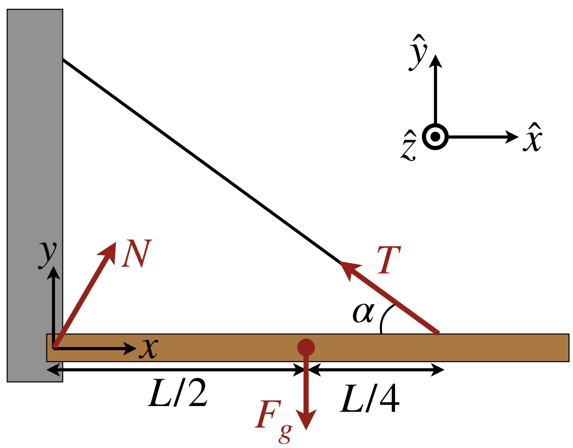
Solutions to Problem Set 11

Rigid body rotation and static equilibrium

PHYS-101(en)

1. The beam

The are three forces acting on the beam. The gravitational force (which acts at the beam's center of mass), the tension force from the rope (which acts at the point of connection between the rope and the beam), and the normal force from the wall (which acts at the point of contact between the wall and the beam). These are illustrated in the figure below.



The beam is in equilibrium when

$$\sum \vec{F} = 0 \tag{1}$$

and

$$\sum \vec{\tau} = 0, \tag{2}$$

where we will choose to calculate the torque about an axis of rotation in the $\hat{z}$ direction passing through the origin (i.e. the leftmost end of the beam). Given the forces in the problem, equation (1) gives

$$\vec{N} + \vec{T} + \vec{F}_g = 0, \tag{3}$$

where

$$\vec{N} = N_x \hat{x} + N_y \hat{y} \tag{4}$$

$$\vec{T} = -T \cos\alpha \hat{x} + T \sin\alpha \hat{y} \tag{5}$$

$$\vec{F}_g = -Mg\hat{y}. \tag{6}$$

Thus, the x component of equation (3) is

$$N_x - T\cos\alpha = 0 \quad \Rightarrow \quad N_x = T\cos\alpha \tag{7}$$

and the y component is

$$N_y + T\sin\alpha - Mg = 0 \quad \Rightarrow \quad N_y = Mg - T\sin\alpha. \tag{8}$$

To calculate the torques $\vec{\tau} = \vec{r} \times \vec{F}$ for use in equation (2), we must consider the position vector $\vec{r}$ from the origin to the point of application of the force. For the normal force from the wall this is $\vec{r}_N = 0$ as the force is applied at the origin. For the gravitational force this is $\vec{r}_g = (L/2)\hat{x}$ as it acts at the center of mass, which is in the middle (given that the beam is uniform). The point of application of the tension is shown in the problem statement to be $\vec{r}_T = (3L/4)\hat{x}$. Thus, equation (2) becomes

$$\sum \vec{\tau} = \vec{r}_N \times \vec{N} + \vec{r}_g \times \vec{F}_g + \vec{r}_T \times \vec{T} = \frac{L}{2}\hat{x} \times (-Mg\hat{y}) + \frac{3}{4}L\hat{x} \times (-T\cos\alpha\hat{x} + T\sin\alpha\hat{y}) = 0, \tag{9}$$

where we have used equations (5) and (6). Using the right-hand rule to simplify the cross products (e.g. $\hat{x} \times \hat{x} = 0$), we find

$$-\frac{MgL}{2}\hat{z} + \frac{3}{4}LT\sin\alpha\hat{z} = 0 \quad \Rightarrow \quad T = \frac{2Mg}{3\sin\alpha}. \tag{10}$$

Plugging this result into equation (5) gives

$$\vec{T} = \frac{2Mg}{3}\left(\frac{-1}{\tan\alpha}\hat{x} + \hat{y}\right). \tag{11}$$

To determine the normal force, we substitute equation (10) into equations (7) and (8) to find

$$N_x = \frac{2Mg}{3\tan\alpha} \tag{12}$$

$$N_y = Mg - \frac{2Mg}{3} = \frac{Mg}{3}. \tag{13}$$

From equation (4) we see that

$$\vec{N} = \frac{Mg}{3}\left(\frac{2}{\tan\alpha}\hat{x} + \hat{y}\right). \tag{14}$$

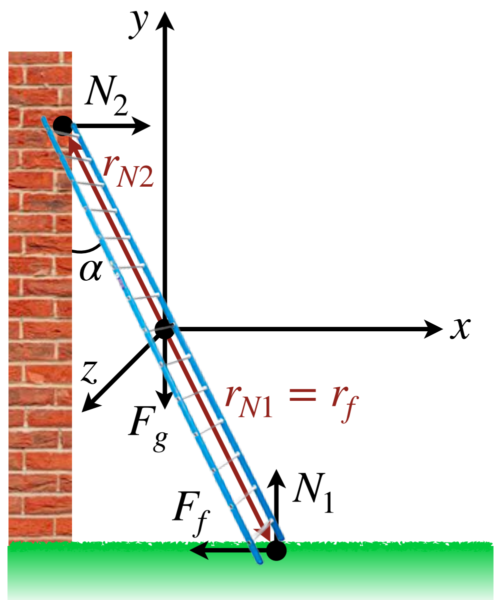
Thus, the vector expressions for the forces are given by equations (6), (11), and (14), while the points of application are indicated in the above plot.

2. The leaning ladder

The forces acting on the ladder are

- its weight $\vec{F}_g = -mg\hat{y}$ applied at the ladder's center of mass,
- the normal force of the ground $\vec{N}_1 = N_1\hat{y}$ applied at the point of contact between the ladder and the ground,
- the normal force of the wall $\vec{N}_2 = N_2\hat{x}$ applied at the point of contact between the ladder and the wall, and
- the friction force between the ladder and the ground $\vec{F}_f = -F_f\hat{x}$ applied at the point of contact between the ladder and the ground.

These are shown in the figure below, where we have define a Cartesian coordinate system with its origin at the center of mass of the ladder.



We know that when $\alpha < \alpha_m$, the ladder is in equilibrium. In this case, Newton's second law (for the extended system composed of the entire ladder) can be written as

$$\sum \vec{F} = m\vec{a} \quad \Rightarrow \quad m\vec{g} + \vec{N}_1 + \vec{N}_2 + \vec{F}_f = 0. \tag{1}$$

Projecting this into the $\hat{x}$ and $\hat{y}$ directions gives

$$N_2 - F_f = 0 \quad \Rightarrow \quad N_2 = F_f \tag{2}$$

and

$$N_1 - mg = 0 \quad \Rightarrow \quad N_1 = mg \tag{3}$$

respectively.

Equations (2) and (3) represent a system of two equations, but we have three unknowns: N_1, N_2, and F_f (we know the maximum value of the friction force is $F_f^{max} = \mu N_1$, but we don't know how F_f depends on α). To solve the problem, we must consider the torque on the ladder $\vec{\tau} = \vec{r} \times \vec{F}$. In equilibrium, the ladder does not turn, so the net torque must be zero. We choose the center of mass of the ladder (i.e. the origin in the above diagram) to be the pivot point. Then, we write down the net torque about this point from all the forces in the problem

$$\vec{\tau}_{net} = \sum \vec{\tau} = \vec{r}_g \times \vec{F}_g + \vec{r}_{N1} \times \vec{N}_1 + \vec{r}_{N2} \times \vec{N}_2 + \vec{r}_f \times \vec{F}_f = 0, \tag{4}$$

where the position vectors go *from* the pivot point *to* the point of application of the force (as shown in the above diagram). Given that the position vector for the gravitational force is $\vec{r}_g = 0$, we find

$$\vec{r}_{N1} \times \vec{N}_1 + \vec{r}_{N2} \times \vec{N}_2 + \vec{r}_f \times \vec{F}_f = 0. \tag{5}$$

The magnitude of all the other position vectors is $|\vec{r}_{N1}| = |\vec{r}_{N2}| = |\vec{r}_f| = L/2$. Using the right-hand rule for the cross product, we can determine the direction of each term in equation (5). Each one is in the $\hat{z}$ direction, however the sign is positive for $\vec{N}_1$ and negative for $\vec{F}_f$ and $\vec{N}_2$. Next, we can use the cross product definition that $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta = AB \sin\theta$ (where θ is the angle between $\vec{A}$ and $\vec{B}$). This gives

$$\frac{L}{2} N_1 \sin\alpha - \frac{L}{2} N_2 \sin\left(\frac{\pi}{2} - \alpha\right) - \frac{L}{2} F_f \sin\left(\frac{\pi}{2} - \alpha\right) = 0. \tag{6}$$

Using the trigonometric identity that $\sin(\pi/2 - \alpha) = \cos\alpha$ gives

$$N_1 \sin\alpha - N_2 \cos\alpha - F_f \cos\alpha = 0. \tag{7}$$

This is the third equation that we need to solve our system of equations. Thus, we substitute equations (2) and (3) to obtain

$$mg \sin\alpha - F_f \cos\alpha - F_f \cos\alpha = 0 \quad \Rightarrow \quad F_f = \frac{mg}{2} \tan\alpha. \tag{8}$$

The static friction will be able to restrain the ladder as long as

$$F_{Fr} < F_{Fr}^{max} = \mu N_1 = \mu mg, \tag{9}$$

where we have used equation (3). Substituting equation (8) into equation (9) allows us to determine the maximum angle for which the ladder does not fall down to be

$$\tan\alpha_m = 2\mu \quad \Rightarrow \quad \alpha_m = \arctan(2\mu). \tag{10}$$

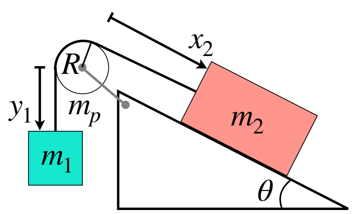
This condition is independent of the mass of the ladder m.

3. Frictionless funicular

Since there are no nonconservative forces in the problem, we can impose conservation of mechanical energy on the entire system (i.e. car, counterweight, and pulley) and write

$$E_{mi} = E_{mf}, \tag{1}$$

where the i subscript indicates the state just before the funicular is released and the f subscript indicates the state after the car has moved a distance d. We will quantify the positions of the car and counterweight (x_2 and y_1 respectively) using the coordinate systems shown in the diagram below.



We will choose the reference point for the gravitational potential energy to be at the height of the center of the pulley, which we see is at $x_2 = 0$ and $y_1 = 0$.

The initial mechanical energy is only the gravitational potential energy of the car and counterweight (as the center of mass of the pulley is at the reference point of the gravitational potential). Thus, we have

$$E_{mi} = U_{gi} = -m_1 g y_{1i} - m_2 g x_{2i} \sin\theta, \tag{2}$$

where y_{1i} is the initial position of the counterweight and x_{2i} is the initial position of the car.

After the cart has moved by a distance d, the mechanical energy is composed of the gravitational potential energy of the car and counterweight, the translational kinetic energy of the car and counterweight, and the rotational kinetic energy of the pulley. This can be written as

$$E_{mf} = U_{gf} + K_f = (-m_1 g y_1 - m_2 g x_2 \sin\theta) + \left(\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 + \frac{I_p}{2} \omega^2\right), \tag{3}$$

where x_2 and y_1 are the locations of the car and counterweight respectively, v_1 and v_2 are the speeds of the counterweight and car respectively, and ω is the angular speed of the pulley.

Since the rope is inextensible and directly connects the two blocks, the car and the counterweight (as well as every point of the rope) move at the same speed. This is the constraint condition, which is expressed as

$$v = v_1 = v_2. \tag{4}$$

Since the rope does not slip on the pulley, the points on the outer rim of the pulley move with a tangential speed v_ϕ equal to the speed of the rope, car, and counterweight

$$v_\phi = R\omega = v \quad \Rightarrow \quad \omega = \frac{v}{R}. \tag{5}$$

Thus, using equations (4) and (5), we can write equation (3) as

$$E_{mf} = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{m_1}{2} v^2 + \frac{m_2}{2} v^2 + \frac{I_p}{2} \frac{v^2}{R^2}. \tag{6}$$

Substituting equations (2) and (6) into the conservation of mechanical energy given by equation (1) yields

$$-m_1 g y_{1i} - m_2 g x_{2i} \sin\theta = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{m_1}{2} v^2 + \frac{m_2}{2} v^2 + \frac{I_p}{2} \frac{v^2}{R^2}, \tag{7}$$

which simplifies to

$$-m_1 g \left(y_{1i} - y_1\right) - m_2 g \left(x_{2i} - x_2\right) \sin\theta = \frac{1}{2}\left(m_1 + m_2 + \frac{I_p}{R^2}\right) v^2. \tag{8}$$

From the geometry of the problem and the fact that d is the distance the car travels *down* the inclined plane (and must be a positive number), we see that

$$d = x_2 - x_{2i}, \tag{9}$$

while

$$d = y_{1i} - y_1. \tag{10}$$

Substituting these gives

$$gd\left(-m_1 + m_2 \sin\theta\right) = \frac{1}{2}\left(m_1 + m_2 + \frac{I_p}{R^2}\right) v^2. \tag{11}$$

We can now solve the for the speed as a function of the distance, which gives

$$v(d) = \sqrt{2gd \frac{m_2 \sin\theta - m_1}{m_1 + m_2 + I_p/R^2}}. \tag{12}$$

Since the problem statement tells us that $I_p = m_p R^2/2$, we can also write this as

$$v(d) = \sqrt{2gd \frac{m_2 \sin\theta - m_1}{m_1 + m_2 + m_p/2}}. \tag{13}$$

4. Pendulum and disk

1. The total moment of inertia of a system about the axis of rotation P is defined to be

$$I_P^{tot} = \int_M \rho^2 dm, \tag{1}$$

where ρ is the distance from the axis P and the integral is performed over the entire mass of the object. Since an integral is just a sum of infinitesimal elements, we can divide the integral into the sum of integrals over the two parts of the system according to

$$I_P^{tot} = I_P^{rod} + I_P^{disk}, \tag{2}$$

where

$$I_P^{rod} = \int_{rod} \rho^2 dm \tag{3}$$

and

$$I_P^{disk} = \int_{disk} \rho^2 dm. \tag{4}$$

We will first calculate I_P^{disk}. Using the parallel axis theorem, we can relate the moment of inertia of the disk about point P I_P^{disk} to its moment of inertia about its center of mass I_{CM}^{disk} according to

$$I_P^{disk} = I_{CM}^{disk} + m_2 L_2^2. \tag{5}$$

The moment of inertia of a disk about its center of mass can be found from a table, which is perfectly acceptable. However, here we will show how to derive it from the definition of the moment of inertia

$$I_{CM}^{disk} = \int_{disk} \rho^2 dm, \tag{6}$$

where ρ is the distance from the center of mass of the disk. Since the disk is uniform, we know that its center of mass is at its geometric center and it has an areal density of

$$\sigma = \frac{m_2}{\pi r_2^2} = \frac{\Delta m}{\Delta A}, \tag{7}$$

where ΔA is a differential element of area. We can use σ to rewrite the integral over mass in equation (6) as an integral over area according to

$$I_{CM}^{disk} = \int_{disk} \rho^2 \sigma dA. \tag{8}$$

While the differential area is $dA = dxdy$ in Cartesian coordinates, we would like to use polar coordinates to reflect the geometry of the disk. In polar coordinates $dA = \rho d\phi d\rho$, which you can see by calculating the area of a differential element with a small extent in radius ρ and angle ϕ. Thus, we can write

$$I_{CM}^{disk} = \int_0^{r_2} \int_0^{2\pi} \rho^2 \sigma \rho d\phi d\rho, \tag{9}$$

where we've chosen the bounds such that the integrals span the entire disk. Since the argument of the integral has no ϕ dependence, we find

$$I_{CM}^{disk} = 2\pi\sigma \int_0^{r_2} r^3 dr. \tag{10}$$

The integral over radius is also straightforward, giving

$$I_{CM}^{disk} = 2\pi\sigma \left(\frac{r_2^4}{4} - 0 \right) = \frac{m_2}{2} r_2^2, \tag{11}$$

where we have substituted in equation (7). Combining this with equation (5) gives the final expression for the moment of inertia of the disk about P

$$I_P^{disk} = \frac{m_2}{2} r_2^2 + m_2 L_2^2. \tag{12}$$

To calculate the moment of inertia of the rod, we can use the result from a table, which again is perfectly acceptable. However, here we will again show how to derive it from the definition of the moment of inertia

$$I_P^{rod} = \int_{rod} \rho^2 dm, \tag{13}$$

where ρ is the distance from the axis P passing through the end of the rod. Since the rod is uniform, we know that it has an linear density of

$$\lambda = \frac{m_1}{L_1} = \frac{\Delta m}{\Delta \rho}, \tag{14}$$

where $\Delta\rho$ is a linear differential element along the length of the rod. We can use λ to rewrite the integral over mass in equation (13) as an integral over length according to

$$I_P^{rod} = \int_{rod} \rho^2 \lambda d\rho. \tag{15}$$

Thus, the integral over the entire rod is just

$$I_P^{rod} = \lambda \int_0^{L_1} \rho^2 d\rho = \lambda \left(\frac{L_1^3}{3} - 0 \right) = \frac{m_1}{3} L_1^2. \tag{16}$$

Substituting this and equation (12) into equation (2) gives the final answer of

$$I_P^{tot} = \frac{m_1}{3} L_1^2 + \frac{m_2}{2} r_2^2 + m_2 L_2^2. \tag{17}$$

2. The center of mass of any system is given by

$$\vec{R}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}. \tag{18}$$

Here our system has two objects: the rod and the disk. Given that the rod is uniform, we know that its center of mass is at its midpoint, which is a distance $L_1/2$ from the point P. Similarly, since the disk is uniform, its center of mass is at its geometric center, which is a distance L_2 from the axis P. The calculation is one dimensional as both of these distances are in the same direction (i.e. along the length of the rod). Plugging this information into equation (18) gives a distance of

$$R_{CM} = \frac{m_1(L_1/2) + m_2(L_2)}{m_1 + m_2} = \frac{m_1 L_1/2 + m_2 L_2}{m_1 + m_2}. \tag{19}$$

3. As all of the forces acting on the pendulum system are conservative, we can impose conservation of mechanical energy

$$E_{mi} = E_{mf} \tag{20}$$

between the initial state described in the problem (denoted by the subscript i) and the final state when the pendulum is at the bottom of its swing (denoted by the subscript f). The only forces involved are gravity, so equation (20) is

$$K_i + U_{gi} = K_f + U_{gf}. \tag{21}$$

The system is released from rest, so $K_i = 0$. Additionally, we will define the reference point for the gravitational potential energy to be the location of the center of mass when the pendulum is at the bottom of its swing, so $U_{gf} = 0$. Thus, we can use the forms of the rotational kinetic energy and gravitational potential energy to write equation (21) as

$$0 + (m_1 + m_2)\, g\Delta h = \frac{I_P^{tot}}{2} \omega_f^2 + 0, \tag{22}$$

where Δh is the change in height of the center of mass from its final position to its initial position and ω_f is the angular speed that we are trying to find. From the picture below, we see that $\Delta h = R_{CM} - R_{CM}\cos\theta_0$. Substituting this into equation (22) along with equations (17) and (19) gives

$$(m_1 + m_2)\, g\, (R_{CM} - R_{CM}\cos\theta_0) = \frac{1}{2}\left(\frac{m_1}{3}L_1^2 + \frac{m_2}{2}r_2^2 + m_2 L_2^2\right)\omega_f^2 \tag{23}$$

$$\Rightarrow \quad g\left(\frac{m_1}{2}L_1 + m_2 L_2\right)(1 - \cos\theta_0) = \frac{1}{2}\left(\frac{m_1}{3}L_1^2 + \frac{m_2}{2}r_2^2 + m_2 L_2^2\right)\omega_f^2 \tag{24}$$

$$\Rightarrow \quad \omega_f = \sqrt{\frac{2g\left(\frac{m_1}{2}L_1 + m_2 L_2\right)(1 - \cos\theta_0)}{\frac{m_1}{3}L_1^2 + \frac{m_2}{2}r_2^2 + m_2 L_2^2}}. \tag{25}$$