceleration can be found by integration (as we have done for the vertical direction of projectile motion) to be

$$x_h(t) = \frac{a}{2}t^2 + v_{h0}t + x_{h0}.$$

From our analysis of the first phase, we know the initial position of the second phase is $x_{h0} = L'v_h/v_t$, while the initial velocity of the second phase is $v_{h0} = v_h$. Therefore, the equation of motion becomes

$$x_h(t) = \frac{a}{2}t^2 + v_h t + L'\frac{v_h}{v_t}.$$

Note that in this equation we have adopted a new time coordinate system, defined such that the time $t = 0$ corresponds to the start of the second phase of the race. Given this, we know that the hare must arrive at the finish line $x = L$ at a time $t = t_2 - t_1$ after the start of the second phase in order to tie the tortoise. This corresponds to the condition

$$x_h(t_2 - t_1) = \frac{a}{2}(t_2 - t_1)^2 + v_h(t_2 - t_1) + L'\frac{v_h}{v_t} = L.$$

Using equations (8) and (9) to replace t_1 and t_2, we find

$$\frac{a}{2}\left(\frac{L}{v_t} - \frac{L'}{v_t}\right)^2 + v_h\left(\frac{L}{v_t} - \frac{L'}{v_t}\right) + L'\frac{v_h}{v_t} = L.$$

This equation contains only a and known quantities. To produce the simplest expression, we will first multiply the entire equation by $2v_t^2$ to find

$$a\left(L - L'\right)^2 + 2v_t v_h\left(L - L'\right) + 2v_t L' v_h = 2v_t^2 L.$$

Rearranging produces

$$a\left(L - L'\right)^2 = 2v_t^2 L - 2v_t v_h L.$$

Finally, simplifying further yields the solution of

$$a = \frac{2v_t L\left(v_t - v_h\right)}{\left(L - L'\right)^2}.$$

Therefore, in order for the hare to win the race, it must accelerate faster than this, producing the condition that

$$a > \frac{2v_t L\left(v_t - v_h\right)}{\left(L - L'\right)^2}.$$

To verify our result, we can check the units

$$\left[\frac{\text{m}}{\text{s}^2}\right] = \frac{\left[\frac{\text{m}}{\text{s}}\right][\text{m}]\left[\frac{\text{m}}{\text{s}}\right]}{[\text{m}]^2}.$$

We can also check the following limiting cases.

Limiting case 1: The tortoise has an initial velocity that is much greater than that of the hare. For this, we expect that the acceleration of the hare must be very large in order to beat the tortoise. Mathematically, we write this as $v_t \to \infty$. In this case the acceleration becomes

$$\lim_{v_t \to \infty} a = \lim_{v_t \to \infty} \frac{2v_t L\left(v_t - v_h\right)}{\left(L - L'\right)^2} = \infty$$

Limiting case 2: The tortoise and the hare have an equal velocity before the bridge. For this we expect that the hare does not need to accelerate at all in order to tie the race. Mathematically, we write this as $v_t = v_h$. We see that

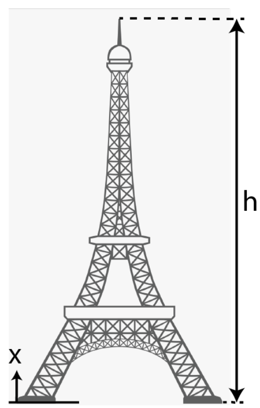
$$a = \frac{2v_t L\left(v_t - v_h\right)}{\left(L - L'\right)^2} = 0.$$

Limiting case 3: The bridge is situated close to the finish line, making the second phase of the race very short. For this we expect that the hare's acceleration will need to be very large in order to win the race. Mathematically, we write this as $L' \to L$. We see that

$$\lim_{L' \to L} a = \lim_{L' \to L} \frac{2v_t L\left(v_t - v_h\right)}{\left(L - L'\right)^2} = \infty.$$

3. Sherlock Holmes

a. For Sherlock to hear the object hit the ground, the object must reach the ground and then the sound from the impact must travel back to him. Let t_1 be the time it takes for the magnifying glass to hit the ground, t_2 be the time it takes for the sound to travel back to Sherlock, and $t_3 = t_1 + t_2$ be the total elapsed time between Sherlock releasing the magnifying glass and the noise reaching his ears. Let h be the height of the Eiffel tower. We take the x axis to go upwards, with the origin at the ground as shown in the figure below.



b. The magnifying glass undergoes one-dimensional motion under constant acceleration, so we use

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0,$$

where $a = -g$. Here the sign of the acceleration must be negative since we defined the positive x direction to be upwards. The initial velocity of the magnifying glass can be assumed to be $v_0 = 0$, while the initial position is $x_0 = h$. Thus, we find

$$x\left(t\right) = -\frac{1}{2}gt^2 + h.$$

We have defined the time t_1 as the time the magnifying glass hits the ground, so we know that

$$x\left(t_1\right) = -\frac{1}{2}gt_1^2 + h = 0. \quad (10)$$

We can solve this equation to find

$$t_1 = \sqrt{\frac{2h}{g}}. \quad (11)$$

The speed of sound v_s is constant in the Earth's atmosphere. When a noise is emitted it travels outwards at this speed in all directions. Thus, the first sound to reach Sherlock travels in a straight line at a constant speed v_s from the ground to Sherlock's ears. By integrating this constant velocity with respect to time, we see that the position of the sound is given by

$$x\left(t\right) = v_s t + x_0.$$

Given our coordinate system, the initial position of the sound is $x_0 = 0$. From our definition of the time t_2, we know that $x\left(t_2\right) = h$, so we have

$$x\left(t_2\right) = h = v_s t_2.$$

From this we can calculate the elapsed time

$$t_2 = \frac{h}{v_s}. \tag{12}$$

Lastly, we know that the total time is simply the sum of the time for the magnifying glass to reach the ground and the time for the sound to travel back to Sherlock

$$t_3 = t_1 + t_2.$$

Substituting equations (11) and (12), we find

$$t_3 = \sqrt{\frac{2h}{g}} + \frac{h}{v_s},$$

where we are interested in finding the height h.
Rearranging gives a quadratic polynomial in $\sqrt{h}$, which is

$$0 = \frac{1}{v_s}\left(\sqrt{h}\right)^2 + \sqrt{\frac{2}{g}}\sqrt{h} - t_3.$$

Using the quadratic formula, we can solve for $\sqrt{h}$ and find two solutions

$$\sqrt{h} = \frac{-\sqrt{\frac{2}{g}} \pm \sqrt{\frac{2}{g} + 4\frac{1}{v_s}t_3}}{\frac{2}{v_s}}, \tag{13}$$

which correspond to the plus versus minus signs. Squaring this equation gives

$$h = \left(\frac{-\sqrt{\frac{2}{g}} \pm \sqrt{\frac{2}{g} + 4\frac{1}{v_s}t_3}}{\frac{2}{v_s}}\right)^2. \tag{14}$$

This is an acceptable answer, but we will rearrange it further to arrive at something simpler. First we will factor out a factor of $\sqrt{2/g}$ from the numerator and combine it with the denominator to get

$$h = \left(\frac{v_s}{2}\sqrt{\frac{2}{g}}\left(-1 \pm \sqrt{1 + \frac{2gt_3}{v_s}}\right)\right)^2. \tag{15}$$

Then we will take the common factor out of the square to arrive at our final answer of

$$h = \frac{v_s^2}{2g}\left(-1 \pm \sqrt{1 + \frac{2gt_3}{v_s}}\right)^2. \tag{16}$$

c. The units of the equation can be written as

$$[\mathrm{m}] = \frac{\left[\frac{\mathrm{m}}{\mathrm{s}}\right]^2}{\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]}\left([1] \pm \sqrt{[1] + \frac{\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right][\mathrm{s}]}{\left[\frac{\mathrm{m}}{\mathrm{s}}\right]}}\right)^2.$$

From this, we see that the entire contents of the parentheses has no units, so

$$[\mathrm{m}] = [\mathrm{m}]$$

as required.

d. Equation (16) contains two solutions (one with the + and one with the −), only one of which is physically valid. To identify the correct solution, we consider the case where $v_s \to \infty$ is very large. In this case, the term $2gt_3/v_s$ becomes much smaller than 1, so it can be ignored. This yields

$$h = \frac{v_s^2}{2g} \left(-1 \pm 1\right)^2 . \tag{17}$$

We see that if we take the "−" solution, the height h definitely becomes infinite due to the prefactor $v_s^2/(2g)$ growing as $v_s \to \infty$. This does not make physical sense. On the other hand, if we take the "+" solution, as $2gt_3/v_s$ becomes negligibly small, the factor in the parenthesis becomes zero and can counteract the fact that the prefactor $v_s^2/(2g)$ is becoming very large. Thus, it can (and does) give a finite answer. A more sophisticated analysis using a Taylor expansion (beyond the scope of this course) shows that the "+" solution gives $h = gt_3^2/2$ in the limit of $v_s \to \infty$. This is the expected result as t_2 becomes 0 according to equation 12, so the total time $t_3 = t_1$ can be found using just equation (11). Thus, the physical solution is

$$h = \frac{v_s^2}{2g} \left(-1 + \sqrt{1 + \frac{2gt_3}{v_s}}\right)^2 . \tag{18}$$

e. Plugging in $t_3 = 9$ s, $v_s = 320$ m/s, and $g = 10$ m/s^2 into equation (18) gives $h = 320$ m.

4. Reference frames

We choose to use a coordinate system where x points east, y points north, and z is vertical and points to the sky. The origin of the coordinate system coincides with the initial position of the ball.

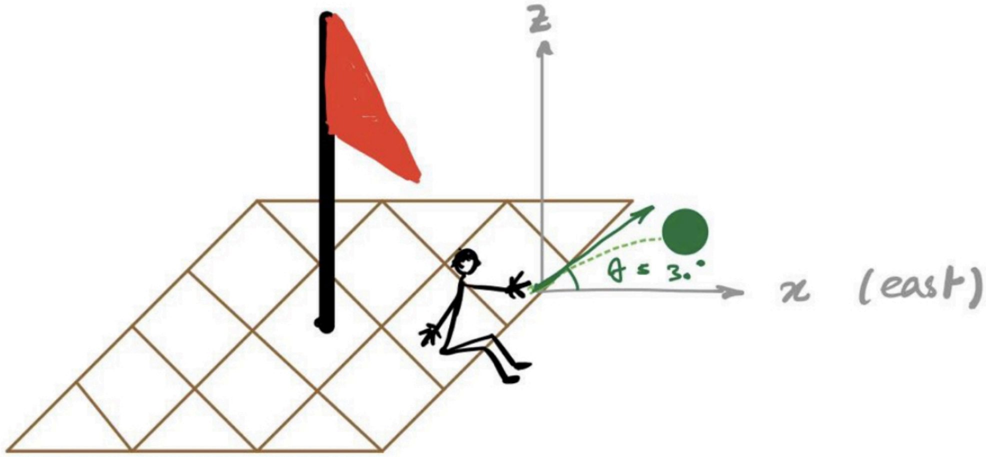


Figure 1: Cartoon of dynamics in the reference frame of the ship

a. The ship sails with a constant velocity, so the only acceleration in the entire system is due to gravity. Thus, in the ship frame the ball undergoes projectile motion, which has equations of motions given by

$$\begin{aligned} x_s(t) &= v_{x0}t + x_0 \\ y_s(t) &= v_{y0}t + y_0 \\ z_s(t) &= -\frac{1}{2}gt^2 + v_{z0}t + z_0 \end{aligned}$$

in the reference frame of the ship. Given the choice of the origin of our coordinate system, $x_0 = y_0 = z_0 = 0$. Additionally, since the ball is thrown from the ship due east, the initial velocity $v_{y0} = 0$. Using the above diagram and trigonometric identities, we see that $v_{x0} = v_b \cos\theta$ and $v_{z0} = v_b \sin\theta$. Substituting these initial conditions yields

$$\begin{aligned} x_s(t) &= v_b t \cos\theta \\ y_s(t) &= 0 \\ z_s(t) &= -\frac{1}{2}gt^2 + v_b t \sin\theta. \end{aligned}$$

These equations represent a parametric form for the trajectory. Alternatively, we can solve the equation in the x direction for time to get

$$t = \frac{x_s}{v_b \cos\theta},$$

which we substitute into the equation in the z direction to find

$$z_s(x_s) = -\frac{g}{2v_b^2 \cos^2\theta}x_s^2 + x_s \tan\theta.$$

This is the equation of a parabola lying in the xz plane (as y is a constant and equal to zero). Substituting numbers, we find

$$z_s = -0.029x_s^2 + 0.58x_s$$

and

$$y_s = 0.$$

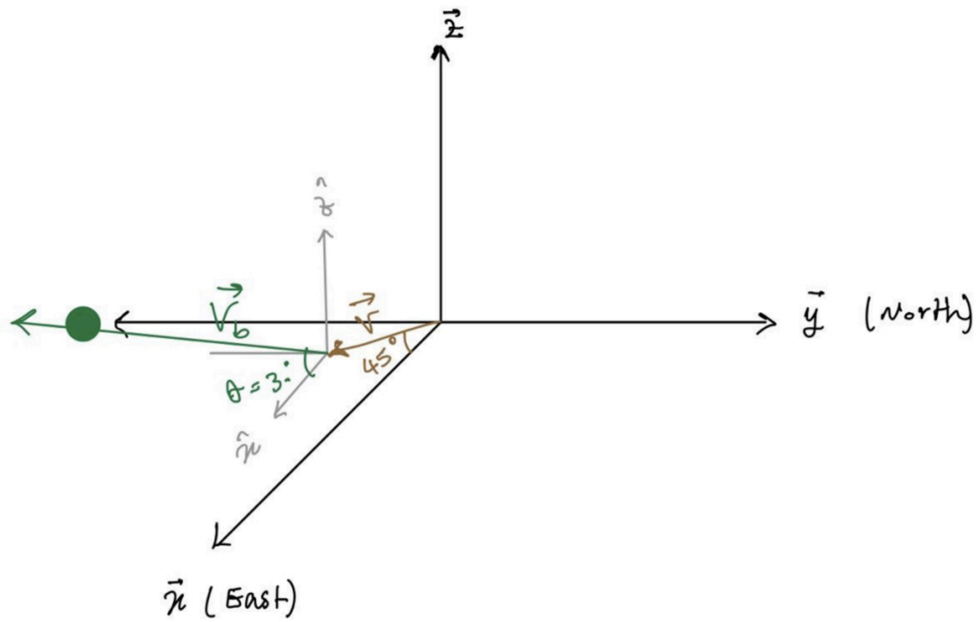


Figure 2: Cartoon of dynamics in the laboratory reference frame

b. In this part, we want to take the motion we found for the ship frame in part a and convert it into the laboratory frame. Later in the course, we will rigorously derive how to convert motion between different coordinate systems. However, here the situation is simple enough that you can visual how the motion of the ship and the ball will combine.

In the laboratory frame, there is no additional acceleration as the ship sails at constant velocity. Thus, the equations of motion remain

$$\begin{aligned} x_l(t) &= v_{x0}t + x_0 \\ y_l(t) &= v_{y0}t + y_0 \\ z_l(t) &= -\frac{1}{2}gt^2 + v_{z0}t + z_0. \end{aligned}$$

Given the choice of the origin of our coordinate system, we also retain $x_0 = y_0 = z_0 = 0$. However, the initial velocities are different. Though the boy throws the ball due east from his perspective, the fact that he is moving on a ship adds a new component to the ball's velocity. Specifically, the initial velocity is the velocity of the ball from the boy's perspective *plus the velocity of the boy.*

In the figure above, the velocity of the ball from the boy's perspective is indicated in green, while the velocity of the boy (which is the same as that of the ship) is indicated in brown. Thus, from trigonometry we see that $v_{x0} = v_b \cos\theta + v\cos\alpha$, $v_{y0} = -v\sin\alpha$, and $v_{z0} = v_b \sin\theta$, where $\alpha = 45°$ is the angle of the ship's velocity relative to due east. Substituting these initial conditions gives

$$\begin{aligned} x_l(t) &= (v_b \cos\theta + v\cos\alpha)\, t \\ y_l(t) &= -vt\sin\alpha \\ z_l(t) &= -\frac{1}{2}gt^2 + v_b t\sin\theta. \end{aligned}$$

These equations are a parametric form of the trajectory of the ball. Alternatively, we can solve the equation in the y direction for time to get

$$t = -\frac{y_l}{v \sin\alpha},$$

which we substitute into the equations in the x and z directions to find

$$\begin{aligned} x_l(y_l) &= -\left(\frac{v_b \cos\theta}{v \sin\alpha} + \frac{\cos\alpha}{\sin\alpha}\right) y_l \\ z_l(y_l) &= -\frac{g}{2v^2 \sin^2\alpha} y_l^2 - \frac{v_b}{v}\frac{\sin\theta}{\sin\alpha} y_l. \end{aligned}$$

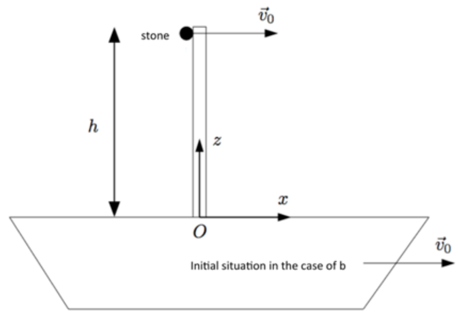
These are equations for a parabola, but one that does not lie in the x-z, nor y-z plane. Substituting numbers for the angles, we find

$$\begin{aligned} x_l(y_l) &= -\left(\sqrt{\frac{3}{2}}\frac{v_b}{v} + 1\right) y_l \\ z_l(y_l) &= -\frac{g}{v^2} y_l^2 - \frac{1}{\sqrt{2}}\frac{v_b}{v} y_l. \end{aligned}$$

Substituting numbers for the speeds (note that $v = 18$ km/hr $= 5$ m/s) and assuming that $g = 9.81$ m/s^2, we find

$$\begin{aligned} x_l(y_l) &= -4.67 y_l \\ z_l(y_l) &= -0.39 y_l^2 - 2.12 y_l. \end{aligned}$$

5. Dropping a stone from a sailboat



For parts 1 and 2 of this problem, we choose a coordinate system at rest with respect to the land when the boat is docked. The z axis is parallel to the mast and the x axis is horizontal (i.e. along to the ground). The origin corresponds to the foot of the mast at time $t = 0$, when the stone is dropped. In part 3, we choose a frame of reference relative to the fixed stars. The idea is to have a frame of reference that is fixed while the sailboat or the stone moves.

In this problem, it is crucial to realize that the initial velocity of the dropped stone is equal to the velocity of the sailboat at the precise moment when the stone is dropped.

1. The dropped stone undergoes constant acceleration due to gravity g in the downwards direction. Thus, we can apply the equations of projectile motion

$$z_{stone}(t) = -\frac{1}{2}gt^2 + v_{z0}t + z_0.$$

Given our coordinate system, the initial conditions of the stone are $v_{z0} = 0$ and $z_0 = h$. Thus, the equation of motion for the stone is

$$z_{stone}(t) = -\frac{1}{2}gt^2 + h.$$

The time it takes the stone to reach the foot of the mast t_{fall} is determined by the condition $z_{stone}(t_{fall}) = 0$, so we find

$$t_{fall} = \sqrt{\frac{2h}{g}}.$$

The dimensions are consistent as $[\mathrm{s}] = \sqrt{[\mathrm{m}]/[\mathrm{m/s^2}]}$ as required.

The distance that the stone lands relative to the foot of the mast is given by

$$d = |x_{stone}(t_{fall}) - x_{foot}(t_{fall})| = 0,$$

where $|\ldots|$ is the absolute value (or magnitude) of the quantity within and $x_{foot}(t)$ is the horizontal location of the foot of the mast. In the horizontal direction, the equation of motion for the stone is

$$x_{stone}(t) = v_{x0}t + x_0. \tag{19}$$

However, since the initial conditions are $v_{x0} = 0$ and $x_0 = 0$, we find simply

$$x_{stone}(t) = 0.$$

Likewise, since the sailboat has no motion in the x direction, the equation of motion for the foot of the mast is also $x_{foot}(t) = 0$. Thus, we find

$$d = 0.$$

2. Since the velocity of the sailboat is solely in the x direction, the equation of motion in z is identical to that of part 1,

$$z_{stone}(t) = h - \frac{1}{2}gt^2.$$

Therefore, the time it takes for the stone to land remains

$$t_{fall} = \sqrt{\frac{2h}{g}}.$$

Now, since the sailboat moves with a constant velocity in the horizontal direction, the equation of motion in x for the foot of the mast is changed to

$$x_{foot}(t) = v_0 t.$$

However, the horizontal motion of the stone is also changed. Since the girl is riding the moving sailboat, when she drops it with no initial velocity *in her reference frame*, it has an velocity $\vec{v}_0$ with respect to the reference frame of the land (in which we are doing the calculation). Thus, equation (19) becomes

$$x_{stone}(t) = v_0 t.$$

Nevertheless, the distance the stone lands relative to the foot of the mast remains unchanged

$$d = |x_{stone}(t_{fall}) - x_{foot}(t_{fall})| = v_0 t_{fall} - v_0 t_{fall} = 0.$$

3. Again, you must determine the equations of motion for the stone and the foot of the mast in the z and x directions.

In this part, we take a frame of reference at rest relative to the stars. As soon as the stone is released, it no longer experiences the constant acceleration. Thus, in the z direction, the stone follows

$$z_{stone}(t) = v_0 t + h,$$

where we have already substituted the initial velocity and position of the stone. In the x direction, the stone has no initial motion or acceleration, so it follows

$$x_{stone}(t) = 0.$$

The foot of the mast stays with the ship and continues to experience constant acceleration, so it follows

$$z_{foot}(t) = \frac{1}{2}at^2 + v_0 t.$$

The mast has no motion in the x direction, so

$$x_{foot}(t) = 0.$$

The stone reaches the foot of the mast when

$$z_{stone}(t_{fall}) = z_{foot}(t_{fall}).$$

Substituting the equations of motion for both sides gives

$$v_0 t_{fall} + h = \frac{1}{2} a t_{fall}^2 + v_0 t_{fall}.$$

Therefore,

$$t_{fall} = \sqrt{\frac{2h}{a}}.$$

Since $a = g$, the time it takes the stone to reach the foot of the mast t_{fall} remains the same as in parts 1 and 2.

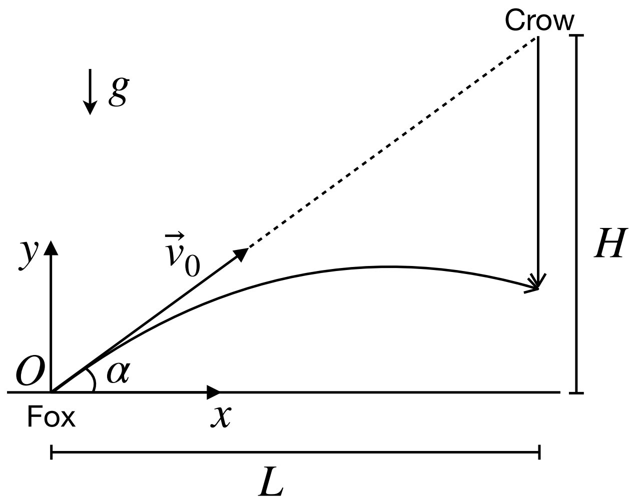
The distance the stone lands relative to the foot of the mast is still given by

$$d = abs\left(x_{stone}\left(t_{fall}\right) - x_{foot}\left(t_{fall}\right)\right),$$

so, as before, the stone falls at the foot of the mast

$$d = 0.$$

6. The fox and the crow



1. We start by taking the diagram provided with the question and defining the angle α as well as a convenient coordinate system. The origin O is the position of the fox, the angle α defines the direction of the initial velocity of the stone relative to the x axis, the y axis is defined to be anti-parallel to $\vec{g}$, and the x axis is defined to be in the direction of the initial horizontal velocity of the stone.

   Both objects experience projectile motion, so we can directly write their equations of motion. For the stone we have
$$x^s\left(t\right) = v^s_{x0}t + x^s_0$$
   and
$$y^s\left(t\right) = -\frac{1}{2}gt^2 + v^s_{0y}t + y^s_0.$$
   Similarly, for the cheese we have
$$x^c\left(t\right) = v^c_{x0}t + x^c_0$$
   and
$$y^c\left(t\right) = -\frac{1}{2}gt^2 + v^c_{0y}t + y^c_0.$$

   The initial conditions of the stone are $x^s_0 = 0$, $y^s_0 = 0$, $v^s_{x0} = v_0 \cos\alpha$, and $v^s_{y0} = v_0 \sin\alpha$, while the initial conditions of the cheese are $x^c_0 = L$, $y^c_0 = H$, $v^c_{x0} = 0$, and $v^c_{y0} = 0$. Substituting these values gives
$$x^s\left(t\right) = v_0 t \cos\alpha \tag{20}$$
$$y^s\left(t\right) = -\frac{1}{2}gt^2 + v_0 t \sin\alpha \tag{21}$$
   for the stone and
$$x^c\left(t\right) = L \tag{22}$$
$$y^c\left(t\right) = -\frac{1}{2}gt^2 + H. \tag{23}$$
   for the cheese.

For the stone and cheese to collide, there must be a single time $t = t_{coll}$ for which they are at the exact same position. We can write this condition as

$$x^s\left(t_{coll}\right) = x^c\left(t_{coll}\right) \tag{24}$$

and

$$y^s\left(t_{coll}\right) = y^c\left(t_{coll}\right). \tag{25}$$

The simplest approach to finding t_{coll} is to recognize that the cheese does not move in the x direction. Substituting the equations of motion for the stone and cheese from above, equation (24) becomes

$$v_0 t_{coll}\cos\alpha = L,$$

which implies that

$$t_{coll} = \frac{L}{v_0\cos\alpha}.$$

Using basic trigonometry and the Pythagorean theorem, the figure above shows that $\cos\alpha = L/\sqrt{L^2+H^2}$. Substituting this gives the collision time

$$t_{coll} = \frac{\sqrt{L^2+H^2}}{v_0}. \tag{26}$$

2. For a collision to always occur (using the assumption that v_0 is sufficiently large to ensure a collision), the collision cannot depend on the value of v_0. To show that this is the case, we can solve for t_{coll} using the equations of motion in the y direction.

   In the y direction we use equation (25), which is

   $$-\frac{1}{2}gt_{coll}^2 + v_0 t_{coll}\sin\alpha = -\frac{1}{2}gt_{coll}^2 + H.$$

   Crucially, we see that the gravitational acceleration terms on both sides cancel and gravity disappears from the problem, leaving

   $$v_0 t_{coll}\sin\alpha = H.$$

   Solving for the time and using trigonometry to show $\sin\alpha = H/\sqrt{L^2+H^2}$, we find that

   $$t_{coll} = \frac{H}{v_0\sin\alpha} = \frac{\sqrt{L^2+H^2}}{v_0}. \tag{27}$$

   We see that the collision times we calculated in equations (26) and (27) are identical. This means we really do have a collision – the stone and cheese are at the same x and y location at the same time. This occurs at time

   $$t_{coll} = \frac{\sqrt{L^2+H^2}}{v_0}.$$

   Note that the fact that there is a collision does not depend on the value of v_0 nor g! Conceptually, this is this case because in the absence of gravity, the stone would collide with the cheese at $y = H$. As gravity is the only force acting on both the stone and the cheese, the displacement in the y direction from H after t_{coll} is the same for the two objects. This ensures that at the time of the collision, the stone and the cheese will have the same y position regardless of v_0.

3. The position of the collision can be found by substituting the collision time t_{coll} back into the equations of motion. Using equation (20) or (22), the x position is simply

   $$x^s\left(t_{coll}\right) = x^c\left(t_{coll}\right) = L. \tag{28}$$

   Using equation (21) or (23) and a bit of algebra, we find that the y location of the collision is

   $$y^s\left(t_{coll}\right) = y^c\left(t_{coll}\right) = -\frac{1}{2}gt_{coll}^2 + H = -\frac{g}{2}\frac{L^2+H^2}{v_0^2} + H. \tag{29}$$

4. Above, we have shown that the stone and cheese always collide, without posing any restriction on the initial speed of the stone. However, in reality there is a restriction to impose – if the initial speed is not large enough, the stone will hit the ground before it hits the cheese. This does not contradict the above derivation – for low initial speeds the above derivation simply predicts that the collision would take place underground (i.e. at $y < 0$). In order for the collision to take place above ground, we must enforce that

$$y^c\left(t_{coll}\right) > 0.$$

Using equation (29), this is equivalent to

$$-\frac{g}{2}\frac{L^2 + H^2}{v_0^2} + H > 0.$$

Rearranging, we see that the constraint on the initial speed is

$$v_0 > \sqrt{g\frac{L^2 + H^2}{2H}}.$$

7. Challenge: Rugby up-and-under play

As indicated in the title, this problem is challenging. We start by defining the coordinate system such that y is upwards in the vertical direction and x is in the horizontal direction of the initial velocity of the ball. The origin is located at the position where the ball is kicked. We will denote the initial speed of the ball by v_{bi}, which we know must be less than v_{bi}^{max}. Using our general solution for projectile motion along with the initial position ($x_0 = 0$ and $y_0 = 0$) and velocity ($v_{x0} = v_{bi} \cos\alpha$ and $v_{y0} = v_{bi} \sin\alpha$), we can write the equations of motion for the ball as

$$\vec{a}_b(t) = -g\hat{y} \tag{30}$$

$$\vec{v}_b(t) = v_{x0}\hat{x} + (-gt + v_{y0})\,\hat{y} = v_{bi}\cos\alpha\hat{x} + (-gt + v_{bi}\sin\alpha)\,\hat{y} \tag{31}$$

$$\vec{r}_b(t) = (v_{x0}t + x_0)\,\hat{x} + \left(-\frac{g}{2}t^2 + v_{y0}t + y_0\right)\hat{y} = v_{bi}t\cos\alpha\hat{x} + \left(-\frac{g}{2}t^2 + v_{bi}t\sin\alpha\right)\hat{y}. \tag{32}$$

1. We want to find the distance at which the player catches the ball. To do so, we must first find the time at which the ball returns to the ground, which we will call t_1. The condition for the ball returning to the ground is $y_b(t_1) = 0$, so we can substitute the $\hat{y}$ component of equation (32) to find

$$y_b(t_1) = 0 = -\frac{g}{2}t_1^2 + v_{bi}t_1\sin\alpha. \tag{33}$$

This equation has two solutions, $t_1 = 0$ and

$$t_1 = \frac{2v_{bi}}{g}\sin\alpha. \tag{34}$$

The first solution corresponds to the time of the kick and the second corresponds to the catch, so the second solution is what we're looking for. By substituting this time into the equation for the horizontal position of the ball from equation (32), we can find the distance at which the ball lands to be

$$x_b(t_1) = v_{bi}t_1\cos\alpha = \frac{2v_{bi}^2}{g}\sin\alpha\cos\alpha. \tag{35}$$

Now we must analyze the player's motion. Since she runs at a constant velocity (that we will call v_p) and her initial position is at the origin, her position is given by

$$x_p(t) = v_p t. \tag{36}$$

Thus, at time $t = t_1$ her position is

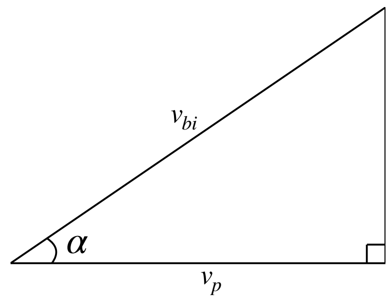
$$x_p(t_1) = v_p t_1 = \frac{2v_{bi}v_p}{g}\sin\alpha, \tag{37}$$

where we have made use of equation (34).

The ball and the player must be at the same location for a catch to occur, which we will call $\ell = x_b(t_1) = x_p(t_1)$. Thus, we require equations (35) and (37) to be equal, which allows us to determine the initial angle of the ball

$$v_{bi}\cos\alpha = v_p \quad\Rightarrow\quad \cos\alpha = \frac{v_p}{v_{bi}} \quad\Rightarrow\quad \alpha = \arccos\left(\frac{v_p}{v_{bi}}\right). \tag{38}$$

To use this information to find a simple expression for ℓ, we can draw the triangle implied by this equation (shown below). After using the Pythagorean theorem to find that the length of the missing



side is $\sqrt{v_{bi}^2 - v_p^2}$, we see that

$$\sin\alpha = \frac{\sqrt{v_{bi}^2 - v_p^2}}{v_{bi}}. \tag{39}$$

Substituting this result into equation (37) (or equation (35)) gives

$$\ell = \frac{2}{g} v_p \sqrt{v_{bi}^2 - v_p^2}. \tag{40}$$

This is the expression for the distance at which the ball lands, which we want to maximize. To do so, we can immediately see that we want to increase the initial velocity of the ball as much as possible by setting

$$v_{bi} = v_{bi}^{max}. \tag{41}$$

This is also intuitively obvious. The harder you kick the ball, the more time it will be in the air and the more time the player will have to run. The dependence on v_p is more complicated. We see that increasing it will increase the multiplying factor in front of the square root (thereby increasing ℓ), but it will also decrease the quantity in the square root (thereby decreasing ℓ). To find the maximum, we can remember our past analysis of projectile motion. The maximum vertical position occurred where the vertical velocity (which is the derivative of the vertical position) went to zero. This is a general technique to find the extrema (i.e. both maxima and minima) of functions: calculate the derivative and solve for the locations at which it is zero. Thus, we take equation (40) and calculate

$$\frac{d\ell}{dv_p} = 0 = \frac{2}{g}\sqrt{v_{bi}^2 - v_p^2} + \frac{2}{g} v_p \left(\frac{1}{2} \frac{-2v_p}{\sqrt{v_{bi}^2 - v_p^2}} \right) = \frac{2}{g}\sqrt{v_{bi}^2 - v_p^2} - \frac{2}{g}\frac{v_p^2}{\sqrt{v_{bi}^2 - v_p^2}} = \frac{2}{g\sqrt{v_{bi}^2 - v_p^2}}\left(v_{bi}^2 - 2v_p^2\right) \tag{42}$$

using the chain rule and product rules. Simplifying this expression, we find that there is only one extrema and it occurs at

$$v_p = \frac{v_{bi}}{\sqrt{2}}. \tag{43}$$

Substituting this result into equation (40) and comparing with any other choice of v_p (e.g. $v_p = 0$), we can verify that this extrema is, in fact, a maxima (as opposed to a minima). Thus, this is the optimal speed that the player would ideally run at. If this isn't possible because $v_p = v_{bi}/\sqrt{2} > v_p^{max}$, the player should run as close as possible to this value, namely at their maximum speed of v_p^{max}. Therefore, we have to explicitly distinguish these two possibilities by writing

$$v_p = \begin{cases} v_p^{max} & \text{if } v_{bi}/\sqrt{2} > v_p^{max} \\ v_{bi}/\sqrt{2} & \text{otherwise} \end{cases}. \tag{44}$$

Combining equations (40), (41), and (44), we find that the maximum distance to catch the ball is

$$\ell = \begin{cases} (2v_p^{max}/g)\sqrt{(v_{bi}^{max})^2 - (v_p^{max})^2} & \text{if } v_{bi}^{max}/\sqrt{2} > v_p^{max} \\ (v_{bi}^{max})^2/g & \text{otherwise} \end{cases}. \tag{45}$$

Combining equations (38), (41), and (44), we find that ideal angle to kick the ball is

$$\alpha = \begin{cases} \arccos\left(v_p^{max}/v_{bi}^{max}\right) & \text{if } v_{bi}^{max}/\sqrt{2} > v_p^{max} \\ \arccos\left(1/\sqrt{2}\right) = \pi/4 = 45° & \text{otherwise} \end{cases}. \tag{46}$$

The interpretation of these results is that if you are sufficiently fast (i.e. v_p^{max} is sufficiently large), the second case in all three equations applies. In this case, you want to kick the ball at $\alpha = 45°$, as this is the angle that maximizes the distance traveled by the ball, and run below your maximum speed, such that you arrive at the same time and place as the ball when it lands. However, the more realistic case is the first, that you can out-kick your running speed. In this case you want to run at your maximum speed and angle your kick higher (i.e. $\alpha > 45°$) so that the ball stays in the air for longer and you have more time to run.

2. The position of the ball is given by equation (32). Using equations (38) and (39), we can write equation (32) as

$$\vec{r}_b(t) = v_p t\hat{x} + \left(-\frac{g}{2}t^2 + t\sqrt{v_{bi}^2 - v_p^2}\right)\hat{y}.$$

Solving the x component of this equation (i.e. $x_b(t) = v_p t$) for time gives $t = x_b/v_p$, which we can substitute into the y component to find

$$y_b(x_b) = -\frac{g}{2}\left(\frac{x_b}{v_p}\right)^2 + \frac{x_b}{v_p}\sqrt{v_{bi}^2 - v_p^2} = -\frac{g}{2v_p^2}x_b^2 + x_b\sqrt{\frac{v_{bi}^2}{v_p^2} - 1}.$$

This is the trajectory of the ball. To find where the defense should be placed, we need to determine at what x position the height of the ball is equal to that of the defense player's hand. Therefore, we set $y_b(x_b) = h$ to find

$$h = -\frac{g}{2v_p^2}x_b^2 + x_b\sqrt{\frac{v_{bi}^2}{v_p^2} - 1} \quad \Rightarrow \quad 0 = \frac{g}{2v_p^2}x_b^2 - x_b\sqrt{\frac{v_{bi}^2}{v_p^2} - 1} + h,$$

which we want to solve for x_b. This is a quadratic equation, which we can solve by first computing the discriminant

$$\Delta = \frac{v_{bi}^2}{v_p^2} - 1 - 4\frac{gh}{2v_p^2} = \frac{v_{bi}^2 - v_p^2 - 2gh}{v_p^2}$$

and then the solution

$$x_b = \frac{v_p}{g}\left(\sqrt{v_{bi}^2 - v_p^2} \pm \sqrt{v_{bi}^2 - v_p^2 - 2gh}\right).$$

We see there are 2 solutions — the shorter distance corresponds to the defense player catching the ball on its way up, and the longer distance corresponds to catching it on the way down. We arrive at the final answer by substituting equations (41) and (44) into the longer distance to get

$$x_b = \begin{cases} (v_p^{max}/g)\left(\sqrt{(v_{bi}^{max})^2 - (v_p^{max})^2} + \sqrt{(v_{bi}^{max})^2 - (v_p^{max})^2 - 2gh}\right) & \text{if } v_{bi}^{max}/\sqrt{2} > v_p^{max} \\ (v_{bi}^{max}/(2g))\left(\sqrt{(v_{bi}^{max})^2} + \sqrt{(v_{bi}^{max})^2 - 4gh}\right) & \text{otherwise} \end{cases}.$$