



1. Blocks, pulleys, and a rope (20 points)

- a. (2.0 points) Make a drawing of the system where you clearly indicate your choice of reference frame. Then, draw free-body diagrams for blocks A , B , and C , including all the forces acting on each object.

First, choose a reference frame and show it clearly (0.5 points):

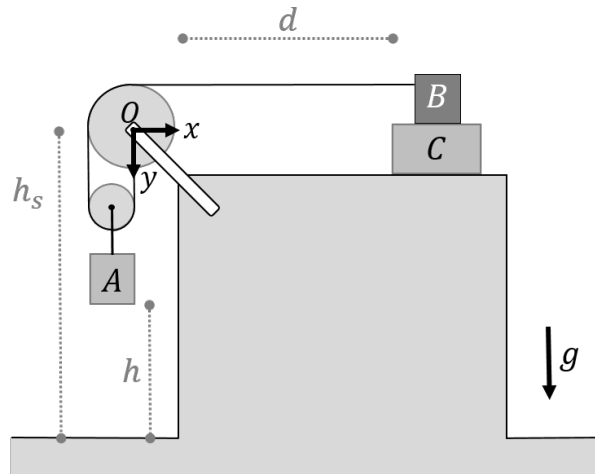


Figure 1: Example reference frame.

Next, draw the free-body diagrams for blocks A , B , and C showing all forces acting on each object (0.5 points each):

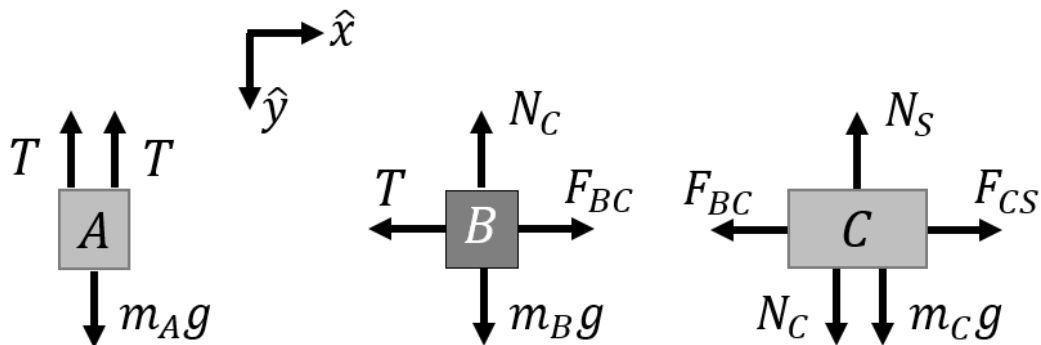


Figure 2: Free-body diagrams for blocks A , B , and C . Note that the force of friction acting between blocks B and C (F_{BC}), and the force of friction acting between block C and the surface (F_{CS}), can be either static or kinetic. Also note the action-reaction pairs in the free-body diagrams for blocks B and C : the normal force, N_C , and the force of friction between blocks B and C , F_{BC} .

- b. **(3.0 points)** Show that the condition for blocks B and C to start moving together as a single object (with a sufficiently large T) is:

$$\mu_{BC}^s > \left(1 + \frac{m_C}{m_B}\right) \mu_{CS}^s$$

Identify the condition for which block C accelerates horizontally (0.5 points):

For blocks B and C to start moving together, the maximum static friction between blocks B and C must be larger than the maximum static friction between block C and the surface. This is required so that $F_{\text{stat},BC}$ can balance T to prevent block B from accelerating until block C accelerates.

From its free-body diagram, we see that block C can only accelerate horizontally when:

$$F_{\text{stat},BC} > F_{\text{stat},CS}^{\text{max}} = \mu_{CS}^s N_S \Rightarrow \mu_{CS}^s N_S < F_{\text{stat},BC} \leq \mu_{BC}^s N_C$$

Correctly write the sum of forces for blocks B and C (0.5 points each):

The values of the normal forces, N_S and N_C , can be obtained from the sum of forces in the y -direction for blocks B and C by noting that the blocks do not accelerate in y :

$$\Sigma F_y^B : m_B g - N_C = 0 \Rightarrow N_C = m_B g$$

$$\Sigma F_y^C : m_C g + N_C - N_S = 0 \Rightarrow N_S = m_C g + N_C = (m_B + m_C) g$$

Combine the expressions for blocks B and C to find the condition required for their acceleration as a single object (0.5 points):

Using the expressions for the normal forces, we find:

$$\mu_{CS}^s N_S = \mu_{CS}^s (m_B + m_C) g < \mu_{BC}^s m_B g \Rightarrow \left(1 + \frac{m_C}{m_B}\right) \mu_{CS}^s < \mu_{BC}^s$$

Then, argue that this condition is sufficient to guarantee that blocks B and C will continue moving as a single object once in motion.

Recognize that the presence of kinetic rather than static friction between block C and the surface guarantees that the blocks will continue to move together as a single object (1 point):

Once in motion, the friction between block C and the surface is kinetic. The friction between blocks B and C , however, must still be static. This means that:

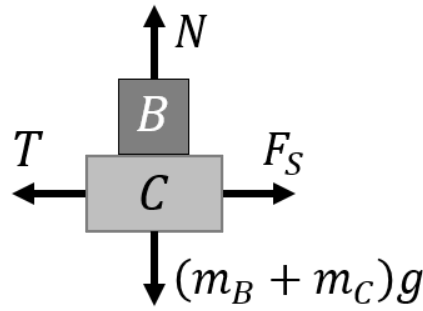
$$F_{\text{kin},CS} < F_{\text{stat},BC}^{\text{max}} \Rightarrow \mu_{CS}^k N_S < \mu_{BC}^s N_C \Rightarrow \left(1 + \frac{m_C}{m_B}\right) \mu_{CS}^k < \mu_{BC}^s$$

Since $\mu_{CS}^k < \mu_{CS}^s$, this inequality is already fulfilled by the condition that the blocks started moving with no motion between them.

- c. **(3.0 points)** Assuming that the condition above is met (i.e., blocks B and C move together as a single object BC), find the maximum value of m_A (call it m_A^{sta}) for which BC stays static.

Correctly write the sum of forces in x and y for the object BC and in y for block A (0.5 points each):

If blocks B and C move as a single object, BC , we can model it with a single body of mass $m_{BC} = m_B + m_C$ subject to the pull of the rope and friction with the surface. Here, we are interested in the case of static friction, as specified in the problem statement. We can then draw the free-body diagram for object BC (Fig. 3).

Figure 3: Free-body diagram for object BC .

From this, the sum of forces on object BC in the x- and y-directions can be written, where F_{stat} refers to the force of static friction between BC and the surface:

$$\begin{aligned}\Sigma F_y^{BC} : (m_B + m_C)g - N &= 0 \Rightarrow N = (m_B + m_C)g \\ \Sigma F_x^{BC} : F_{\text{stat}} - T &= 0 \Rightarrow T = F_{\text{stat}} \leq \mu_{CS}^s N = \mu_{CS}^s (m_B + m_C)g\end{aligned}$$

From its free-body diagram, we can also write the sum of forces acting on block A :

$$\Sigma F_y^A : m_A g - 2T = 0 \Rightarrow T = \frac{1}{2} m_A g$$

Combine the expressions for object BC and block A to find the condition for the system to stay static (1 point):

If we substitute in the expression for T found from object BC , we see that:

$$\frac{1}{2} m_A g \leq \mu_{CS}^s (m_B + m_C)g \Rightarrow m_A \leq 2\mu_{CS}^s (m_B + m_C)$$

Correctly identify the maximum value of m_A (0.5 points):

So the maximum value of m_A for which the object BC stays stationary is:

$$m_A^{\text{sta}} = 2\mu_{CS}^s (m_B + m_C)$$

Note that the same solution can be found by examining all forces in blocks B and C independently:

$$\begin{aligned}\Sigma F_x^B : F_{\text{stat},BC} - T &= 0 \\ \Sigma F_x^C : F_{\text{stat},CS} - F_{\text{stat},BC} &= 0 \\ T = F_{\text{stat},CS} &\leq \mu_{CS}^s N_S \Rightarrow T \leq \mu_{CS}^s (m_B + m_C)g\end{aligned}$$

Where the tension can be found from the sum of forces on block A , giving the same answer as if BC was treated as a single object.

- d. **(4.0 points)** Assume now that $m_A > m_A^{sta}$, so that BC starts moving. Show that the acceleration of BC , \vec{a}_{BC} , and the acceleration of block A , \vec{a}_A , are related by $|\vec{a}_{BC}| = 2|\vec{a}_A|$.

Recognize that the constant length of the rope allows x_{BC} to be related to y_A (0.5 points):

First, we use the fact the length of the rope, L , is constant. This allows us to relate x_{BC} (the horizontal component of the position of BC) to y_A (the vertical component of block A). The length of the rope can be expressed as:

$$L = y_A + \pi R_S + y_A + \frac{\pi}{2} R_L + x_{BC}$$

Take the time derivative of L to relate the accelerations of block A and object BC (1 point):

We know that L does not change in time, so:

$$0 = \frac{dL}{dt} = \dot{y}_A + \dot{y}_A + \dot{x}_{BC} = 2\dot{y}_A + \dot{x}_{BC}$$

To relate the accelerations of block A and BC , we can differentiate this equation again:

$$0 = \frac{d^2L}{dt^2} = 2\ddot{y}_A + \ddot{x}_{BC} \Rightarrow 2\ddot{y}_A = -\ddot{x}_{BC}$$

Noting that $\ddot{y}_A = a_y^A$ and $\ddot{x}_{BC} = a_x^{BC}$, we find that:

$$2a_y^A = -a_x^{BC}$$

Find the relationship requested in the problem statement (0.5 points):

Considering that the acceleration of block A only occurs in the vertical direction, while the acceleration of BC is strictly horizontal, we find the requested relationship:

$$|\vec{a}_{BC}| = |a_x^{BC}| = 2|a_y^A| = 2|\vec{a}_A|$$

Find the acceleration of block A in terms of the parameters of the problem.

Use the sum of forces in x and y for object BC in the moving case to solve for the tension (1 point):

To find the value of a_x^{BC} , we can return to the free-body diagrams of block A and object BC for the moving case, i.e., when kinetic friction is acting on BC :

$$\begin{aligned} \Sigma F_y^{BC} : (m_B + m_C)g - N &= 0 \Rightarrow N = (m_B + m_C)g \\ \Sigma F_x^{BC} : F_{\text{kin}} - T &= (m_B + m_C)a_x^{BC} \end{aligned}$$

Solving for the tension:

$$\begin{aligned} T &= -(m_B + m_C)a_x^{BC} + \mu_{CS}^k N \\ T &= -(m_B + m_C)a_x^{BC} + \mu_{CS}^k (m_B + m_C)g \\ T &= (m_B + m_C) \left[-a_x^{BC} + g\mu_{CS}^k \right] \end{aligned}$$

Use the sum of forces in y for block A to find its acceleration (1 point). Subtract 0.25 points if \vec{a}_A is not written as a vector:

Using the free-body diagram for block A in the moving case, we find:

$$\Sigma F_y^A : m_A g - 2T = m_A a_y^A \Rightarrow a_y^A = g - \frac{2T}{m_A}$$

Substituting in the expression for T and using the previous result that $2a_y^A = -a_x^{BC}$, we can solve for the acceleration of block A :

$$\begin{aligned} a_y^A &= g - \frac{2}{m_A}(m_B + m_C) \left[-a_x^{BC} + g \mu_{CS}^k \right] \\ a_y^A &= g - \frac{2}{m_A}(m_B + m_C) \left[2a_y^A + g \mu_{CS}^k \right] \\ a_y^A + \frac{4}{m_A}(m_B + m_C) a_y^A &= g \left[1 - \frac{2}{m_A}(m_B + m_C) \mu_{CS}^k \right] \\ a_y^A &= g \left[\frac{m_A - 2 \mu_{CS}^k (m_B + m_C)}{m_A + 4(m_B + m_C)} \right] \end{aligned}$$

Finally, we note that the acceleration of block A is a vector:

$$\vec{a}_A = a_y^A \hat{y} = g \left[\frac{m_A - 2 \mu_{CS}^k (m_B + m_C)}{m_A + 4(m_B + m_C)} \right] \hat{y}$$

- e. **(4.0 points)** Find a condition on d and h for block A to hit the floor before BC touches the large pulley.

Find the equations of motion for object BC and block A (1 point each):

Block A and object BC accelerate from rest with constant, though different, magnitudes. If BC starts to move at $t = 0$, then in our choice of reference frame, we have:

$$\begin{aligned} x_{BC}(0) &= d + R_L \\ x_{BC}(t) &= (d + R_L) + \frac{1}{2} a_x^{BC} t^2 \\ y_A(0) &= h_S - h \\ y_A(t) &= (h_S - h) + \frac{1}{2} a_y^A t^2 \end{aligned}$$

Find the condition on d and h requested in the problem statement (1 point):

For object BC to reach the pulley at a time t_p :

$$x_{BC}(t_p) = R_L = (d + R_L) + \frac{1}{2} a_x^{BC} t_p^2 \Rightarrow t_p = \sqrt{\frac{-2d}{a_x^{BC}}} = \sqrt{\frac{d}{a_y^A}}$$

For block A to reach the floor at a time t_f :

$$y_A(t_f) = h_S = (h_S - h) + \frac{1}{2} a_y^A t_f^2 \Rightarrow t_f = \sqrt{\frac{2h}{a_y^A}}$$

For block A to arrive at the floor before BC reaches the pulley, $t_f < t_p$. Therefore:

$$\sqrt{\frac{2h}{a_y^A}} < \sqrt{\frac{d}{a_y^A}} \Rightarrow h < \frac{d}{2}$$

If that condition is met, how long does it take A to hit the floor?

Solve for t_f (1 point):

If the above condition on h and d is met, the amount of time it will take block A to hit the floor can be expressed as:

$$t_f = \sqrt{\frac{2h}{a_y^A}} = \sqrt{\frac{2h}{g} \left[\frac{m_A + 4(m_B + m_C)}{m_A - 2\mu_{CS}^k (m_B + m_C)} \right]}$$

- f. **(4.0 points)** Imagine now that the whole system is placed in an elevator (see Fig. 2) and the elevator accelerates downward with a constant acceleration of magnitude a , as seen from a fixed reference frame outside the elevator. Assume that $a < g$, and $m_A > m_A^{sta}$. Find \vec{a}_A as seen from the fixed reference frame.

Write the sum of forces accounting for a for blocks B and C (1 point each):

If the system is in an elevator, the free-body diagrams remain unchanged, however, the sum of the vertical forces is no longer zero for blocks B and C . Now, there is an acceleration, a , in the y -direction. Recalling that $a_x^B = a_x^C = a_x^{BC}$ for BC to stick together, we can write:

$$\Sigma F_y^B : m_B g - N_C = m_B a \Rightarrow N_C = m_B (g - a)$$

$$\Sigma F_x^B : F_{stat,BC} - T = m_B a_x^B = m_B a_x^{BC}$$

$$\Sigma F_y^C : m_C g + N_C - N_S = m_C a \Rightarrow N_S = m_C (g - a) + N_C = (m_B + m_C)(g - a)$$

$$\Sigma F_x^C : F_{kin,CS} - F_{stat,BC} = m_C a_x^C = m_C a_x^{BC}$$

Use the sum of forces for blocks B and C to solve for the tension (1 point):

Adding the above equations in x , we find that:

$$F_{kin,CS} - T = (m_B + m_C) a_x^{BC}$$

From the sum of forces, we can also find an expression for $F_{kin,CS}$:

$$F_{kin,CS} = \mu_{CS}^k N_S = \mu_{CS}^k (m_B + m_C)(g - a)$$

Using this to solve for the tension:

$$T = (m_B + m_C) \left[\mu_{CS}^k (g - a) - a_x^{BC} \right]$$

Use the sum of forces in y for block A to find its acceleration (1 point). Subtract 0.25 points if \vec{a}_A is not written as a vector:

For block A , we again only have forces acting in the y -direction:

$$\Sigma F_y^A : m_A g - 2T = m_A a_y^A \Rightarrow m_A a_y^A = m_A g - 2(m_B + m_C) \left[\mu_{CS}^k (g - a) - a_x^{BC} \right]$$

Now, we apply the earlier constraint ($2a_y^A = -a_x^{BC}$) on the accelerations of block A and object BC , since the rope is still inextensible. As a result, we find:

$$\begin{aligned} m_A a_y^A &= m_A g - 2(m_B + m_C) \left[\mu_{CS}^k (g - a) - a_x^{BC} \right] \\ a_y^A [m_A + 4(m_B + m_C)] &= m_A g - 2\mu_{CS}^k (m_B + m_C)(g - a) \\ \vec{a}_A = a_y^A \hat{y} &= g \left[\frac{m_A - 2\mu_{CS}^k (m_B + m_C)(1 - \frac{a}{g})}{m_A + 4(m_B + m_C)} \right] \hat{y} \end{aligned}$$

Compared to the magnitude of \vec{a}_A from part d, we see that being in the elevator reduces the retarding effect of friction on the motion of object BC .