General Physics: Mechanics

Swiss Plasma Center

PHYS-101(en)

Lecture 7a:
Conservation of momentum, center of mass and continuous mass transfer

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Reminder

- Mini mock exam tomorrow
  - In-class (SG1) during normal lecture hours (10:15-11:00)
  - The format is very similar to the exam. You will be given a “booklet” with the questions and space to write your solutions
  - “Cheat” sheet, notes, etc., are all allowed *this time*, just **no talking with neighbors**
  - Turn in at the end if you want exam to be graded (optional). Graded exams will be returned on Monday November 10th

Midterm course feedback

Swiss Plasma Center

- In general positive.

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  - my handwriting is too small

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EPFL Swiss Plasma Center

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  - long lecture on Mondays. Possibility of doing more examples
  - live-streaming of lectures

Today’s agenda (Serway 6, 9 and MIT 8 -12)

Swiss Plasma Center

1. **Brief review of last week**

2. Some examples
   - Conservation of momentum
   - Motion of Center of mass

3. Systems with variable mass
   - Rocket science and interstellar space travel!

DEMO (113)

Recoil of a cart

Review of last week — Momentum

Swiss Plasma Center

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- **Impulse**: Integral of net force over a time interval, or change in momentum over a time interval $\vec{I} = \int_{t_0}^{t_f} \vec{F}\, dt = \Delta\vec{p}$

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Swiss Plasma Center

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- **Center of mass**: For systems and extended bodies, the one point that would move in the same path as a point mass subjected to the same net force

$$\vec{F}_{net}^{ext} = \frac{d\vec{p}_{sys}}{dt} = M\vec{A}_{CM}$$

Conceptual question

You drop a stone from the top of a high cliff. Consider the earth and stone as a system (neglecting the effects of the sun, moon, and other astronomical objects). As the stone falls, the momentum of this system…

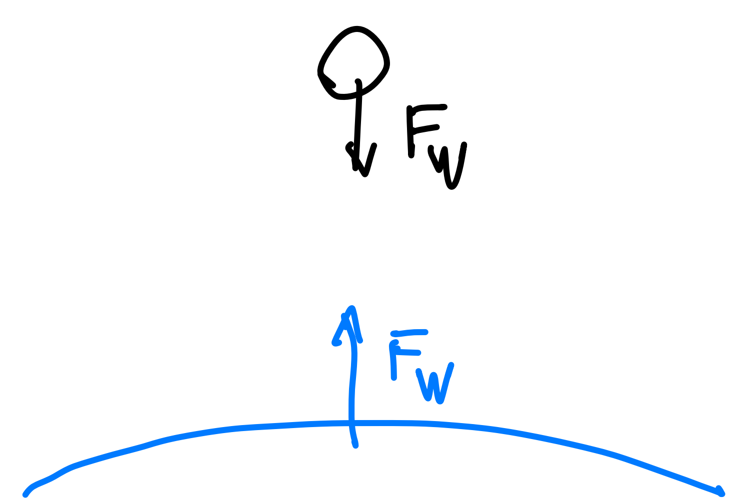
A. increases in the downwards direction.
B. decreases in the downwards direction.
C. stays the same.

$\vec{F}_{ext} \approx 0$

$$\vec{p}(t_0) = \vec{p}_S(t_0) + \vec{p}_E(t_0) = m_S \cancelto{0}{\vec{v}_S}(t_0) + m_E \cancelto{0}{\vec{v}_E}(t_0) = 0$$

$$\vec{p}(t) = m_S \vec{v}_S(t) + m_E \vec{v}_E(t) = 0$$

$$\Rightarrow \vec{v}_E(t) = -\frac{m_S}{m_E} \vec{v}_S(t)$$



Conceptual question

responseware.eu
Session ID: epflphys101en

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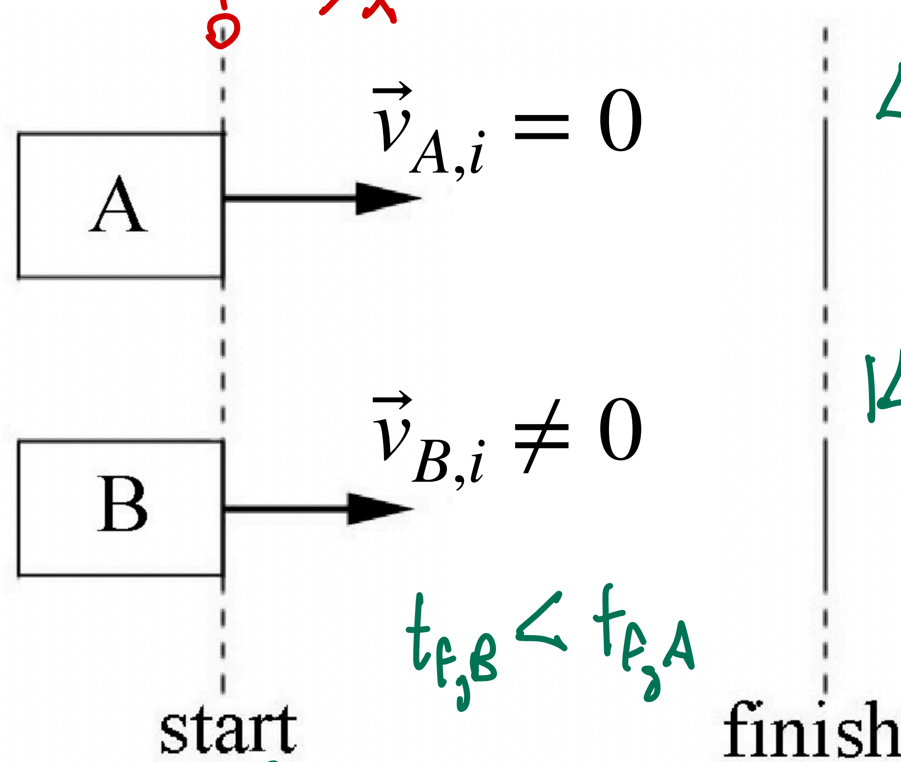
Identical constant forces push two identical objects A and B continuously from a start line to a finish line. If A is initially at rest and B is initially moving to the right… $\vec F = F\hat x$

$0 \rightarrow \hat x$

$F = m a_A$

$\Rightarrow a_A = a_B$

$F = m a_B$



$t_0 = 0$ t_f

$t_{f,B} < t_{f,A}$

$$\Delta\vec p = \vec I = \int_{t_0}^{t_f} \vec F\, dt = \int_0^{t_f} F\hat x\, dt$$
$$= \hat x F \int_0^{t_f} dt = \hat x F t_f$$

$|\Delta\vec p_A| = F t_{f,A}$ $|\Delta\vec p_B| = F t_{f,B}$

$|\Delta\vec p_A| > |\Delta\vec p_B|$

A. object A has the larger change in momentum.
B. object B has the larger change in momentum.
C. Both objects have the same change in momentum.

Conceptual question

responseware.eu
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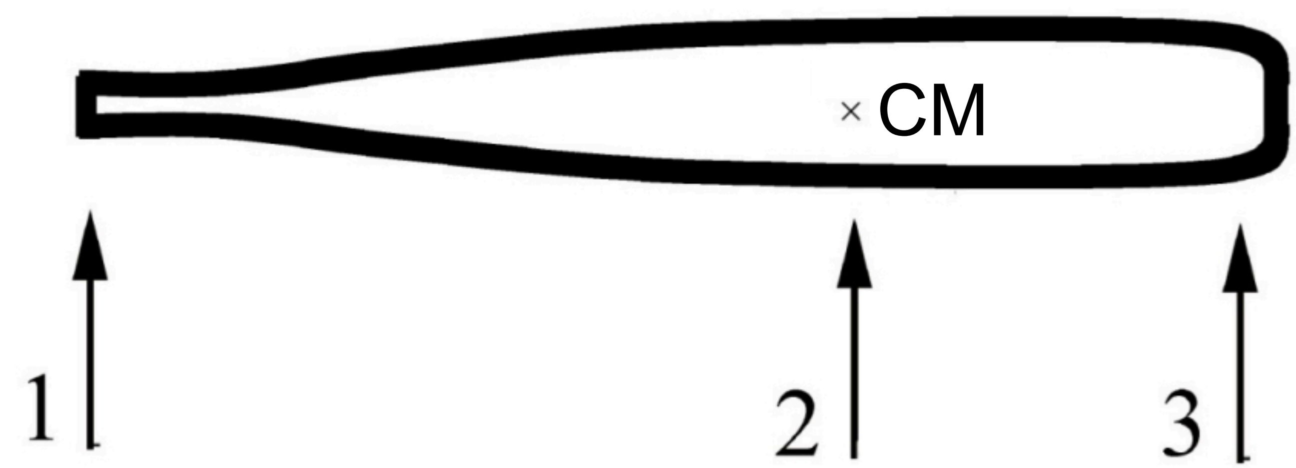
The greatest acceleration of the center of mass (CM) of the baseball bat will be produced by pushing with a force F at

A. Position 1
B. Position 2
C. Position 3
D. All are the same.



$$\sum \vec{F}_{ext} = \vec{F} = M\vec{A}_{CM} \Rightarrow \vec{A}_{CM} = \frac{1}{M}\vec{F}$$

Today’s agenda (Serway 6, 9 and MIT 8 -12)

EPFL Swiss Plasma Center

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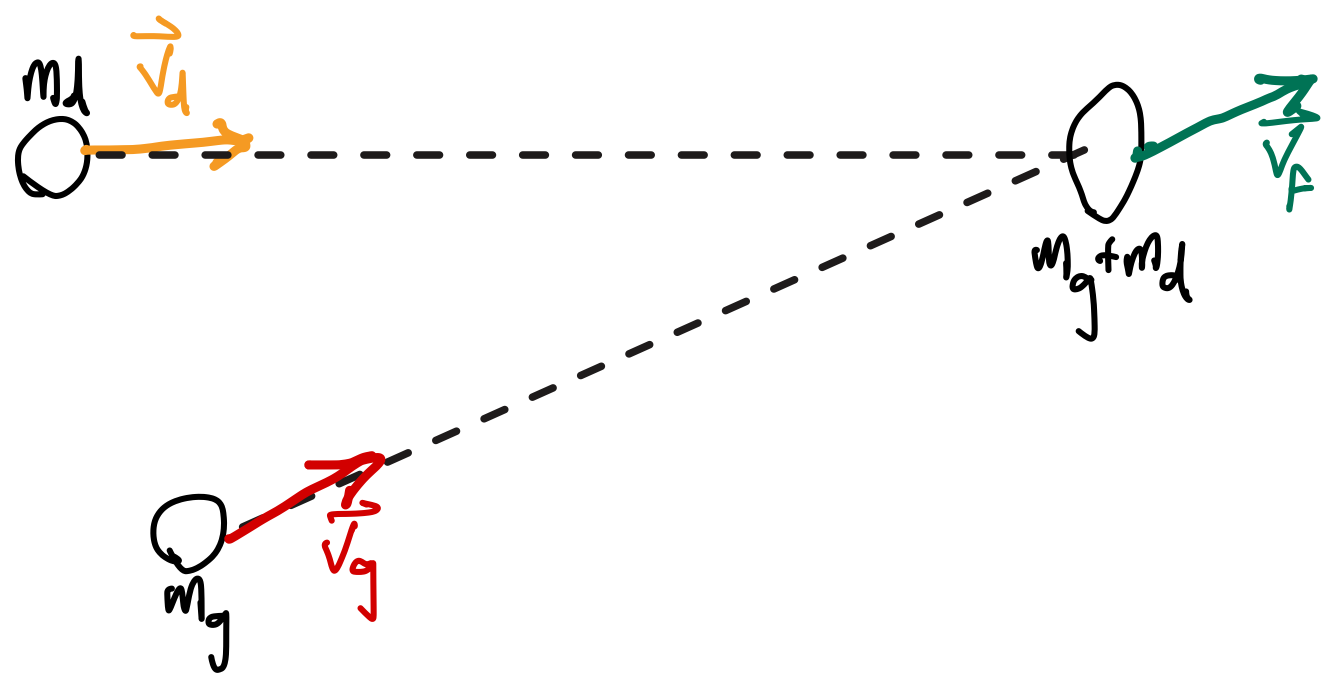
   - **Conservation of momentum**
   - Motion of Center of mass

3. Systems with variable mass
   - Rocket science and interstellar space travel!

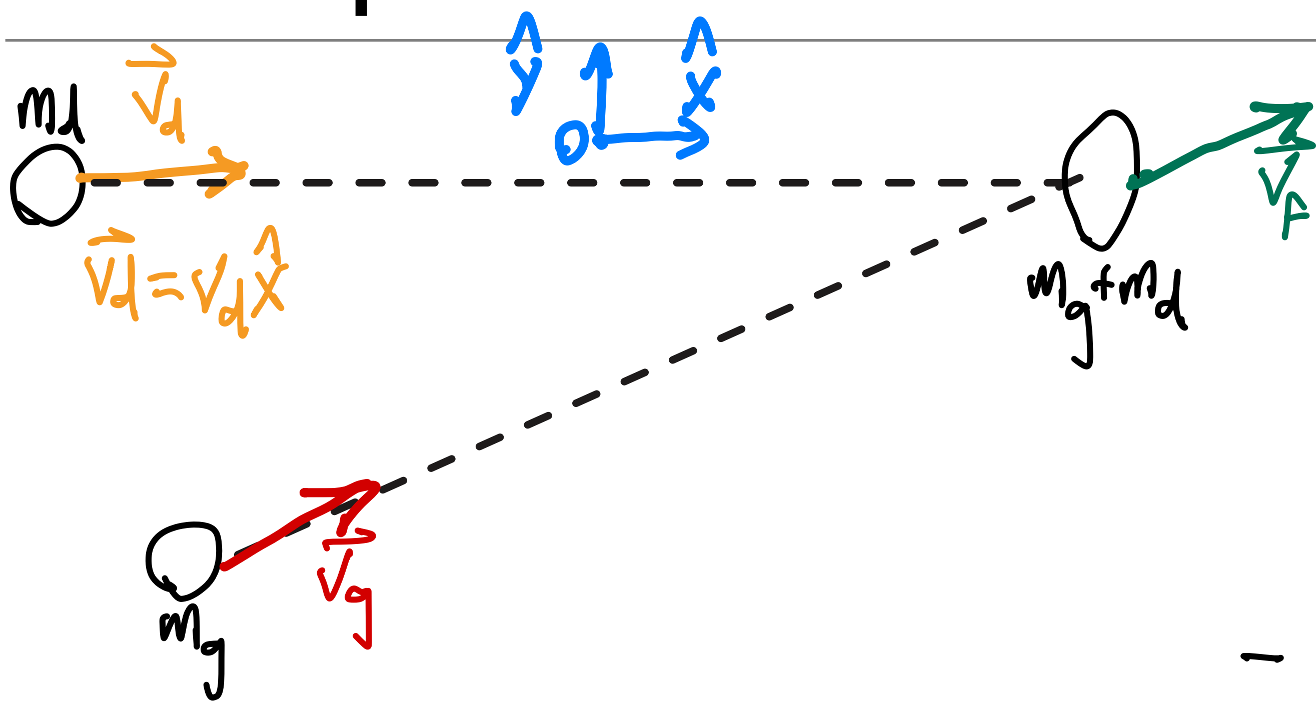
Example: Conservation of momentum

A girl of mass m_g sees that her dog (mass m_d) is playing on a flat icy surface. The dog suddenly falls and starts sliding with velocity $\vec{v}_d$. The girl runs after the dog, but upon touching the icy surface she also falls and slides with velocity $\vec{v}_g$. Fortunately, she has aimed right and reaches the dog.

Upon touching the dog the girl holds on to it and both keep sliding together. What is their velocity $\vec{v}_f$?



Example: Conservation of momentum



- Before collision:

$\vec p_{d,b} = m_d \vec v_d \qquad \vec p_{g,b} = m_g \vec v_g$

$\vec p_b = m_d \vec v_d + m_g \vec v_g$

- After collision:

$\vec p_a = (m_g + m_d)\, \vec v_f$

Cons. of momentum: $\vec p_b = \vec p_a \Rightarrow m_d \vec v_d + m_g \vec v_g = (m_g + m_d)\, \vec v_f$

$$\Rightarrow \boxed{\vec v_f = \frac{1}{m_g + m_d}\left(m_d \vec v_d + m_g \vec v_g\right)}$$

If $\vec v_g = v_g \hat x$ then $\vec v_f = \frac{1}{m_g + m_d}\left[m_d v_d \hat x + m_g v_g \hat x\right] = \hat x \frac{1}{m_g + m_d}\left(m_d v_d + m_g v_g\right)$

If now I also assume $m_g = m_d$ and $\vec v_d = 0$ $\Rightarrow \vec v_f = \frac{m_g}{m_g + m_d} v_g \hat x = \frac{1}{2} v_g \hat x$

DEMO (766)

Inelastic collision

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Swiss Plasma Center

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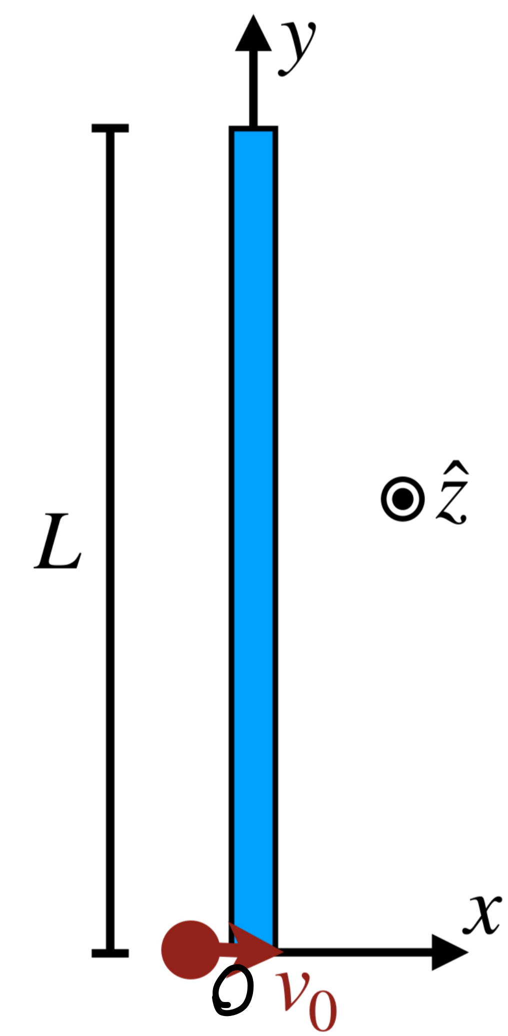
Example: Motion of center of mass (CM)

A slender rod of length L and mass M rests along the y-axis on a flat icy surface. The linear mass density of the rod $\lambda(y)$ varies quadratically with the distance from the origin. $\lambda(y) = Ky^2$

A particle of mass $2M$ that moves along the x-axis with speed v_0 strikes the rod at the instant of time $t = t_0$.

Find the position $\overrightarrow{R}_{CM}$ of the center of mass as a function of t .



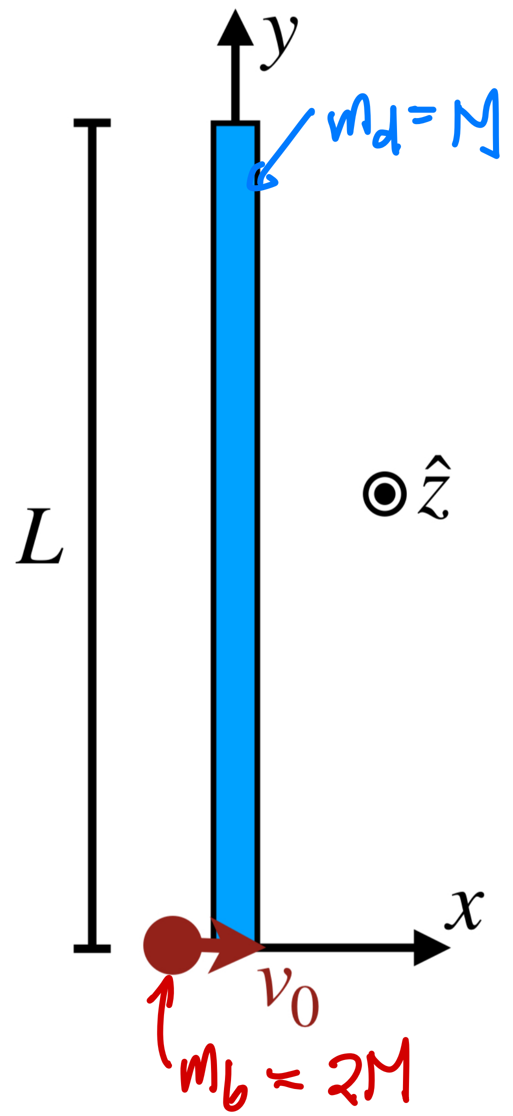
1) What the system?

It is composed of (a) a rod and (b) a particle

2) $\sum \vec{F}_{ext} = 0 = M\vec{A}_{CM} \Rightarrow \vec{A}_{CM} = 0 \Rightarrow \vec{V}_{CM} = \text{constant}$

$\frac{d\vec{R}_{CM}}{dt} = \vec{V}_{CM} \Rightarrow \vec{R}_{CM}(t) - \vec{R}_{CM}(t_0) = \int_{t_0}^{t} \vec{V}_{CM}\, dt = \vec{V}_{CM} \int_{t_0}^{t} dt = (t - t_0)\vec{V}_{CM}$

Example: Motion of center of mass (CM)



$$\vec{R}_{CM}(t) = (t-t_0)\vec{V}_{CM} + \vec{R}_{CM}(t_0)$$

First find $\vec{V}_{CM}$:

Choose $t_b < t_0$, then

$$\vec{V}_{CM}(t_b) = \frac{m_b \underbrace{\vec{v}_b(t_b)}_{v_0\hat{x}} + m_d \cancelto{0}{\vec{V}_{CM,d}(t_b)}}{m_b + m_d} = \frac{m_b}{m_b+m_d} v_0 \hat{x}$$

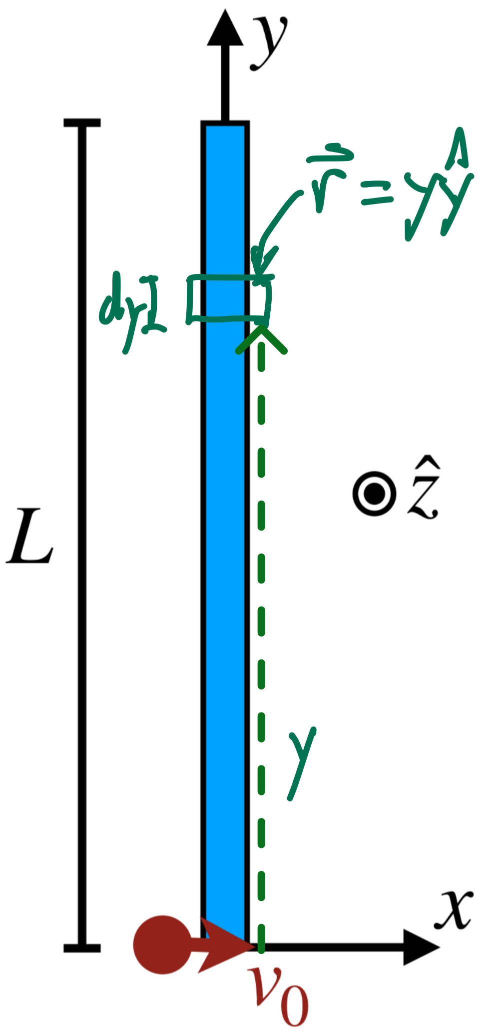
$$= \frac{2M}{2M+M} v_0 \hat{x} = \frac{2}{3} v_0 \hat{x} = \vec{V}_{CM}$$

Next we find $\vec{R}_{CM}(t_0)$

$$\vec{R}_{CM}(t_0) = \frac{m_b \cancelto{0}{\vec{r}_b(t_0)} + m_d \vec{R}_{CM,d}(t_0)}{m_b + m_d} = \frac{m_d}{m_b+m_d} \vec{R}_{CM,d}(t_0)$$

$$= \frac{1}{3} \vec{R}_{CM,d}(t_0)$$

Example: Motion of center of mass (CM)



Now we compute $\vec{R}_{CM,d}(t_0)$

$$\frac{dm}{dy} = \lambda(y) = Ky^2 \Rightarrow dm = Ky^2 dy$$

$$M = m_d = \int_0^L dm = \int_0^L Ky^2 dy = K\int_0^L y^2 dy = K\left[\frac{y^3}{3}\right]_0^L = K\frac{L^3}{3}$$

$$\Rightarrow M = K\frac{L^3}{3} \Rightarrow K = \frac{3M}{L^3}$$

$$\vec{R}_{CM,d}(t_0) = \frac{1}{m_d}\int_0^L \vec{r}\, dm = \frac{1}{m_d}\int_0^L y\hat{y}\cdot Ky^2 dy$$

$$= \hat{y}\frac{1}{m_d}K\int_0^L y^3 dy = \hat{y}\frac{1}{m_d}K\left[\frac{y^4}{4}\right]_0^L = \frac{1}{m_d}K\frac{L^4}{4}\hat{y}$$

$$= \hat{y}\frac{1}{\cancel{M}}\frac{3\cancel{M}}{\cancel{L^3}}\frac{L^{\cancel{4}}}{4} = \hat{y}\frac{3}{4}L$$

$$\vec{R}_{CM}(t) = (t-t_0)\vec{V}_{CM} + \vec{R}_{CM}(t_0) = (t-t_0)\frac{2}{3}v_0\hat{x} + \frac{1}{3}\frac{3}{4}L\hat{y}$$

$$= \boxed{(t-t_0)\frac{2}{3}v_0\hat{x} + \frac{1}{4}L\hat{y}}$$

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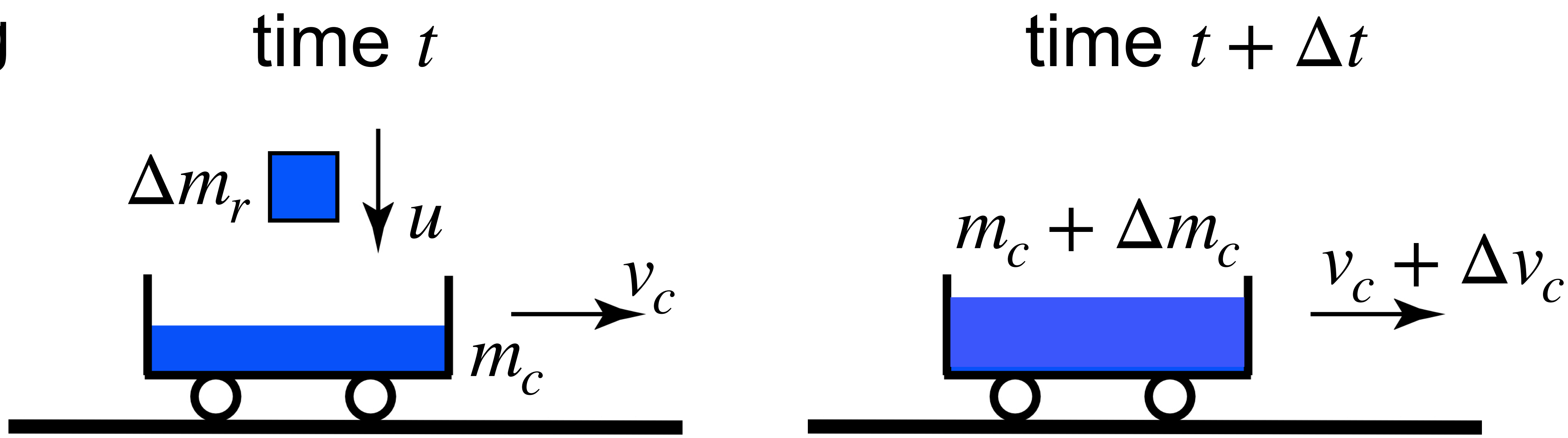
DEMO (178)

Falling chain

Types of systems with changing mass

1. Transfer of mass to an object, but no transfer of momentum along the axis of motion

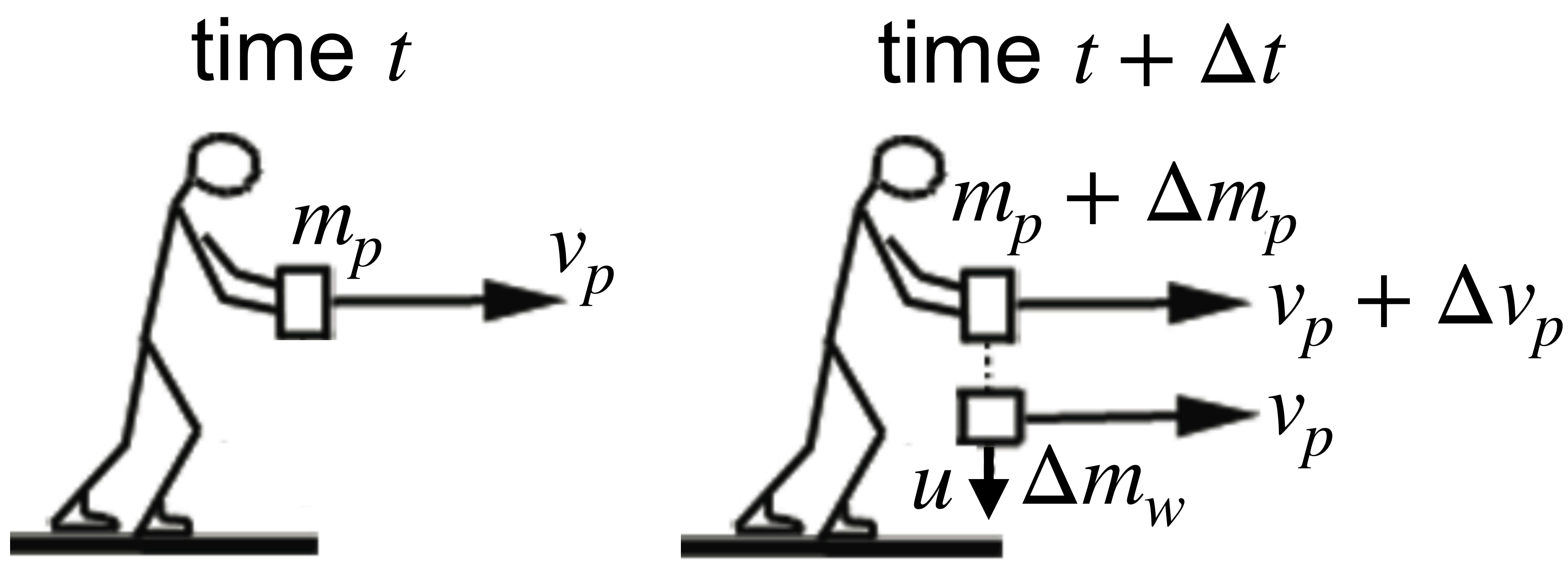
Rain falling into a cart:



Types of systems with changing mass

2. Transfer of mass <u>from</u> an object, but <u>no transfer of momentum</u> along the axis of motion

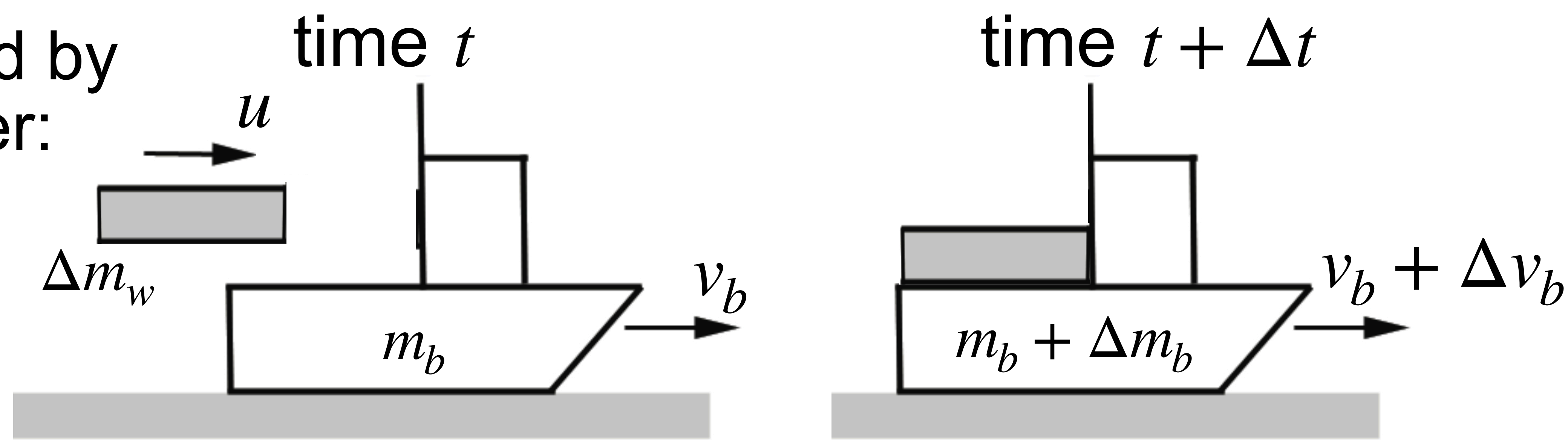
Ice skater with a leaky water bottle:



Types of systems with changing mass

3. Mass <u>hits</u> an object, <u>providing an impulse</u> that transfers momentum along the direction of motion

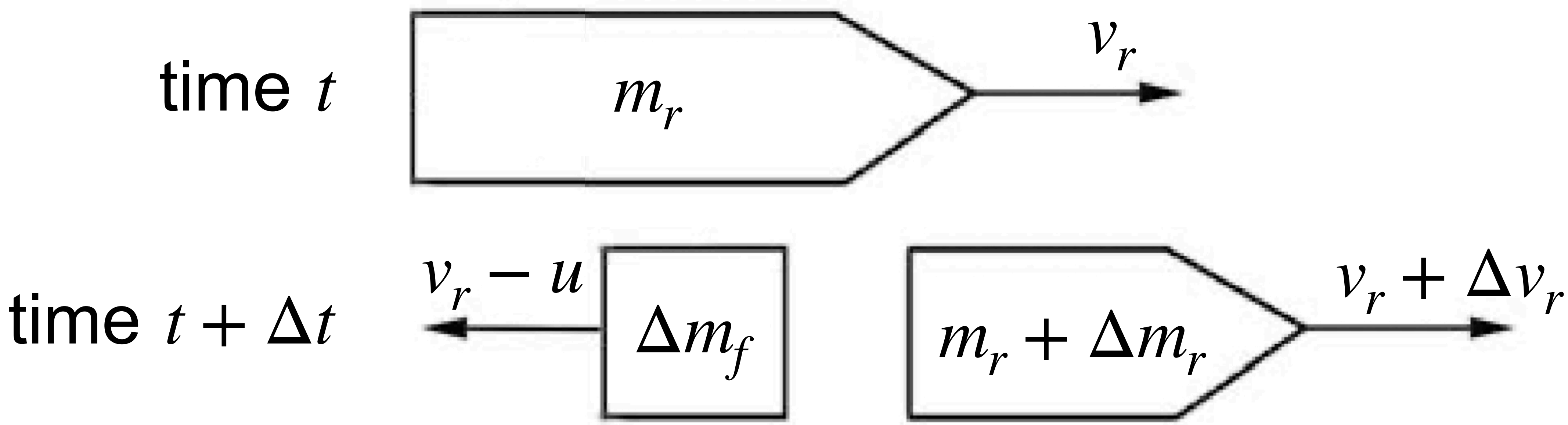
Boat pushed by a jet of water:



Types of systems with changing mass

4. Mass is <u>ejected</u> from an object, <u>resulting in a recoil</u> along the direction of motion

Rocket:



DEMO (113)

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Recoil of a cart
(ejection of gas)

Tackling systems with changing mass

- Determine the velocity of an object with changing mass

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1. Choose reference frame and define system

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Tackling systems with changing mass

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1. Choose reference frame and define system
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4. Use generalized version of Newton's 2nd law

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$$\vec{F}^{ext}_{net} = \frac{d\vec{p}_{sys}}{dt} \boxed{= \lim_{\Delta t \to 0} \frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t}}$$

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5. Find and solve the resulting differential equation

Momentum diagrams

- A simple graphical summary of ALL the masses and velocities in the system
- Similar to free body diagrams
- Draw them at important moments in time or at arbitrary moments

At some time "t"

Δm_R

$\vec{v}_R(t)$

$m_c(t)$

$\vec{v}_c(t)$

A "short" time Δt later (i.e. at $t+\Delta t$)

$m_c(t+\Delta t)$

$\vec{v}_c(t+\Delta t)$

Conserv. of mass

$$m_c(t+\Delta t) = m_c(t) + \Delta m_R$$

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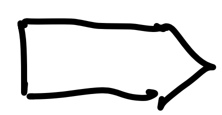
Derivation of rocket equation

A rocket of dry mass M_0 starts from rest in our Solar System with a fuel mass M_f. It burns all the fuel, ejecting it backwards with velocity u relative to the rocket. This exhaust velocity u is independent of the velocity of the rocket. What is the final speed of the rocket? How long will it take to get to Proxima Centauri?

$\longmapsto \hat{x}$



Everything happens along x ⇒ 1D problem

No external forces

At some "t"

$\vec{v}_r(t) = v_r(t)\hat{x}$

A short Δt later

$$\vec{v}_e = \vec{v}_r(t) + \vec{u} = v_r(t)\hat{x} - u\hat{x} = (v_r(t) - u)\hat{x}$$

Cons. of mass

$$m_r(t) = m_r(t+\Delta t) + \Delta m_e$$

$$\Rightarrow \underbrace{m_r(t+\Delta t) - m_r(t)}_{=\Delta m_r} = -\Delta m_e$$

Derivation of rocket equation

$$p_{sys}(t) = m_r(t)\,v_r(t)$$

$$p_{sys}(t+\Delta t) = \overbrace{m_r(t+\Delta t)}^{m_r(t)-\Delta m_e}\,v_r(t+\Delta t) + \Delta m_e\,[v_r(t) - u]$$

$$= m_r(t)\,v_r(t+\Delta t) \underbrace{- \Delta m_e\,v_r(t+\Delta t) + \Delta m_e\,v_r(t)}_{= -\Delta m_e\,[v_r(t+\Delta t) - v_r(t)] \approx 0} - \Delta m_e\,u$$

$$\approx m_r(t)\,v_r(t+\Delta t) \underbrace{- \Delta m_e}_{-\Delta m_e = \Delta m_r}\,u$$

$$\frac{1}{\Delta t}[p_{sys}(t+\Delta t) - p_{sys}(t)] = \frac{1}{\Delta t}[m_r(t)\,v_r(t+\Delta t) + \Delta m_r\,u - m_r(t)\,v_r(t)]$$

$$\xrightarrow{\Delta t \to 0} \frac{dp_{sys}}{dt} = F_{ext} = 0$$

in this case

$$= m_r(t)\cdot\frac{1}{\Delta t}[v_r(t+\Delta t) - v_r(t)] + u\,\frac{\Delta m_r}{\Delta t}$$

$$\xrightarrow{\Delta t \to 0} m_r(t)\,\frac{dv_r}{dt} + u\,\frac{dm_r}{dt}$$

$$\Rightarrow\; 0 = m_r\,\frac{dv_r}{dt} + u\,\frac{dm_r}{dt} \;\Rightarrow\; \boxed{\frac{1}{u}\,\frac{dv_r}{dt} = -\frac{1}{m_r}\,\frac{dm_r}{dt}}$$

Derivation of rocket equation

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$$\frac{1}{\underbrace{u}_{\text{const}}}\frac{dv_r}{dt} = -\frac{1}{m_r}\frac{dm_r}{dt}$$

$$\frac{1}{u}\frac{dv_r}{dt} = \frac{d}{dt}\left(\frac{1}{u}v_r\right) \quad \text{because } u = \text{const}$$

$$\frac{d}{dt}\left[\ln(m_r(t))\right] = \frac{1}{m_r}\frac{dm_r}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{u}v_r\right] = -\frac{d}{dt}\left[\ln(m_r(t))\right] \Rightarrow \frac{1}{u}v_r = -\ln(m_r(t)) + C_1$$

$$\Rightarrow v_r(t) = -u\cdot\ln(m_r(t)) + C_2$$

Now we use initial conditions:

At $t=0$ $m_r(t) = M_0 + M_F$ and $v_r(0) = 0$

$$\Rightarrow 0 = v_r(0) = -u\cdot\ln(M_0+M_F) + C_2 \Rightarrow C_2 = u\cdot\ln(M_0+M_F)$$

Therefore $v_r(t) = -u\cdot\ln(m_r(t)) + u\cdot\ln(M_0+M_F) = u\ln\left(\frac{M_0+M_F}{m_r(t)}\right)$

When fuel is over $m_r = M_0 \Rightarrow v_{rf} = u\ln\left(\frac{M_0+M_F}{M_0}\right) = \boxed{u\cdot\ln\left(1+\frac{M_F}{M_0}\right)}$

DEMO (172)

Rockets!

Significance of the rocket equation

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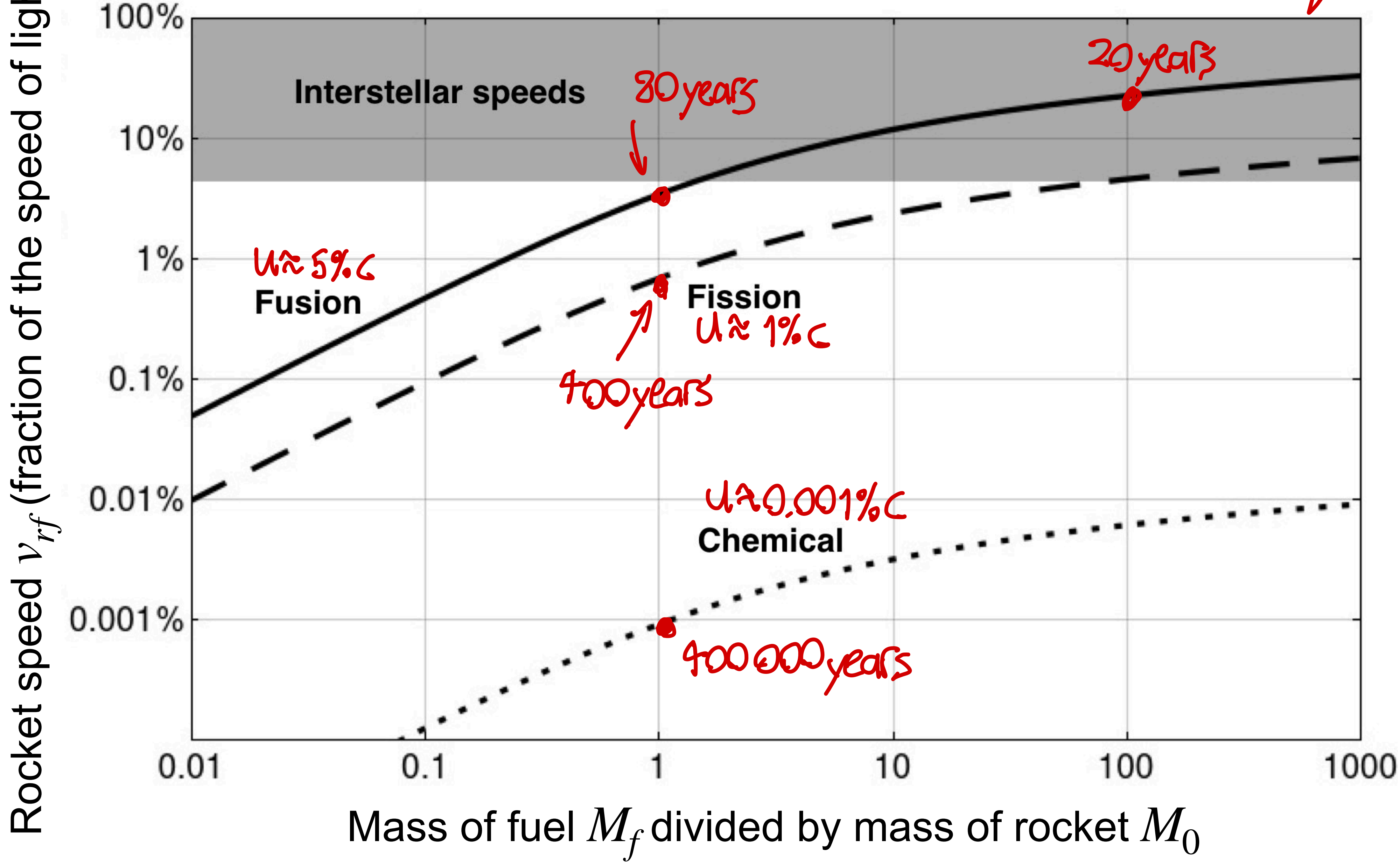
$$v_{rf} = u \ln\left(1 + \frac{M_f}{M_0}\right)$$

Alpha Centauri is ≈ 4 light years away



Conceptual question

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A rocket is moving in outer space without gravity. It is burning fuel so that its "thrust" is constant (i.e. $u\dfrac{dm}{dt} = \text{const}$) at all times. Its acceleration is…

A. constant.
B. decreasing.
C. increasing.

$$-m_r \frac{dv_r}{dt} = u \overbrace{\frac{dm_r}{dt}}^{\text{negative}}$$

$$\Rightarrow \frac{dv_r}{dt} = -\frac{1}{m_r} u \frac{dm_r}{dt} > 0$$

Summary

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- Tackling systems with changing mass

1. Choose reference frame and define system
2. Apply conservation of mass
3. Use generalized Newton's 2nd law: $\vec{F}^{ext}_{net} = \lim_{\Delta t \to 0} \frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t}$
4. Find and solve the resulting differential equation

- Rocket equation: $v_{rf} = u \ln \left(1 + \frac{M_f}{M_0} \right)$

Mock exam tomorrow 😲