General Physics: Mechanics

PHYS-101(en)

Lecture 8a: Kinetic energy and work

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Émilie du Châtelet
(circa 1740)

Today’s agenda (Serway 7-8, MIT 13-14)

1. Energy and work
   - Kinetic energy
   - Work done by a force
   - Work-kinetic energy theorem
   - To be continued next week…

DEMO (172)

From last week: Rockets!

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$$v_{rf} = u \ln\left(1 + \frac{M_f}{M_0}\right)$$

DEMO (556)

The skiers

DEMO (556)

Swiss Plasma Center

The skiers

1) What are the final speeds of the skiers?

2) Which skier would win in a race?

Energy

- **Energy** is one of the most fundamental and important concepts in physics
- It is possessed by objects, in various forms, and can be transferred between them
- Can be understood intuitively from its Greek origin ἐνέργεια, meaning “activity”
- It is given an exact and rigorous definition in the language of mathematics
- It is so important because it is conserved (as we will see next lecture)
- Similar to momentum, but more complex as it has many different forms (kinetic, potential, etc.)

Kinetic energy

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$$K = \frac{m}{2} v^2$$

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- Émilie du Châtelet used observations of brass weights, dropped into clay from various heights, to motivate the importance of a quantity proportional to mv^2

Derived quantity

- All forms of energy have units of Joules (J) = [kg·m^2/s^2] = [N·m]

Review

Conceptual question

responseware.eu
Session ID: epflphys101en

Consider two carts, of mass ① m and ② $2m$, at rest on an air track. If you push the first cart for 3 s and then the other for the same length of time, exerting equal force on each, the final **momentum** of the light cart is…

A. 4 times…
B. 2 times…
[✓ C.] equal to…
D. half…
E. one-fourth…

the momentum of the heavy cart.

$$\vec{I} = \int_{t_0}^{t_f} \vec{F}\,dt = \Delta\vec{p} = \vec{p}(t_f) - \cancelto{0}{\vec{p}(t_0)}$$

$$\Rightarrow \vec{p}(t_f) = \int_{t_0}^{t_f} \vec{F}\,dt$$

Conceptual question

Consider two carts, of mass $\overset{①}{\underset{m_1}{m}}$ and $\overset{②}{\underset{m_2}{2m}}$, at rest on an air track. If you push the first cart for 3 s and then the other for the same length of time, exerting equal force on each, the final **kinetic energy** of the light cart is…

[✓ A.] greater than…
B. equal to…
C. less than…

$$K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m \left(\frac{p}{m}\right)^2 = \frac{1}{2m} p^2$$

$$K_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{\cancel{2}} (\cancel{2}m) \left(\frac{p}{2m}\right)^2 = \frac{1}{4m} p^2 = \frac{1}{2} K_1$$

$$K_2 < K_1$$

the kinetic energy of the heavy cart.

$\vec{p}_1(t_f) = \vec{p}_2(t_f) \Rightarrow$ Define $p = |\vec{p}_1(t_f)| = |\vec{p}_2(t_f)|$

Now $|\vec{p}_1(t_f)| = |m_1 \vec{v}_1| = m_1 |\vec{v}_1| = m_1 v_1 = p \Rightarrow v_1 = \frac{p}{m}$

$|\vec{p}_2(t_f)| = |m_2 \vec{v}_2| = m_2 v_2 = p \Rightarrow v_2 = \frac{p}{2m}$

Work

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- This can be positive if the force is in the direction of motion or negative if the force opposes the direction of motion
- Work is to kinetic energy as impulse is to momentum

Work done by a constant force

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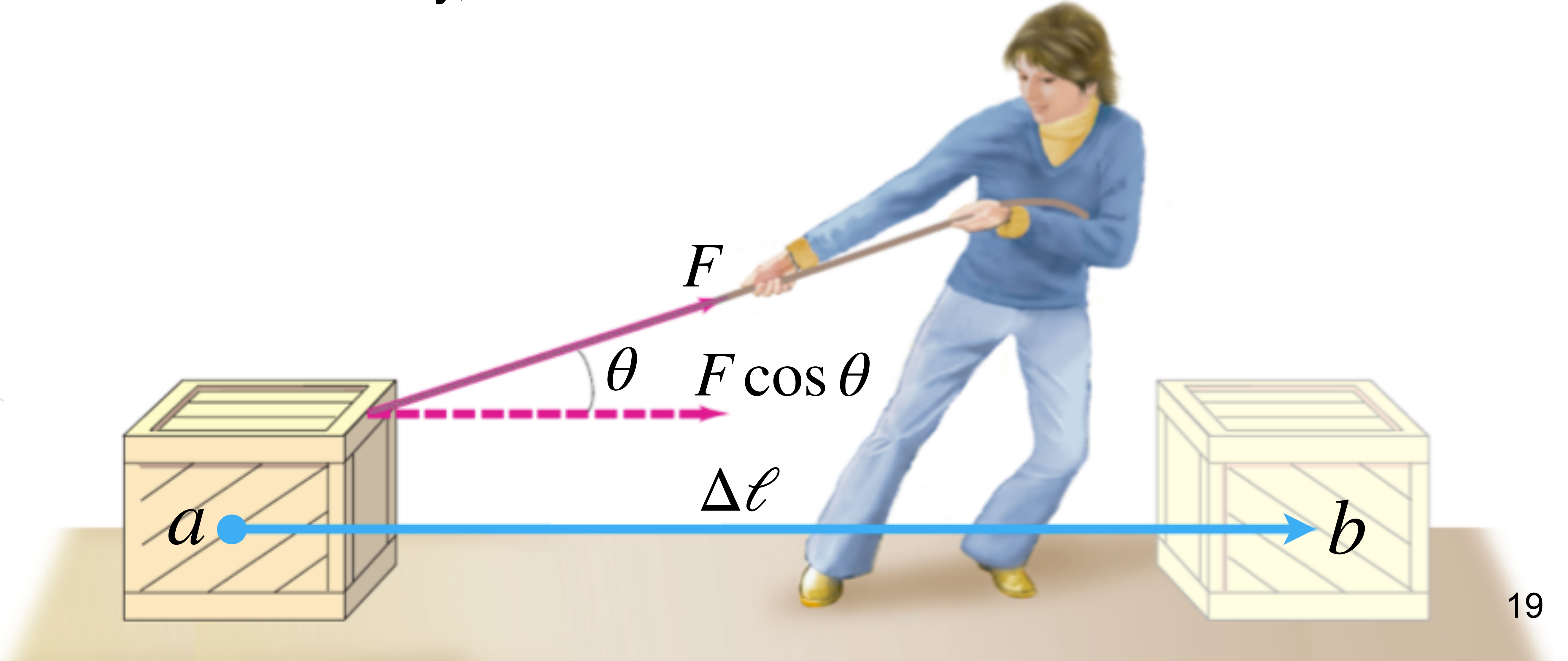
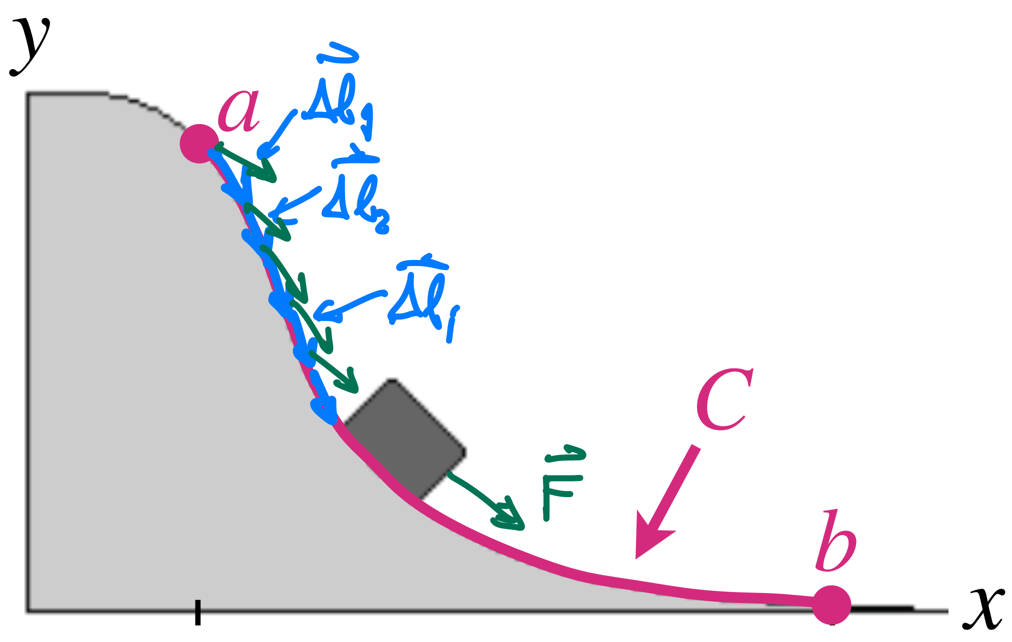
- Mathematically, this is $W = \overrightarrow{F} \cdot \overrightarrow{\Delta \ell} = F\,\Delta \ell \cos\theta$



Work done by a variable force

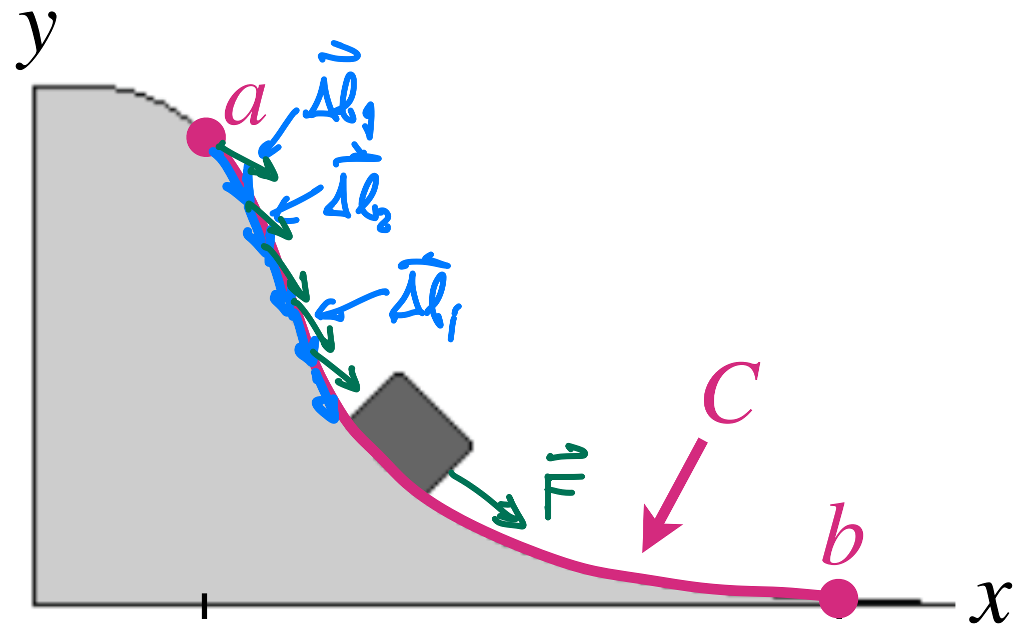
- The **work** done by a variable force and/or along a curved path can be found by considering the force constant over tiny straight displacements

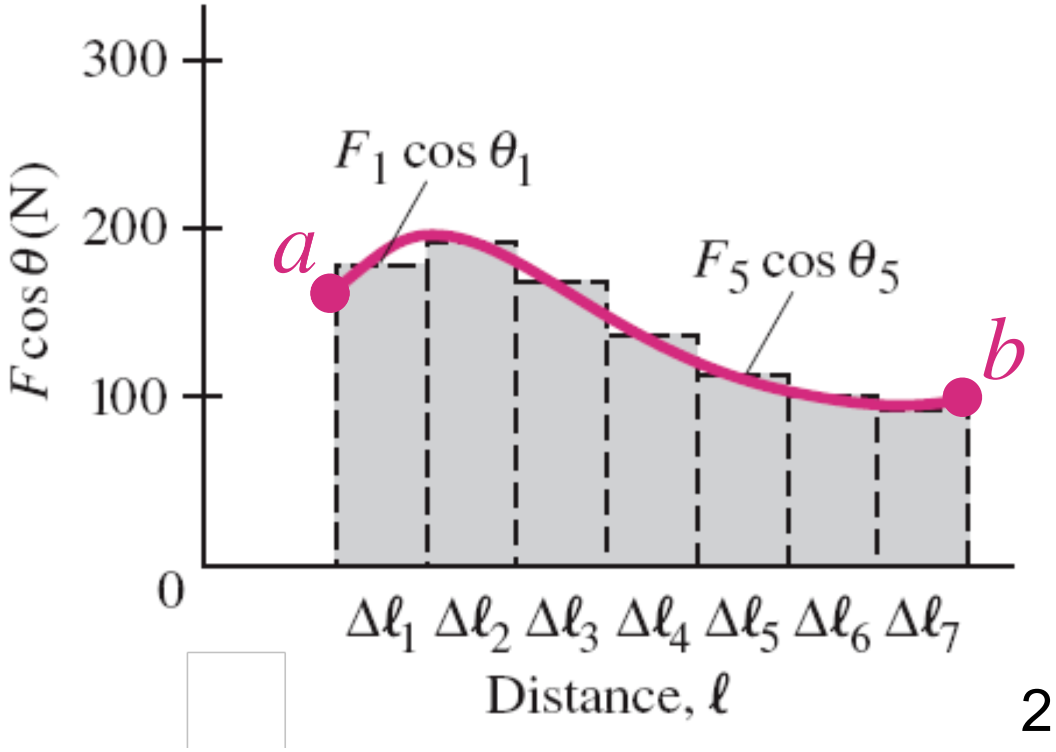


Work done by a variable force

- The **work** done by a variable force and/or along a curved path can be found by considering the force constant over tiny straight displacements
- Mathematically, each segment contributes

$$W_i = \overrightarrow{F}_i \cdot \overrightarrow{\Delta\ell}_i$$





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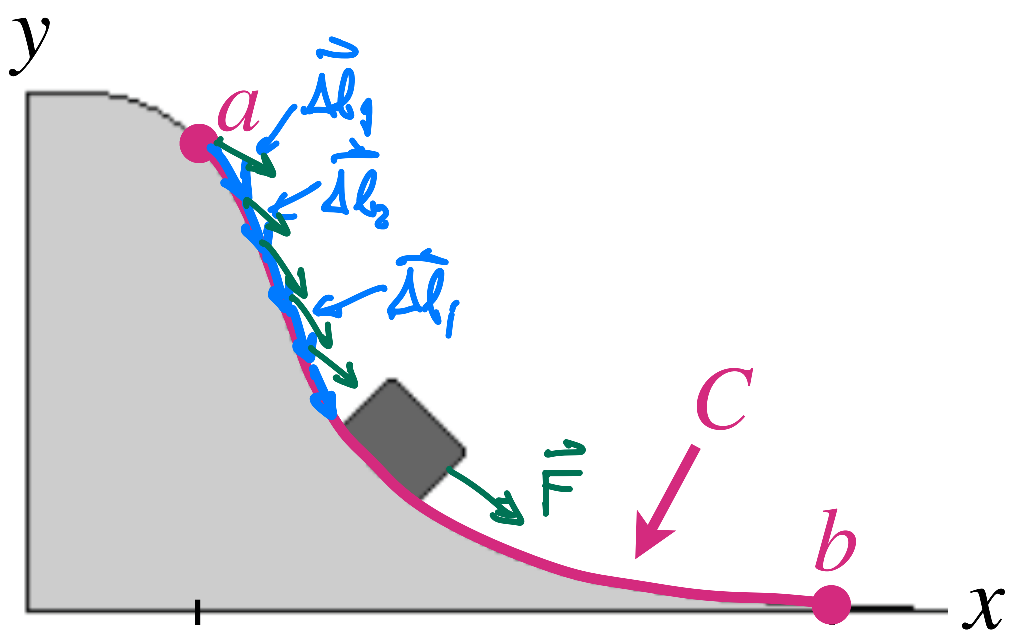


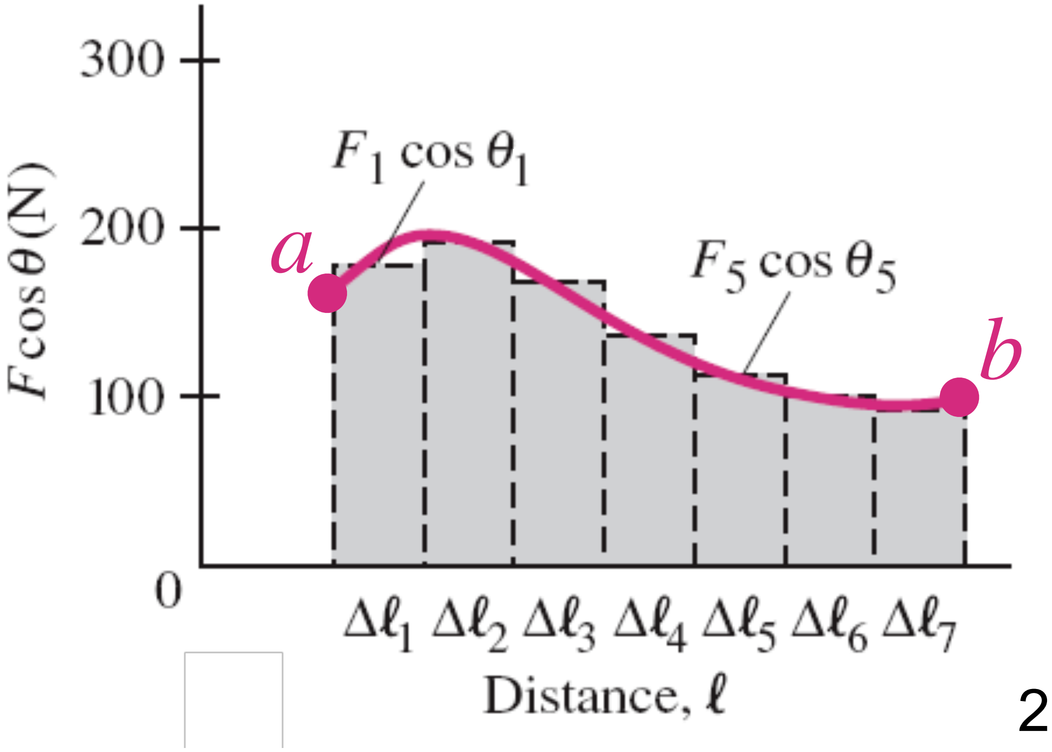
- Mathematically, each segment contributes

$$W_i = \vec{F}_i \cdot \vec{\Delta \ell}_i$$

- The sum is

$$W = \lim_{\Delta \ell_i \to 0} \sum_i \vec{F}_i \cdot \vec{\Delta \ell}_i$$

$$= \int_a^b \vec{F} \cdot \vec{d\ell}$$



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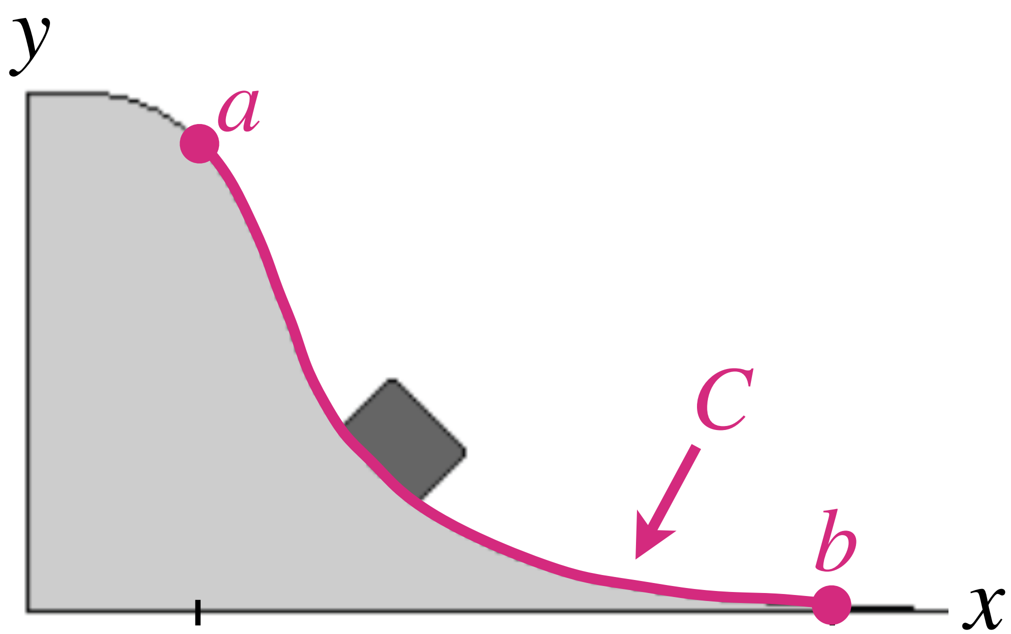

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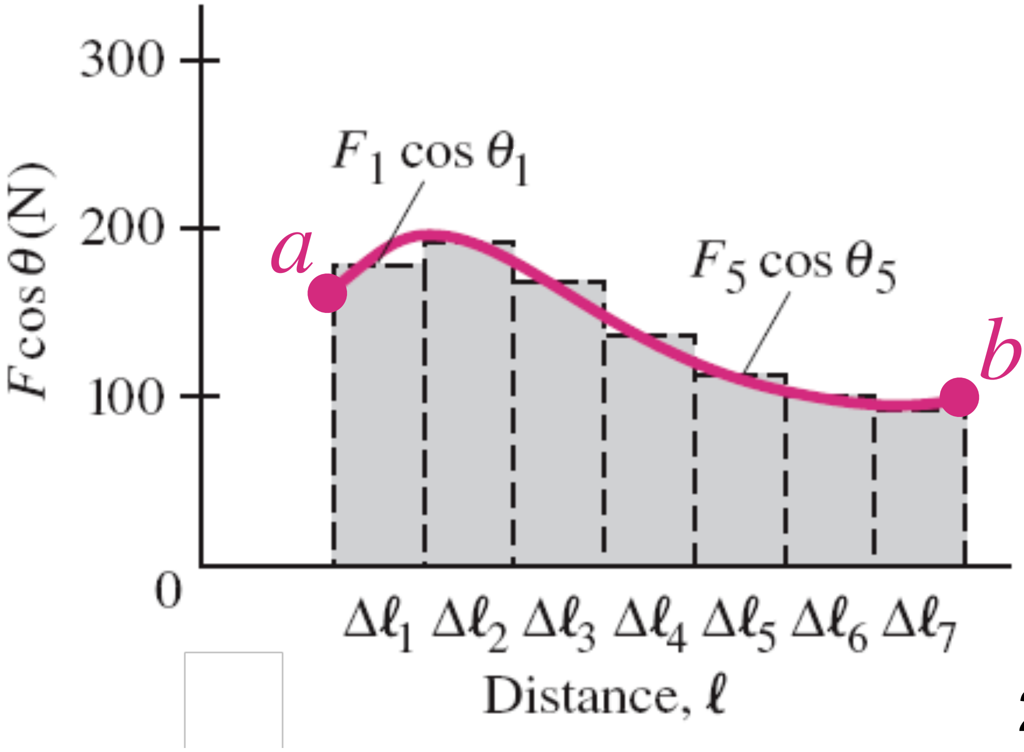
$$W = \lim_{\Delta\ell_i \to 0} \sum_i \vec{F}_i \cdot \overrightarrow{\Delta\ell}_i$$

$$= \int_a^b \vec{F} \cdot \overrightarrow{d\ell} \boxed{= \int_C \vec{F} \cdot \overrightarrow{d\ell}}$$



Work in various coordinate systems

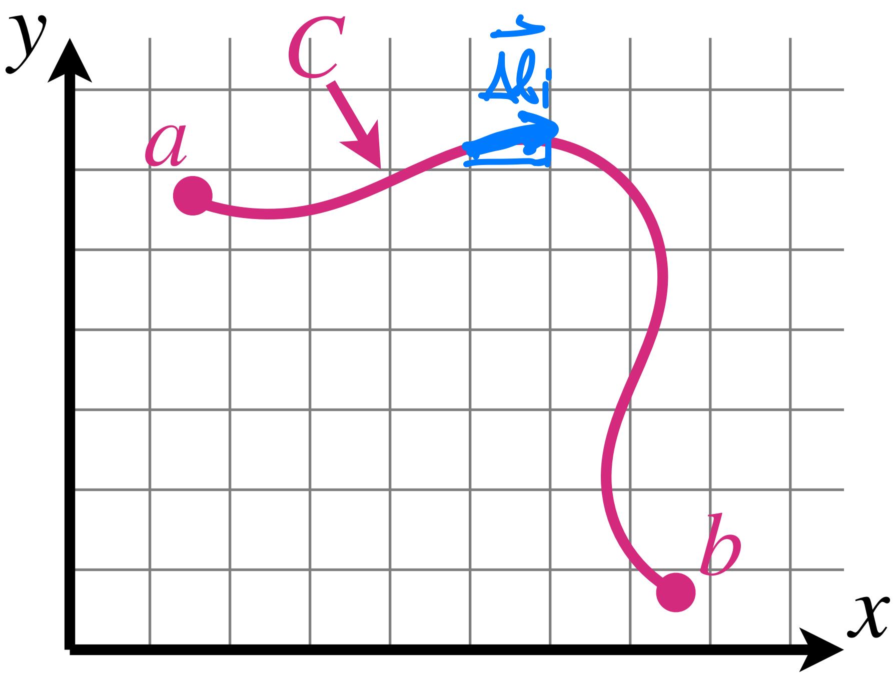
- Cartesian:

$$W = \int_C \vec{F} \cdot d\vec{l}$$

$$= \int_C \vec{F} \cdot (dx\,\hat{x} + dy\,\hat{y})$$

$$= \int_C (F_x \hat{x} + F_y \hat{y}) \cdot (dx\,\hat{x} + dy\,\hat{y})$$

$$= \int_C F_x\,dx + F_y\,dy + F_z\,dz$$



$$\vec{\Delta l_i} = \Delta x_i\,\hat{x} + \Delta y_i\,\hat{y}$$

$$\xrightarrow{\Delta l \to 0} \quad d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

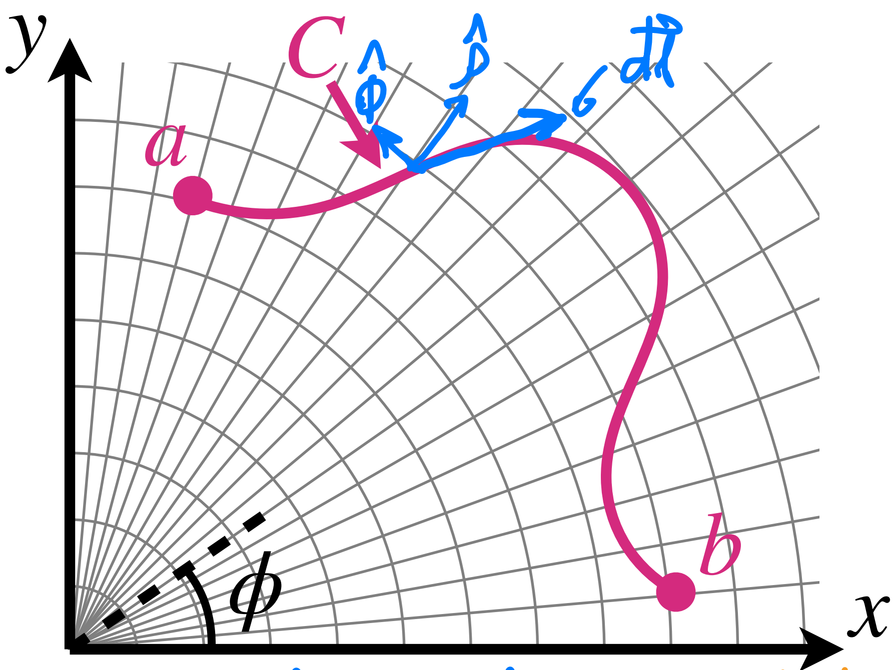
Work in various coordinate systems

- Polar (and cylindrical):

$$W = \int_C \vec{F} \cdot \vec{dl}$$

$$= \int_C \vec{F} \cdot (\hat{\rho}\, d\rho + \hat{\phi}\, \rho\, d\phi)$$

$$= \int_C F_\rho\, d\rho + \int_C F_\phi\, \rho\, d\phi + \int_C F_z\, dz$$

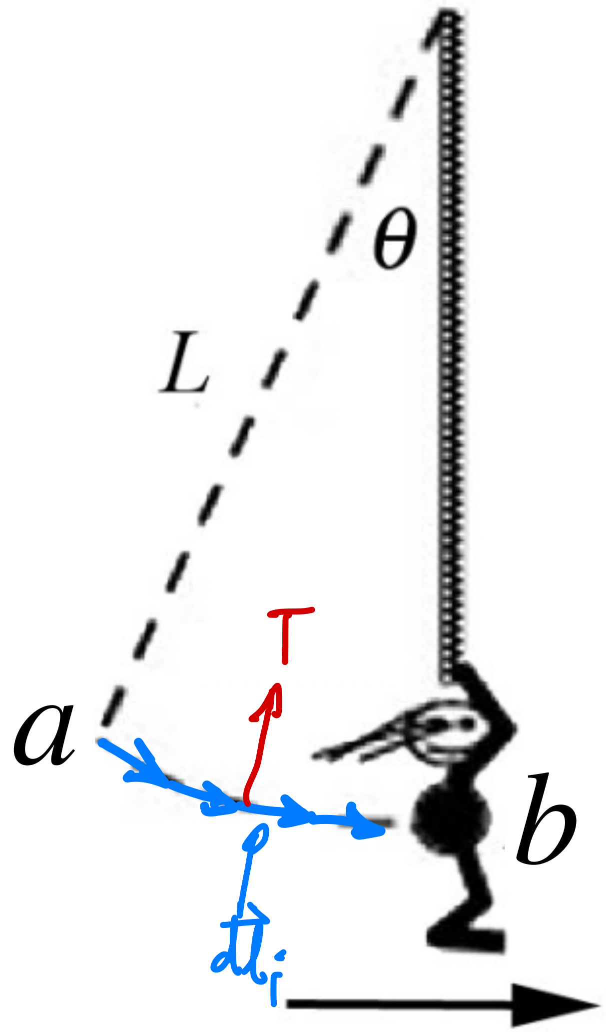


$$\vec{dl} = \hat{\rho}\, d\rho + \hat{\phi}\, \rho\, d\phi + \hat{z}\, dz$$

Conceptual question

A person swings on an inextensible rope that is attached to a fixed point. The rope exerts a tension T on the person. The work done by tension on the person as she moves from a to b is:

- A. T
- B. TL
- C. $TL\theta$
- D. $mgL(1 - \cos\theta)$
- [✓ E.] 0



$\vec{T} \cdot \vec{dl}_i = 0$

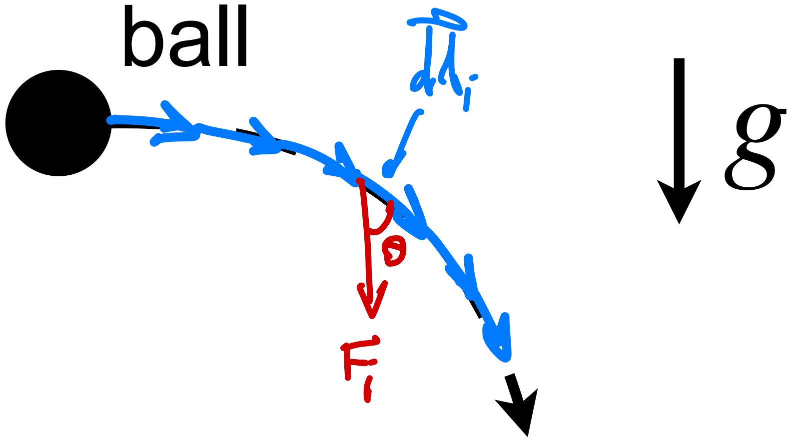
for all "pieces" i of path

Conceptual question

A ball is given an initial horizontal velocity and allowed to fall under the influence of gravity, as shown below.

The work done by the force of gravity on the ball is…

[✓ A.] positive.
B. zero.
C. negative.

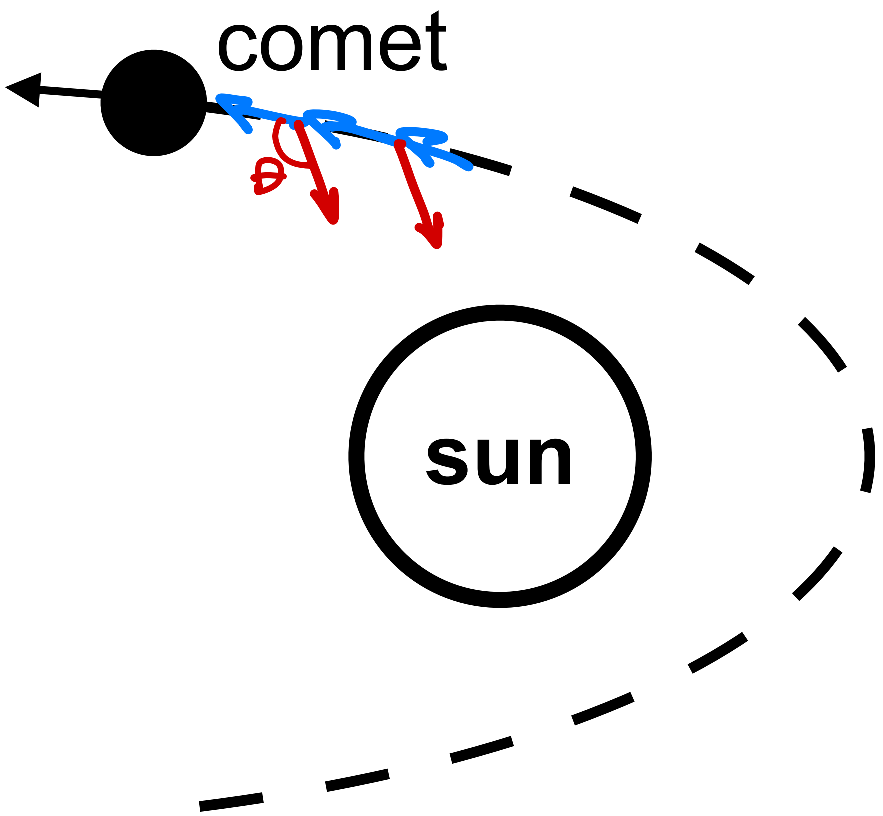


$\vec{F_i}\cdot\vec{dl_i} = F_g\, dl \cos(\theta) > 0$ for all i

Conceptual question

A comet is on a hyperbolic orbit around the Sun. While the comet is moving <u>away</u> from the Sun, the work done by the Sun on the comet is…

A. positive.
B. zero.
[✓ C. negative.]

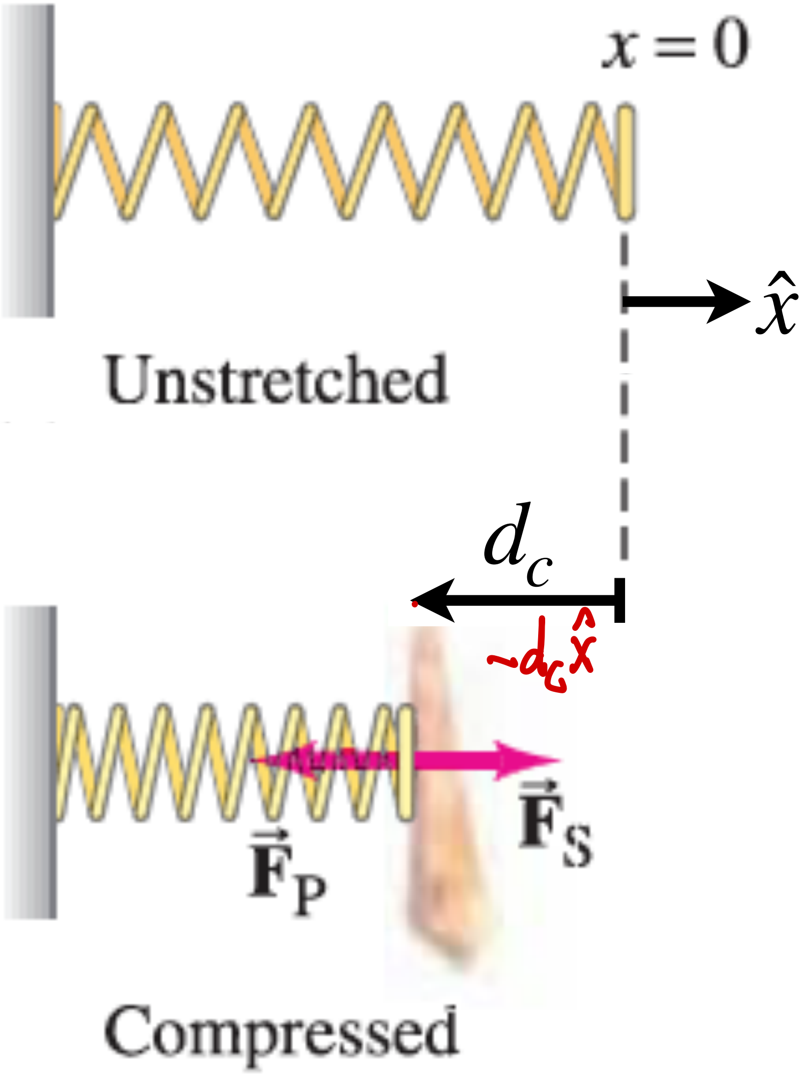


$\frac{\pi}{2} < \theta < \pi$

$\cos(\theta) < 0$

Work done by a spring

- A person compresses a spring by a distance d_c from its equilibrium position. What is the work done by the spring on the person as it stretches back towards the equilibrium position?



$$\vec{F}_S = -k(x - \overset{0}{\cancel{x_0}})\hat{x} = -kx\hat{x}$$

$$\vec{dl} = dx\hat{x}$$

$$W_S = \int_{-d_c}^{0} \vec{F}_S \cdot \vec{dl} = \int_{-d_c}^{0} (-kx\hat{x}) \cdot (\hat{x}\,dx)$$

$$= \int_{-d_c}^{0} -kx\,dx = -k\int_{-d_c}^{0} x\,dx$$

$$= -k\left[\frac{x^2}{2}\right]_{-d_c}^{0} = -k\left(0 - \frac{(-d_c)^2}{2}\right)$$

$$= \boxed{\frac{1}{2}kd_c^2}$$

Work-kinetic energy theorem

- Doing work on a system changes its **kinetic energy**
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For a constant mass system,
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- Work-kinetic energy theorem:

For a constant mass system,
the total work done by all the forces equals
the change in the kinetic energy of the system.

- In mathematics, this is

$$\boxed{W_{net} = \Delta K}$$

Proving the work-kinetic energy theorem

$K = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v}$

$\vec{v}\cdot\vec{v} = |\vec{v}||\vec{v}|\cos(0) = |\vec{v}|^2 = v^2$

$$\frac{d}{dt}K = \frac{d}{dt}\left[\frac{1}{2}mv^2\right] = \frac{d}{dt}\left[\frac{1}{2}m\vec{v}\cdot\vec{v}\right] = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\left(\vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt}\cdot\vec{v}\right)$$

$$= \frac{1}{2}m\left(2\vec{v}\cdot\frac{d\vec{v}}{dt}\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} = m\vec{v}\cdot\vec{a} = m\vec{a}\cdot\vec{v}$$

From Newtons 2nd law, $m\vec{a} = \vec{F}_{net}$

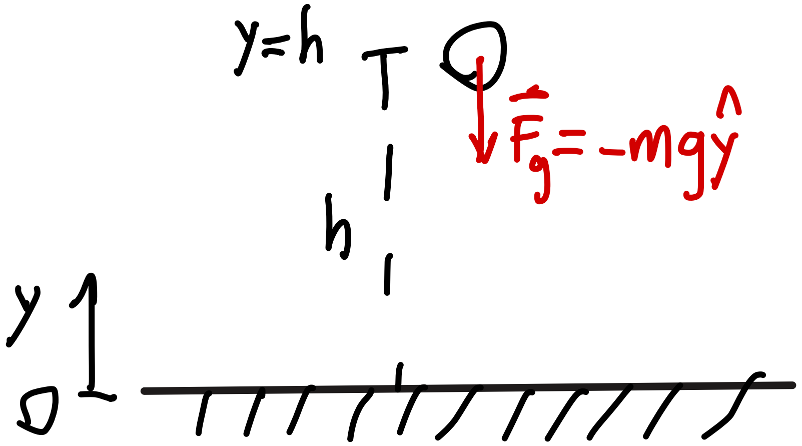
$\frac{d}{dt}K = \vec{F}_{net}\cdot\vec{v}$ But $\vec{v} = \frac{d\vec{l}}{dt}$ ⇒ $\frac{d}{dt}K = \vec{F}_{net}\cdot\frac{d\vec{l}}{dt}$

$$\underbrace{\int_{t_a}^{t_b}\frac{d}{dt}K\,dt}_{K(t_b)-K(t_a)} = \int_{t_a}^{t_b}\vec{F}_{net}\cdot\frac{d\vec{l}}{dt}\,dt = \int_a^b \vec{F}_{net}\cdot d\vec{l} = W_{net}$$

$$\Rightarrow \boxed{\Delta K = W_{net}}$$

Work done by gravity

- An object at rest falls directly downwards from a height $y = h$ to the ground at $y = 0$ under the sole influence of gravity. What is the work done by gravity? What is the change in kinetic energy?



$$\vec{F}_g = -mg\hat{y} \qquad \vec{d\ell} = \hat{y}\,dy$$

$$W_g = \int \vec{F}_g \cdot \vec{d\ell} = \int_h^0 -mg\hat{y}\cdot\hat{y}\,dy$$

$$= \int_h^0 -mg\,dy = -mg\big[y\big]_h^0$$

$$= -mg(0-h) = \boxed{mgh}$$

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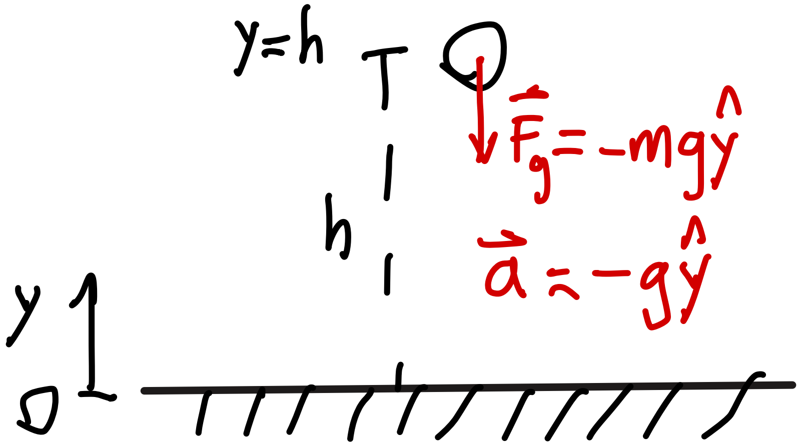


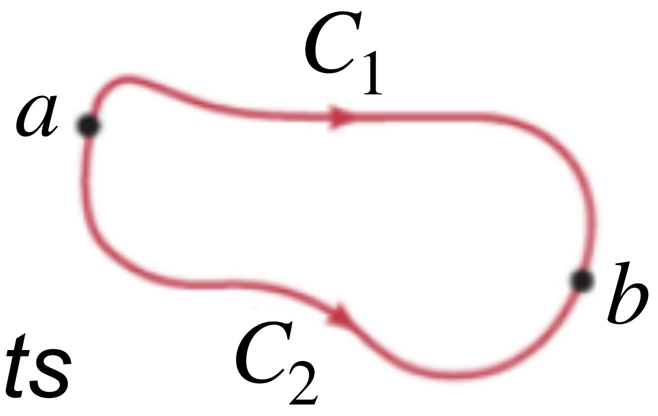
$\vec{F}_g = -mg\hat{y}$

$\vec{a} = -g\hat{y}$

$$v_y(t) = \cancelto{0}{v_0} - gt$$
$$= -gt$$
$$-v_f = -gt_f$$
$$\Rightarrow t_f = \frac{v_f}{g}$$

$$y(t) = \overbrace{y_0}^{h} + \cancelto{0}{v_{y0}}t - \frac{1}{2}gt^2$$
$$= h - \frac{1}{2}gt^2$$
$$0 = h - \frac{1}{2}gt_f^2 = h - \frac{1}{2}g\left(\frac{v_f}{g}\right)^2$$
$$= h - \frac{1}{2g}v_f^2$$
$$\Rightarrow \underbrace{mgh}_{W_g} = \underbrace{m\frac{1}{2}v_f^2 - \overbrace{\frac{1}{2}mv_{y0}^2}^{0}}_{\Delta K}$$

Conservative and nonconservative forces



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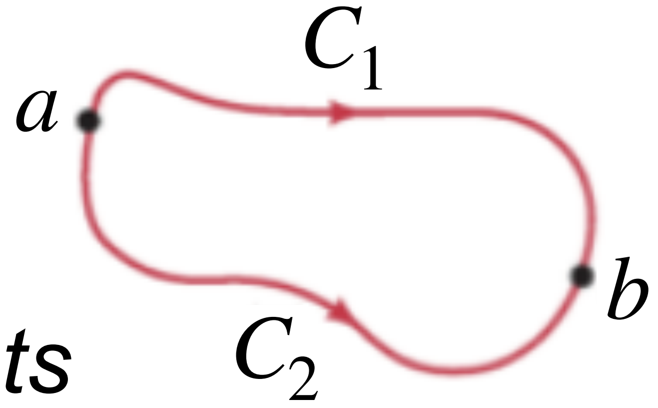

The work done by the force on
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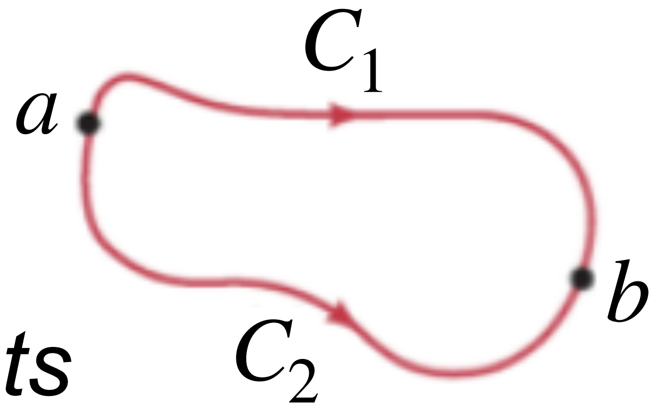
$$W = \underset{\text{by } C_1}{\int_a^b \vec{F}\cdot\vec{dl}} = \underset{\text{by } C_2}{\int_a^b \vec{F}\cdot\vec{dl}} = -\underset{\text{by } C_2}{\int_b^a \vec{F}\cdot\vec{dl}}$$

or equivalently

The net work done by the force on
a particle moving around any closed path is zero.

$$\oint \vec{F}\cdot dl = \underset{\text{by } C_1}{\int_a^b \vec{F}\cdot\vec{dl}} + \underset{\text{by } C_2}{\int_b^a \vec{F}\cdot\vec{dl}} = 0$$

Conservative and nonconservative forces



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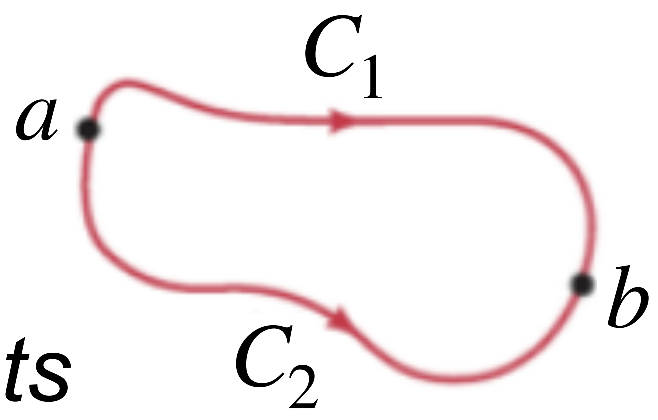

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- Gravity and springs are conservative, while friction and air drag are non-conservative

Path independence of gravity

- Show that the gravitational force $\vec{F}_g = -mg\hat{y}$ is conservative

$$W_{g1} = \int_{C_1} \vec{F}_g \cdot d\vec{l} = \int_{y_a}^{y_b} -mg\overbrace{\hat{y}\cdot\hat{y}}^{1}\,dy + \cancelto{0}{\int_{x_a}^{x_b} -mg\hat{y}\cdot\hat{x}\,dx}$$

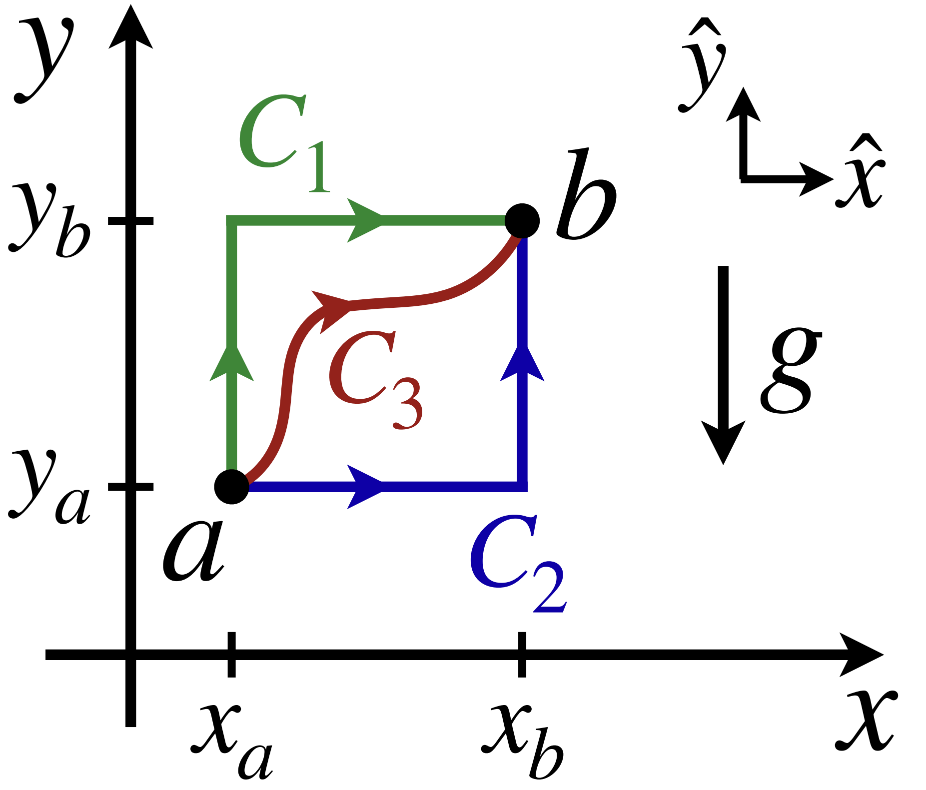
$$= -mg\int_{y_a}^{y_b} dy = -mg(y_b - y_a)$$

$$W_{g2} = \int_{C_2} \vec{F}_g \cdot d\vec{l} = \cancelto{0}{\int_{x_a}^{x_b} -mg\hat{y}\cdot\hat{x}\,dx} + \int_{y_a}^{y_b} -mg\overbrace{\hat{y}\cdot\hat{y}}^{1}\,dy$$

$$= -mg(y_b - y_a)$$

$$W_{g3} = \int_{C_3} \vec{F}_g \cdot d\vec{l} = \int_{C_3} -mg\hat{y}\cdot(\hat{x}\,dx + \hat{y}\,dy)$$

$$= \int_{y_a}^{y_b} -mg\,dy = -mg(y_b - y_a)$$



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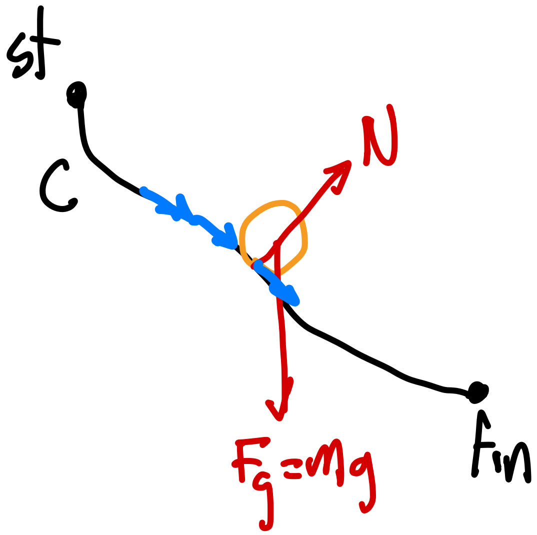


$$W_{net} = \int_C \vec{F}_{net}\cdot d\vec{l} = \int_C (\vec{F}_g + \vec{N})\cdot d\vec{l}$$

$$= \int_C \vec{F}_g\cdot d\vec{l} + \cancelto{0}{\int_C \vec{N}\cdot d\vec{l}} = W_g$$

$$\Rightarrow W_g = \Delta K$$

1) What are the final speeds of the skiers?

$$W_g = mgh, \quad \Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2}\cancelto{0\ \text{(rest)}}{m v_0^2} = \frac{1}{2} m v_f^2 \Rightarrow \cancel{m}gh = \frac{1}{2}\cancel{m} v_f^2$$

$$\boxed{v_f = \sqrt{2gh}}$$

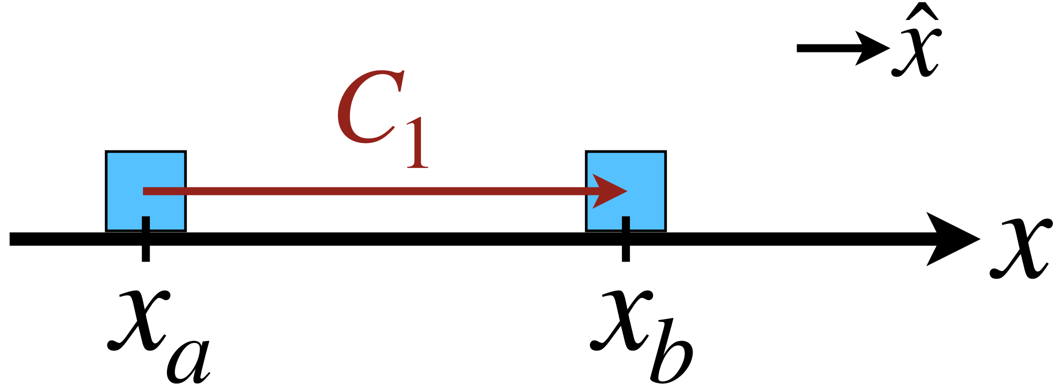
DEMO (556)

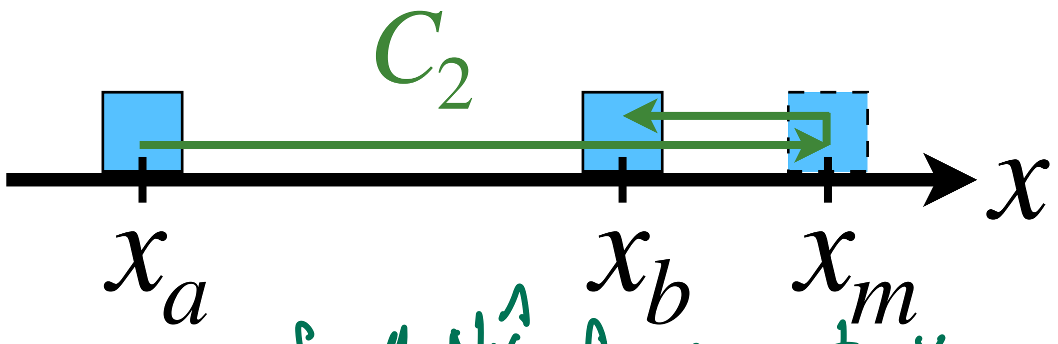
Brachistochrone problem

2) Which skier would win in a race?

Path dependence of kinetic friction

- Show that the kinetic friction force $\vec{F}_f = \pm\,\mu_k N\hat{x}$ is <u>not</u> conservative





$\vec{F}_f = -\mu_K N\hat{x}$ $\quad \vec{dl} = \hat{x}\,dx$

$W_{f_1} = \int_{C_1} \vec{F}_f\cdot\vec{dl} = \int_{x_a}^{x_b} -\mu_K N\hat{x}\cdot\hat{x}\,dx$

$= \int_{x_a}^{x_b} -\mu_K N\,dx = -\mu_K N\int_{x_a}^{x_b} dx$

$= -\mu_K N(x_b - x_a)$

$\vec{F}_f = \begin{cases} -\mu_K N\hat{x} & \text{from } x_a \text{ to } x_m \\ \mu_K N\hat{x} & \text{from } x_m \text{ to } x_b \end{cases}$

$W_{f_2} = \int_{C_2} \vec{F}\cdot\vec{dl} = \int_{x_a}^{x_m} -\mu_K N\hat{x}\cdot\hat{x}\,dx$

$+ \int_{x_m}^{x_b} \mu_K N\hat{x}\cdot\hat{x}\,dx$

$= -\mu_K N(x_m - x_a) + \mu_K N(x_b - x_m)$

$= -\mu_K N(2x_m - x_b - x_a) \neq W_{f_1}$

Rules of thumb for mechanics problems

- Do you need to know information at particular times or at particular locations?
  - Care about timings: Use Newton’s laws (as it requires the equation of motion)
  - Care about locations: Use work and energy
- Are the forces constant?
  - Yes: Use Newton’s laws
  - No: Use work and energy
- Work and energy are scalar quantities, so they cannot determine the direction of velocity or acceleration

Summary

- Kinetic energy is defined by $K = \frac{m}{2}v^2$ and has units of Joules

- The work done by a force $\overrightarrow{F}$ is

$$W = \int_a^b \overrightarrow{F} \cdot d\overrightarrow{\ell}$$

- Work-kinetic energy theorem: The total (net) work done on an object equals the change in its kinetic energy

$$W_{net} = \Delta K = \frac{m}{2}v_b^2 - \frac{m}{2}v_a^2$$

- The work done by conservative forces *does not* depend on the path, while it *does* for nonconservative forces

Competitive marble racing tomorrow