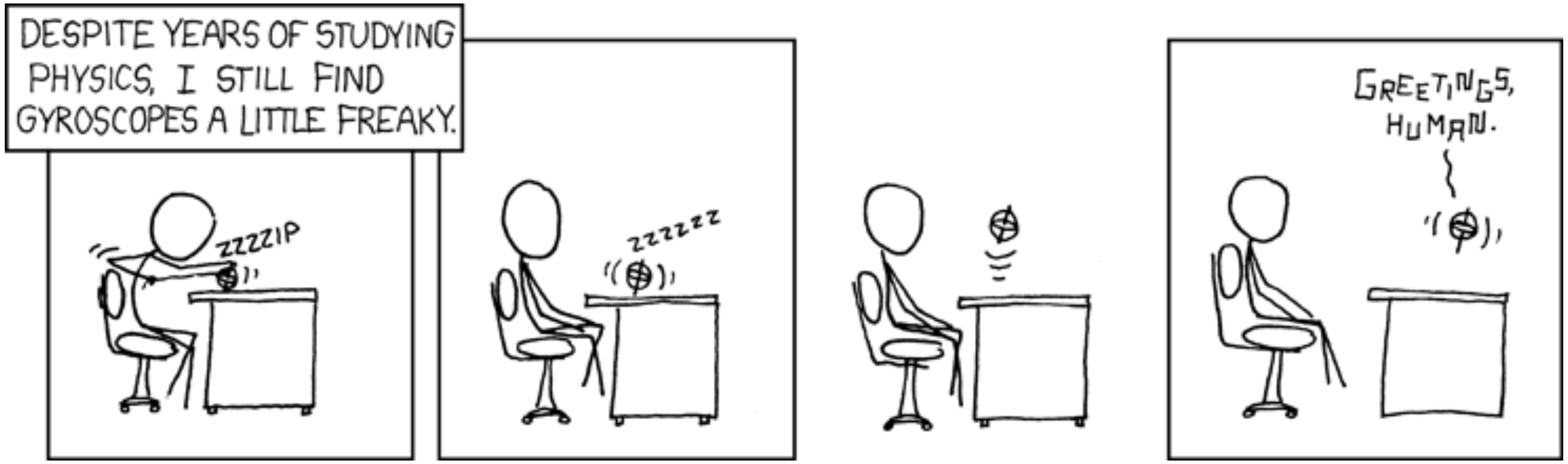
General Physics: Mechanics

Swiss Plasma Center

PHYS-101(en)

Lecture 13a: Kepler's laws, gyroscopes and harmonic motion



xkcd.com/332

Dr. Marcelo Baquero
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December 8th, 2025

Announcements

- Next Monday (December 15th) we will start the lecture with written course feedback

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- Next Monday (December 15th) we will start the lecture with written course feedback
- Next Monday I will give back the graded Mock exams for those of you who turned them in

Announcements

EPFL Swiss Plasma Center

- An agreement was reached among PHYS-101 lecturers regarding the Formula sheet

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  - You will be given a Formula sheet prepared by me
    - I will upload it to Moodle this week

Announcements

Swiss Plasma Center

- An agreement was reached among PHYS-101 lecturers regarding the Formula sheet
  - You will be given a Formula sheet prepared by me
    - I will upload it to Moodle this week
  - Additionally, you are allowed to bring a **one-sided** A4 sheet made by you.
    - Handwritten, no solved exercises.

Today’s agenda (Serway 11,13; MIT 22,23)

1. **Kepler's laws of planetary motion**
2. Gyroscopes
3. Harmonic motion
   - Simple harmonic motion

Kepler's laws of planetary motion

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Kepler's laws of planetary motion

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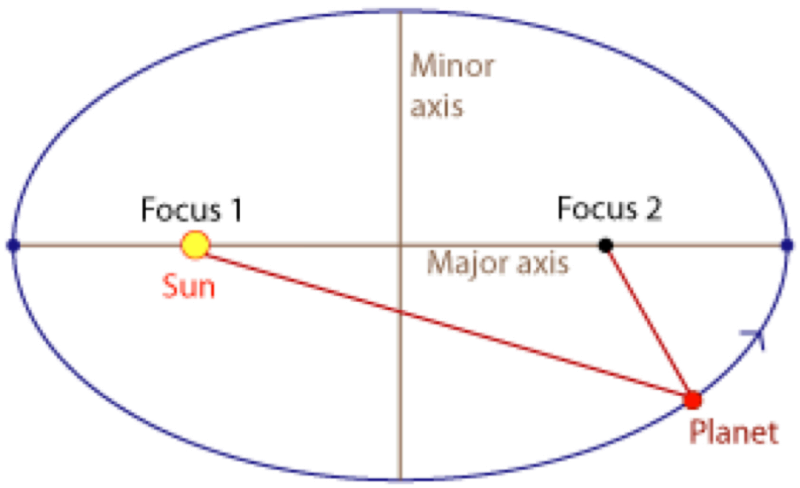
  1. The orbit of each planet is an ellipse, with the Sun at one focus.

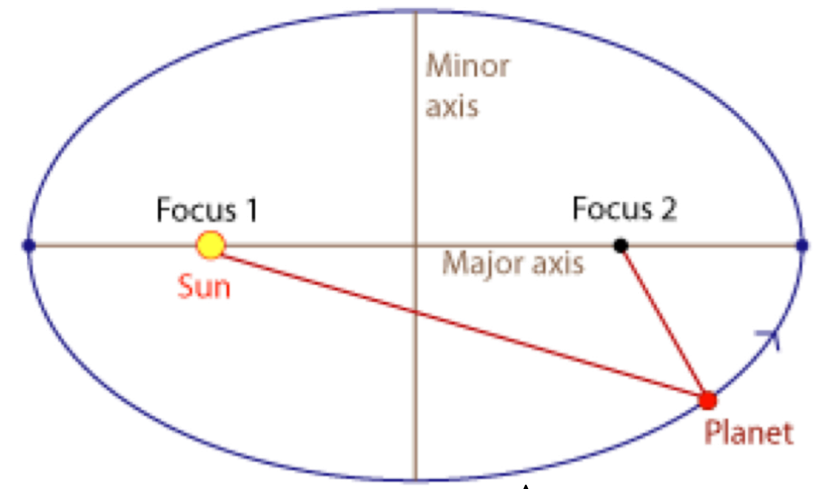


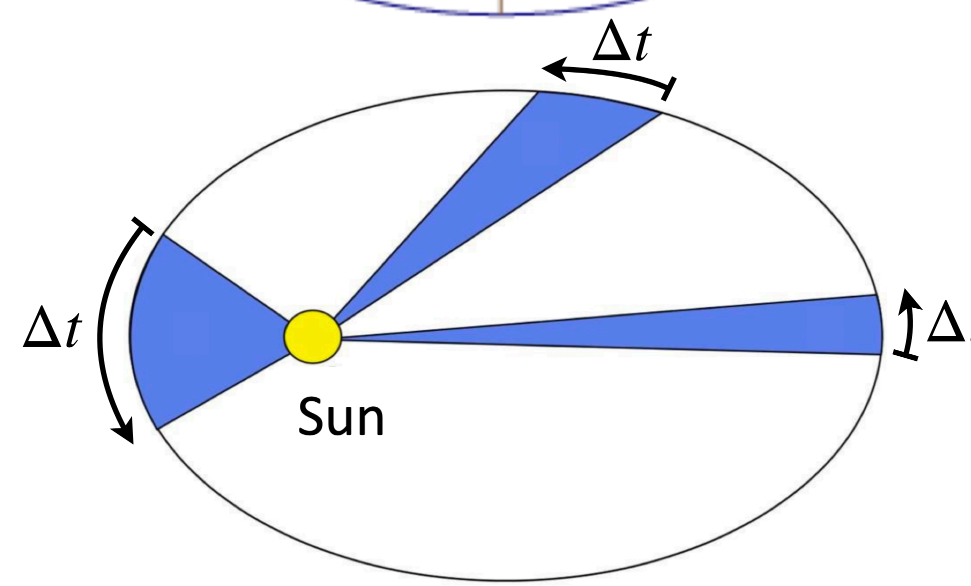
Kepler's laws of planetary motion

- From 1610-1619 Johannes Kepler wrote:

    1. The orbit of each planet is an ellipse, with the Sun at one focus.

    2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.





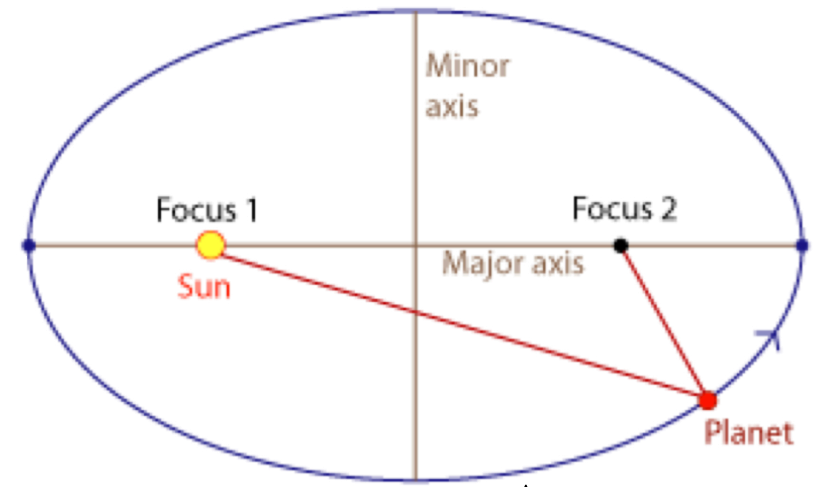
Kepler's laws of planetary motion

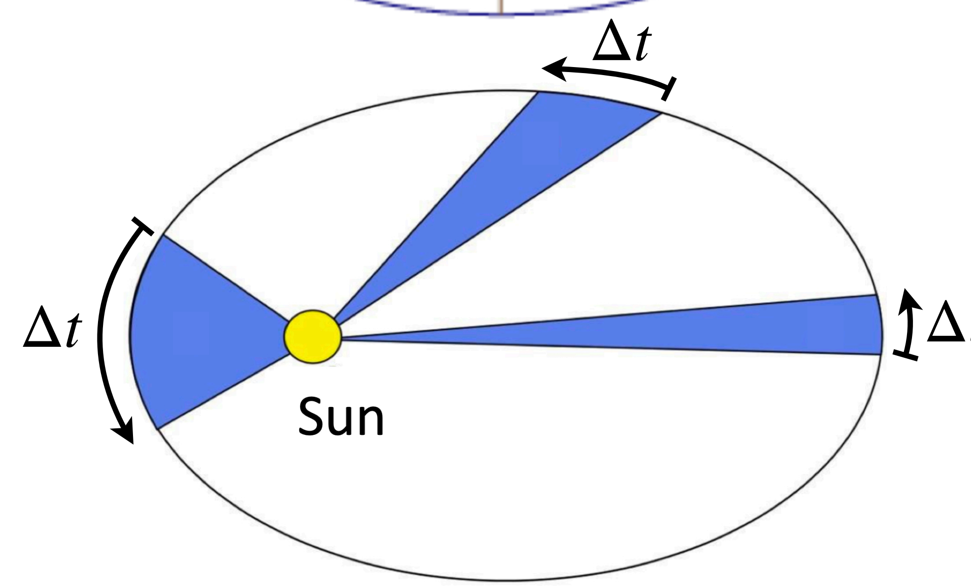
- From 1610-1619 Johannes Kepler wrote:

    1. The orbit of each planet is an ellipse, with the Sun at one focus.

    2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.

    3. The square of a planet's orbital period is proportional to the cube of its mean distance from the Sun.





Planet	Period, T (Earth year)	Avg distance to Sun, r (10^6 km)	T^2/r^3 (10^{-25} yr^2/km^3)
Mercury	0.241	57.9	2.99
Venus	0.615	108.2	2.99
Earth	1	149.6	2.99
Mars	1.88	227.9	2.99
Jupiter	11.86	778.3	2.98
Saturn	29.5	1427	2.99
Uranus	84.0	2870	2.98
Neptune	165	4497	2.99

Kepler’s laws of planetary motion

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3. The square of a planet’s orbital period is proportional to the cube of its mean distance from the Sun.

- Approximate orbits as circular and use

$$\vec{F}_G = -G\frac{m_p m_s}{r^2}\hat{r}$$

Planet	Period, T (Earth year)	Avg distance to Sun, r (10^6 km)	T^2/r^3 (10^{-25} yr^2/km^3)
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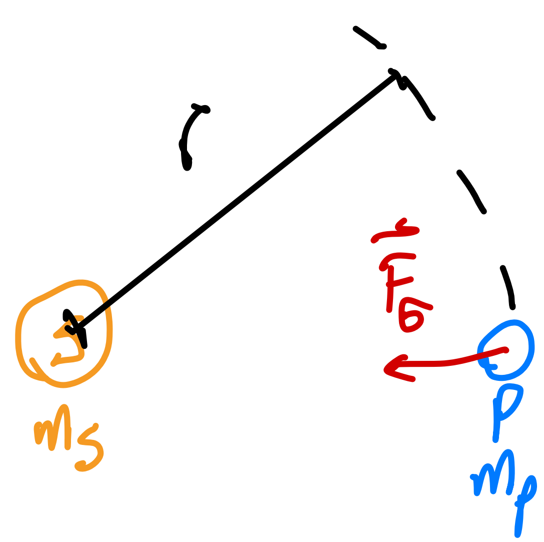


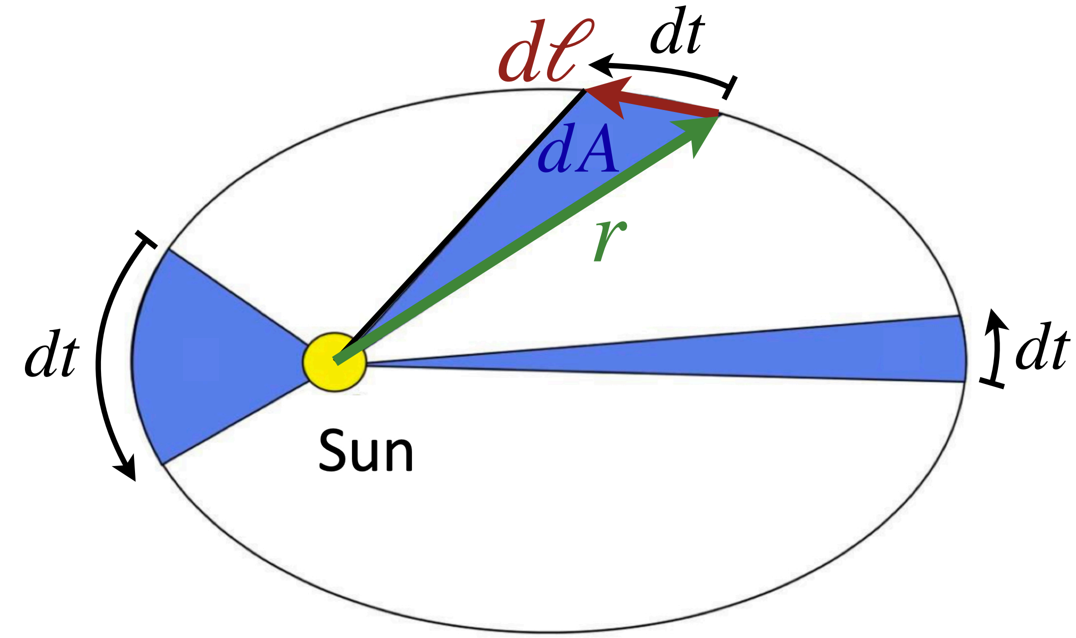
$$\vec{F}_G = \vec{F}_{cent} \Rightarrow -G\frac{m_s m_p}{r^2} = m_p a_{cent} = -m_p r\omega^2$$

$$\Rightarrow G\frac{m_s}{r^2} = r\omega^2 = r\left(\frac{2\pi}{T}\right)^2 = r\frac{4\pi^2}{T^2}$$

$$\Rightarrow \boxed{T^2 = \frac{4\pi^2}{G m_s} r^3}$$

Kepler's laws of planetary motion

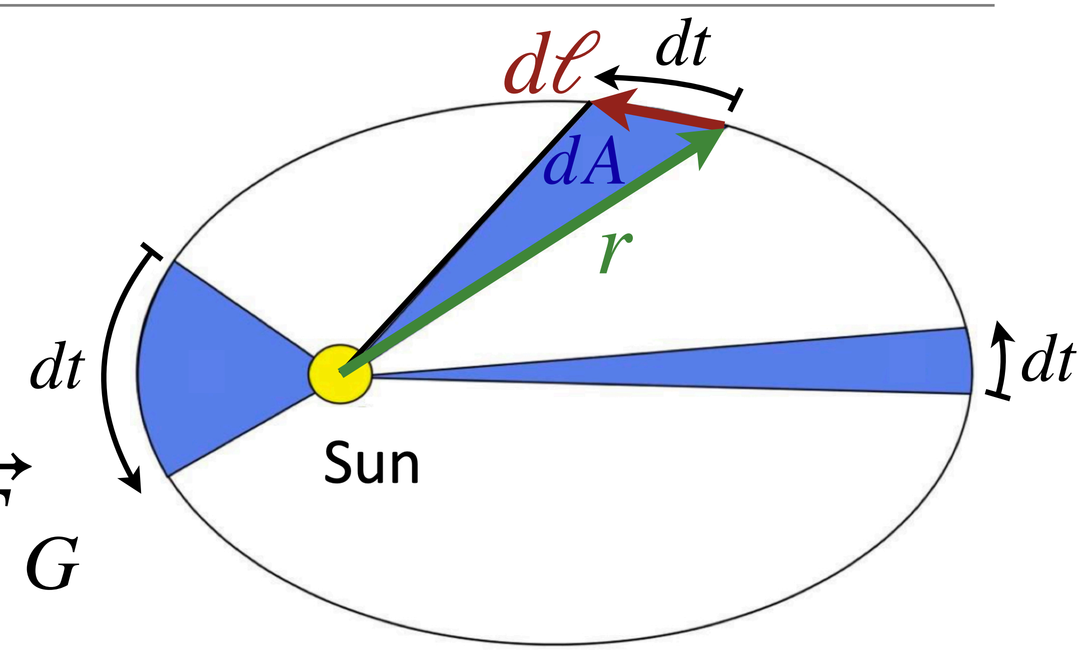
2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.



Kepler's laws of planetary motion

2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.

- Gravity is a central force, so $\vec{r} \,||\, \overrightarrow{F}_G$



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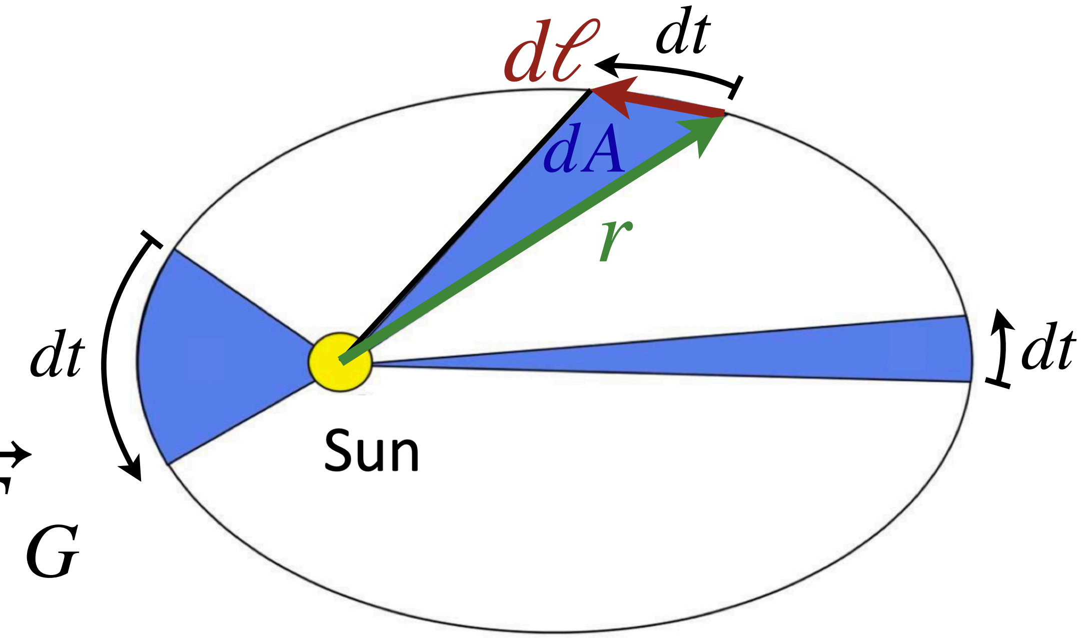

- Gravity is a central force, so $\vec{r} \,||\, \overrightarrow{F}_G$
- Thus, $\vec{\tau}_G = \vec{r} \times \overrightarrow{F}_G = 0$, so $\overrightarrow{L}_p = \vec{r} \times m_p \vec{v}$ is conserved

Kepler's laws of planetary motion

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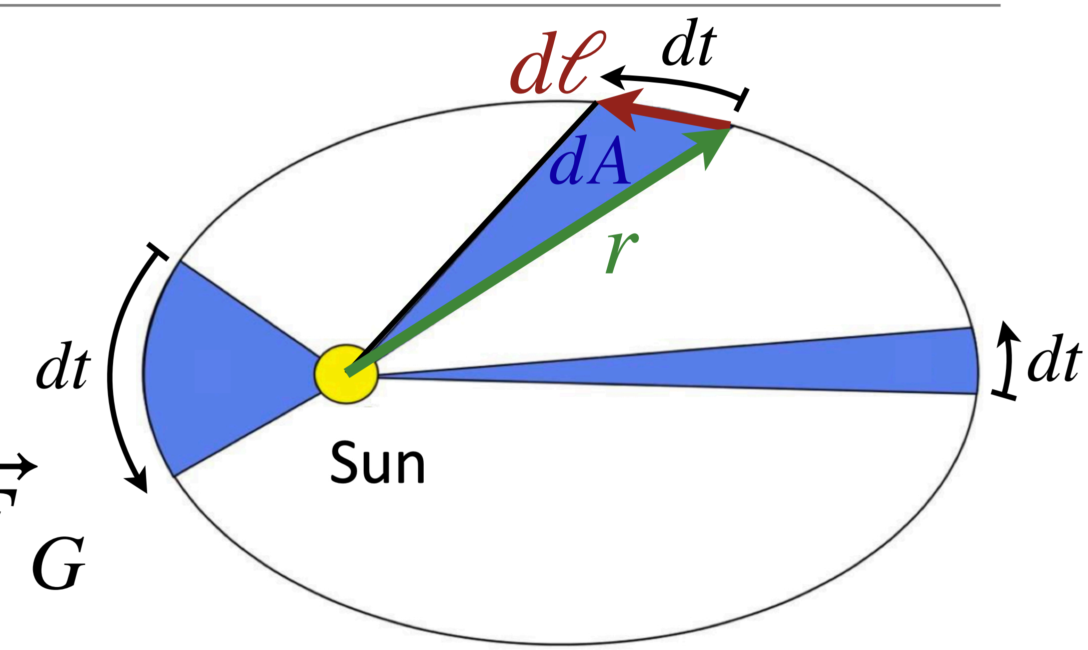

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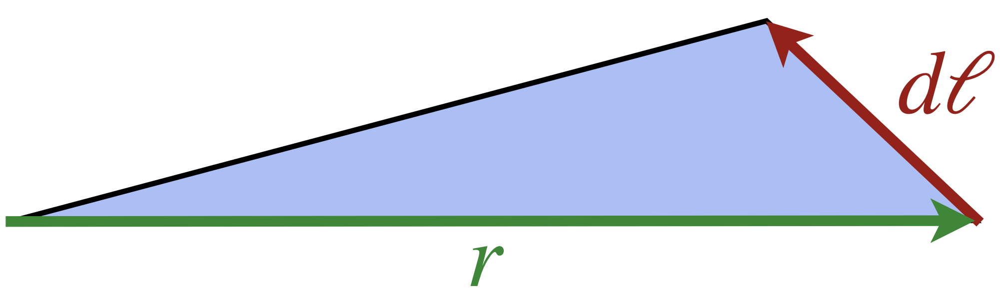
- Thus, $\vec{\tau}_G = \vec{r} \times \overrightarrow{F}_G = 0$, so $\overrightarrow{L}_p = \vec{r} \times m_p \vec{v}$ is conserved
- How is this related to area?



Kepler's laws of planetary motion

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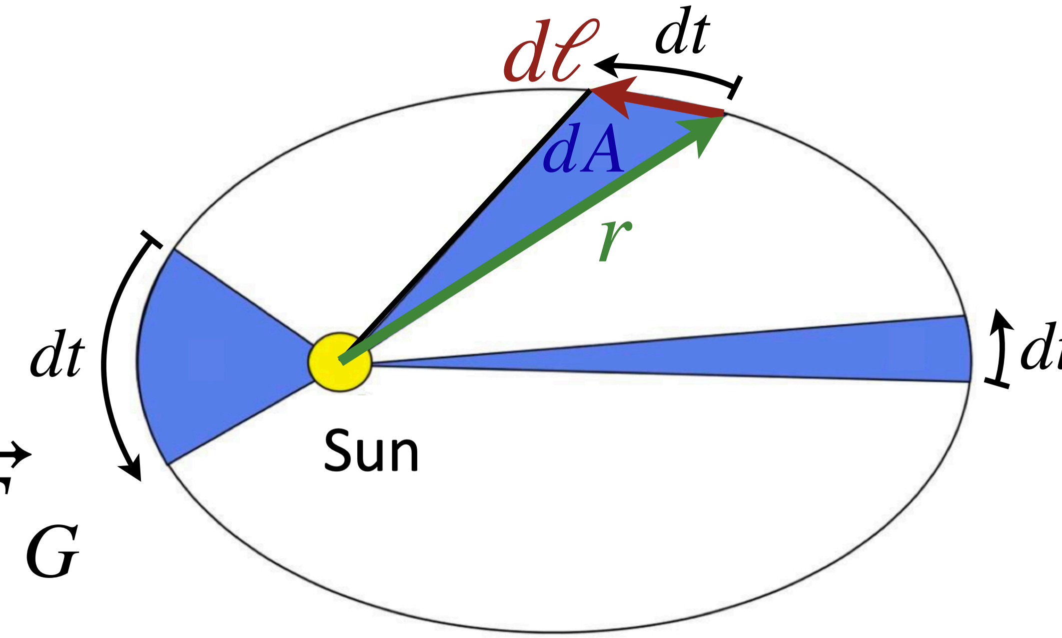

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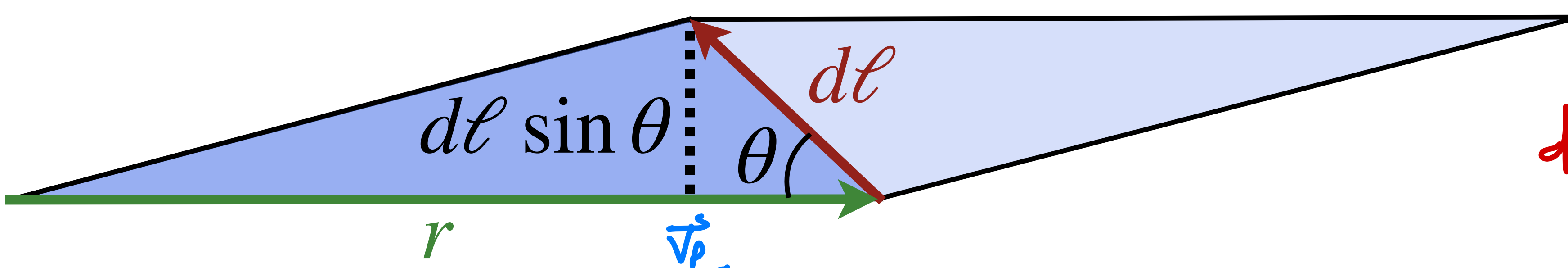

$$A_p = r\, d\ell \sin(\theta)$$
$$dA = \frac{1}{2} A_p = \frac{1}{2} r\, d\ell \sin(\theta) = \frac{1}{2} |\vec{r} \times d\vec{\ell}\,|$$

$$dA = \frac{1}{2}|\vec{r} \times d\vec{\ell}\,| = \frac{1}{2}\left|\vec{r} \times \frac{d\vec{\ell}}{dt}\, dt\right| = \frac{1}{2}|\vec{r} \times \vec{v}_p|\, dt = \frac{1}{2m_p}|\vec{r} \times m_p\vec{v}_p|\, dt = \frac{1}{2m_p}|\vec{L}_p|\, dt$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2m_p}|\vec{L}_p| = \text{constant}$$

Kepler's laws of planetary motion

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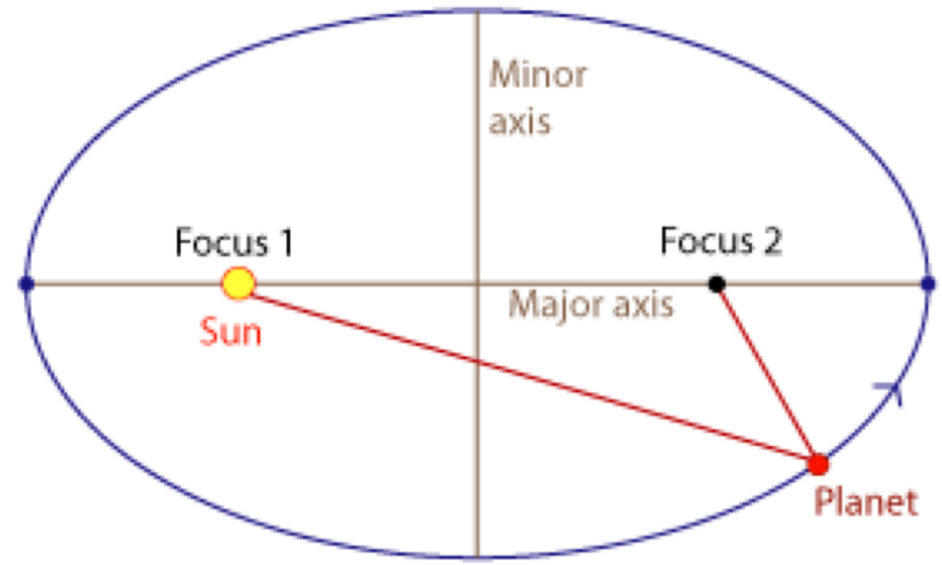


1. The orbit of each planet is an ellipse, with the Sun at one focus.

- Need to know the universal gravitational potential energy

$$U_G = -G\frac{m_p m_s}{r}$$

- Apply mechanical energy conservation:

$$E_m = K + U_G = \frac{1}{2} m v_p^2 - G\frac{m_p m_s}{r} = const$$

- Apply conservation of angular momentum:

$$\vec{L}_p = \vec{r} \times m_p \vec{v}_p = const$$

- Considerable mathematical magic

Today’s agenda (Serway 11,13; MIT 22,23)

Swiss Plasma Center

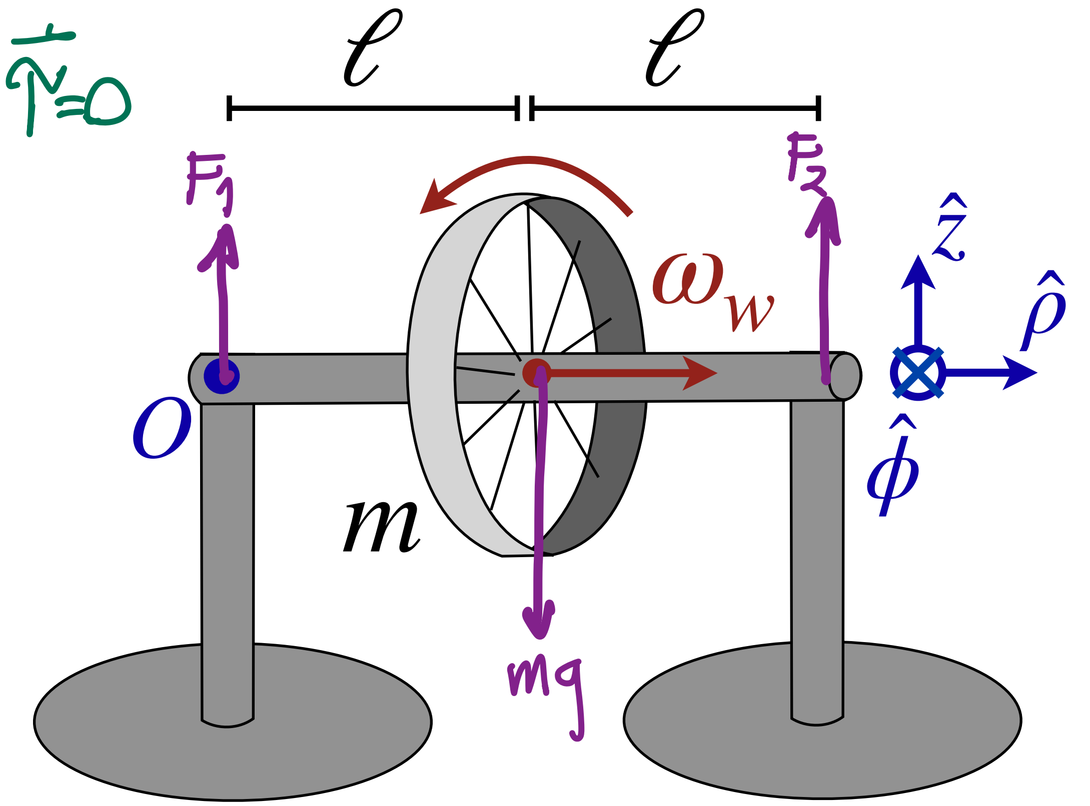
1. Kepler's laws of planetary motion
2. **Gyroscopes**
3. Harmonic motion
   - Simple harmonic motion

DEMO (48)

Bicycle wheel

Analyzing fixed axis rotation

Beam + wheel is in static eq: $\sum \vec{F} = 0$, $\sum \vec{\tau} = 0$



$\sum \vec{F} = 0$

$\sum F_z = 0 = F_1 + F_2 - mg \Rightarrow F_1 = mg - F_2$

$\sum \vec{\tau} = 0$ I choose "O" to be pivot

$\vec{\tau}_1 = 0 \times \vec{F}_1 = 0$ $\quad \vec{\tau}_2 = (2\ell\hat{\rho}) \times (F_2 \hat{z}) = 2F_2\ell(\hat{\rho} \times \hat{z}) = 2F_2\ell(-\hat{\phi}) = -2F_2\ell\hat{\phi}$

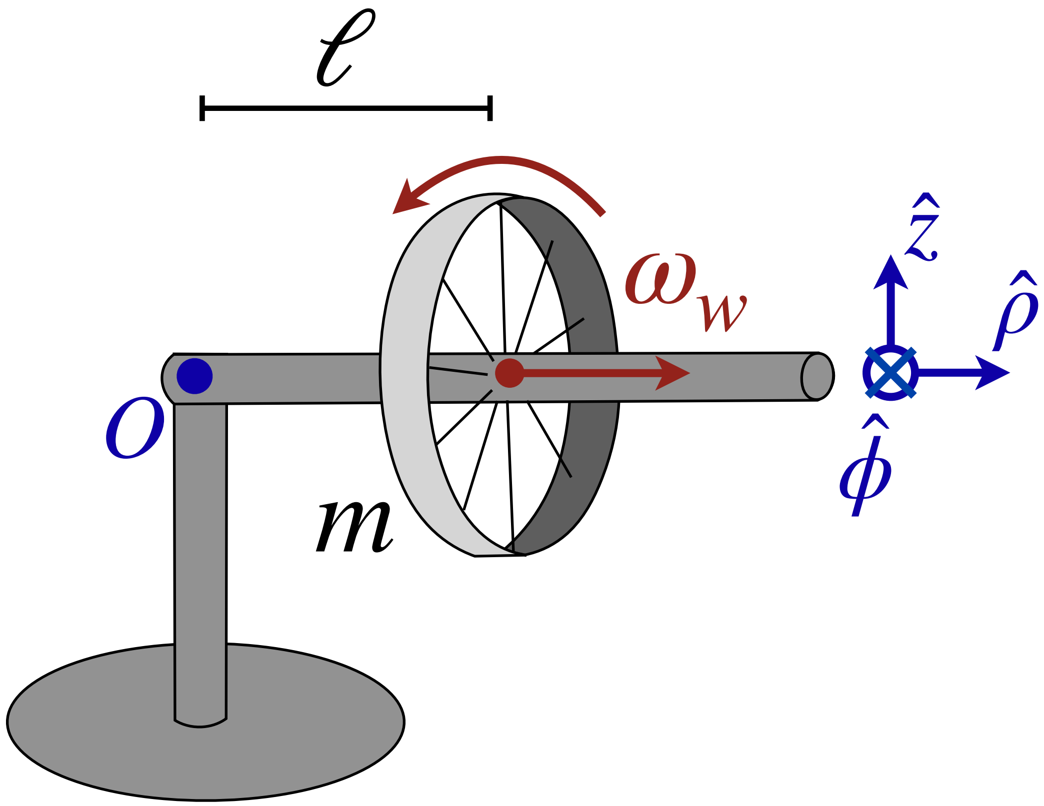
$\vec{\tau}_{mg} = (\ell\hat{\rho}) \times (-mg\hat{z}) = -mg\ell(\hat{\rho} \times \hat{z}) = mg\ell\hat{\phi}$

$\sum \vec{\tau} = -2F_2\ell\hat{\phi} + mg\ell\hat{\phi} = (-2F_2 + mg)\ell\hat{\phi} = 0$

Along $\hat{\phi}$: $-2F_2 + mg = 0 \Rightarrow F_2 = \frac{1}{2}mg \Rightarrow F_1 = mg - \frac{1}{2}mg = \frac{1}{2}mg$

$$\boxed{F_1 = F_2 = \frac{1}{2}mg}$$

DEMO (48): A gyroscope

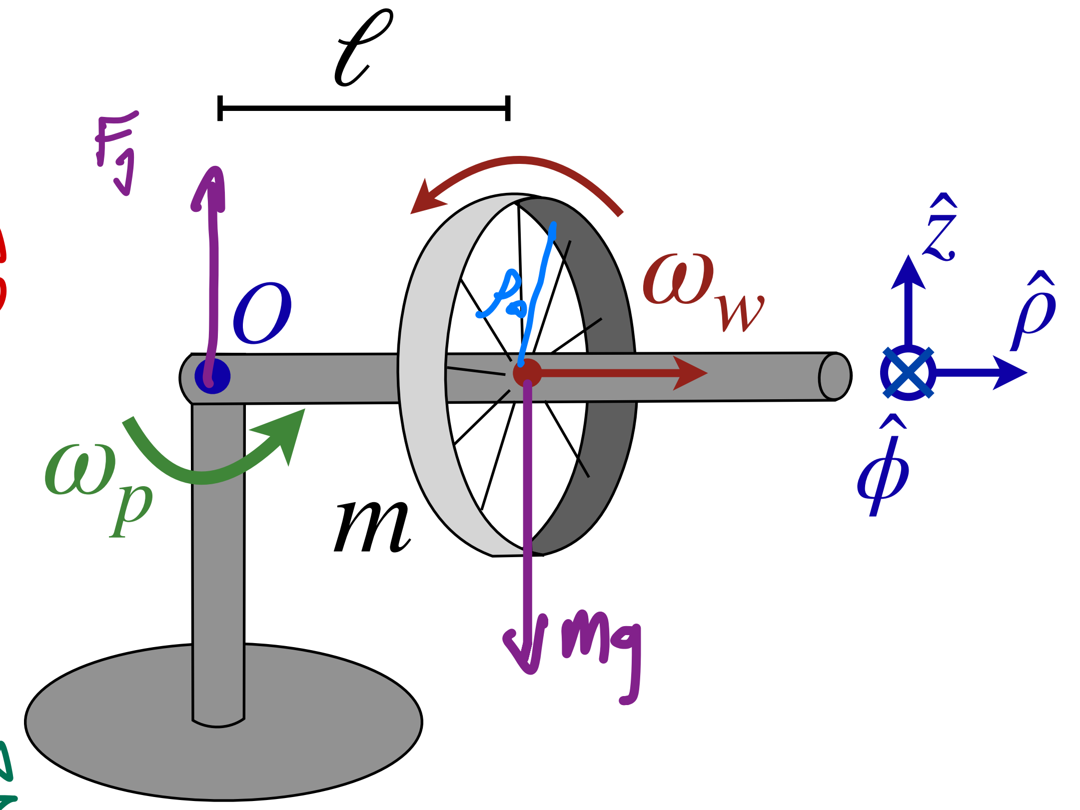


Analyzing a gyroscope

$\sum \vec{\tau}$ (I choose "O" as pivot again)

$\vec{\tau}_{F_1} = 0 \times \vec{F}_1 = 0 \qquad \vec{\tau}_{mg} = (\ell\hat{\rho}) \times (-mg\hat{z}) = mg\ell\hat{\phi}$

$\Rightarrow \sum \vec{\tau} = mg\ell\hat{\phi}$



$\sum \vec{\tau} = \frac{d}{dt}\vec{L}_{tot} = \frac{d}{dt}(\vec{L}_w + \vec{L}_p)$

$\vec{L}_w = I_w \vec{\omega}_w = \overbrace{I_w}^{m\rho_0^2} \omega_w \hat{\rho} \qquad \vec{L}_p = I_p \vec{\omega}_p = I_p \omega_p \hat{z}$

$\frac{d\vec{L}_{tot}}{dt} = \frac{d}{dt}\left[I_w\omega_w\hat{\rho} + I_p\omega_p\hat{z}\right] = \underline{I_w \frac{d}{dt}\left[\omega_w\hat{\rho}\right]} + I_p \frac{d\omega_p}{dt}\hat{z} \qquad \frac{d\hat{\rho}}{dt} = \dot{\phi}\hat{\phi} = \omega_p\hat{\phi}$

$= I_w\left[\frac{d\omega_w}{dt}\hat{\rho} + \omega_w\frac{d\hat{\rho}}{dt}\right] = I_w\frac{d\omega_w}{dt}\hat{\rho} + I_w\omega_w\omega_p\hat{\phi}$

$\Rightarrow mg\ell\hat{\phi} = \sum\vec{\tau} = \frac{d\vec{L}_{tot}}{dt} = I_w\dot{\omega}_w\hat{\rho} + I_w\omega_w\omega_p\hat{\phi} + I_p\dot{\omega}_p\hat{z}$

Analyzing a gyroscope

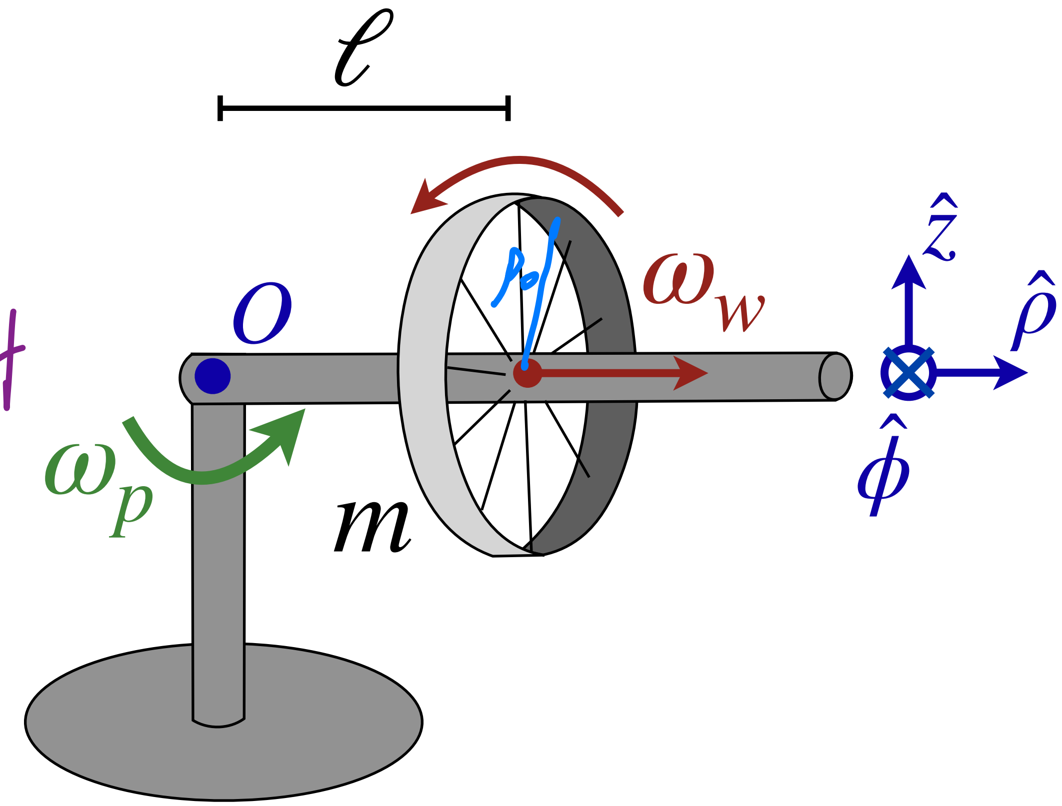
$$mg\ell\,\hat{\phi} = I_w \dot{\omega}_w \hat{\rho} + I_w \omega_w \omega_p \hat{\phi} + I_p \dot{\omega}_p \hat{z}$$

In $\hat{\rho}$: $0 = I_w \dot{\omega}_w \Rightarrow \dot{\omega}_w = 0 \Rightarrow \omega_w = \text{const}$

In $\hat{z}$: $0 = I_p \dot{\omega}_p \Rightarrow \dot{\omega}_p = 0 \Rightarrow \omega_p = \text{const}$

In $\hat{\phi}$: $\cancel{m}g\ell = \underbrace{I_w}_{m\rho_0^2} \omega_w \omega_p = \cancel{m}\rho_0^2 \omega_w \omega_p$

$$\Rightarrow \boxed{\omega_p = \frac{g\ell}{\rho_0^2 \omega_w}}$$



DEMO (501, 40)

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Gyroscopes

Today's agenda (Serway 11,13; MIT 22,23)

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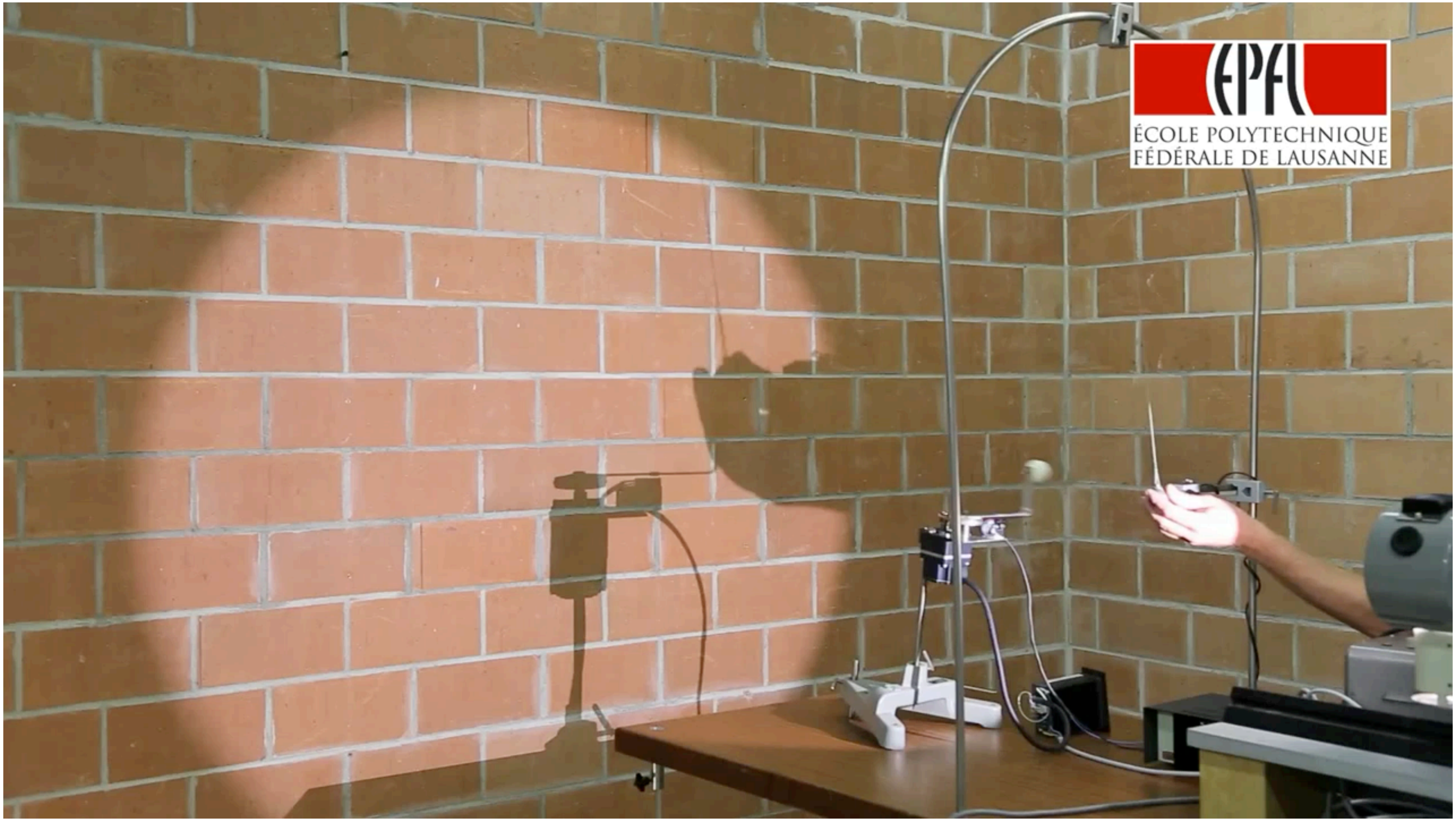
1. Kepler's laws of planetary motion
2. Gyroscopes
3. **Harmonic motion**
    - Simple harmonic motion

DEMO (190): Harmonic motion is like 1D circular motion



Harmonic motion

- Special type of periodic motion caused by forces of the form

$$\overrightarrow{F} = -k\,\underbrace{\Delta\vec{r}}_{\vec{r}-\vec{r}_0}$$

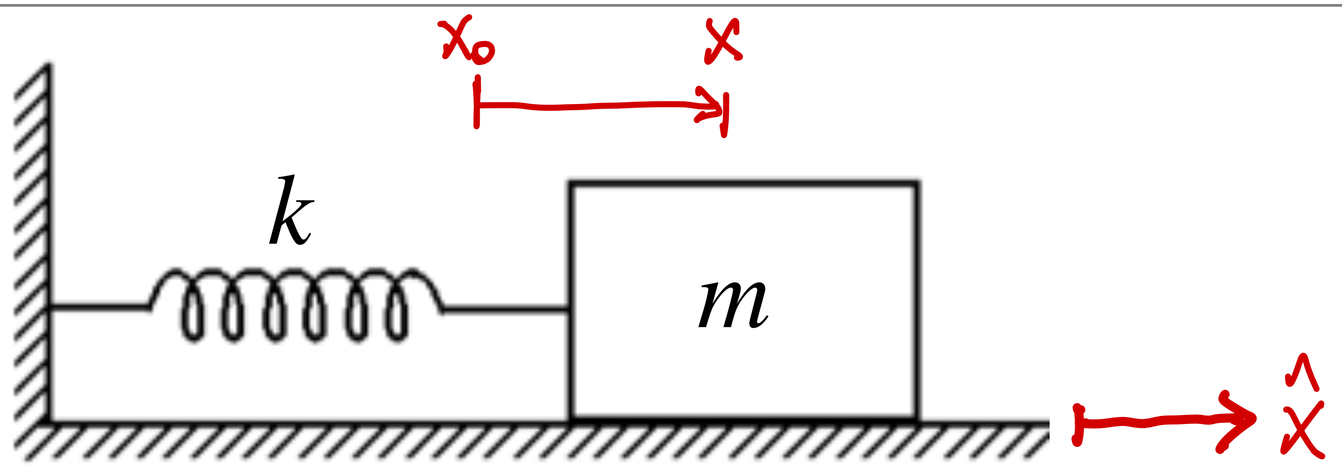
Harmonic motion

- Special type of periodic motion caused by forces of the form

$$\overrightarrow{F} = -k\,\Delta\vec{r}$$

- This is has the same form as the *spring force*, which can represent many systems
    - e.g. atoms in crystals, pendulums, balls rolling in bowls

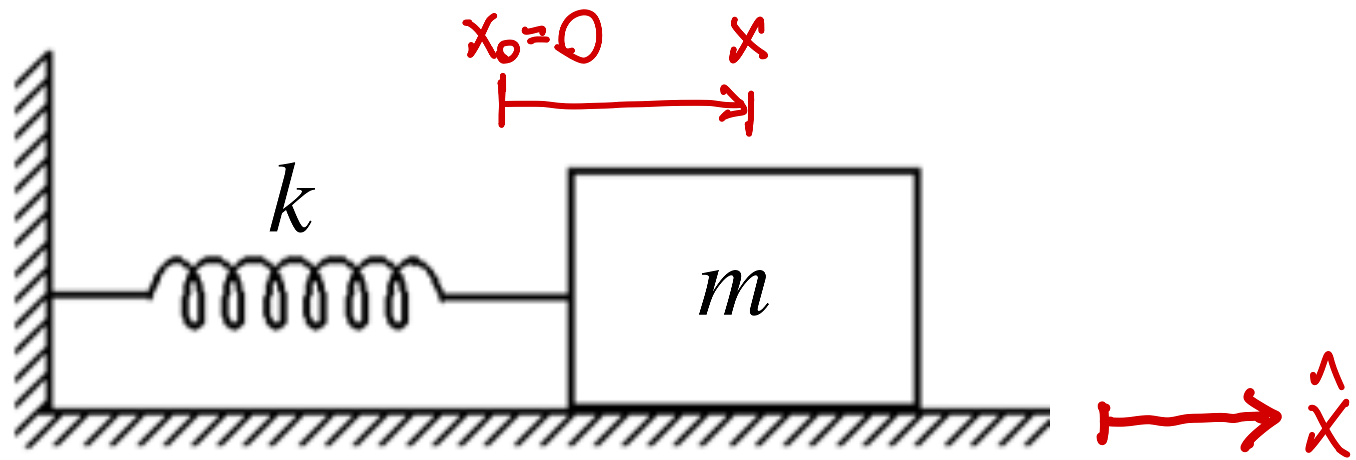
“Simple” harmonic oscillation in 1D



- Consider a frictionless mass-spring system

$$F = -k\Delta x = -k(x - x_0)$$

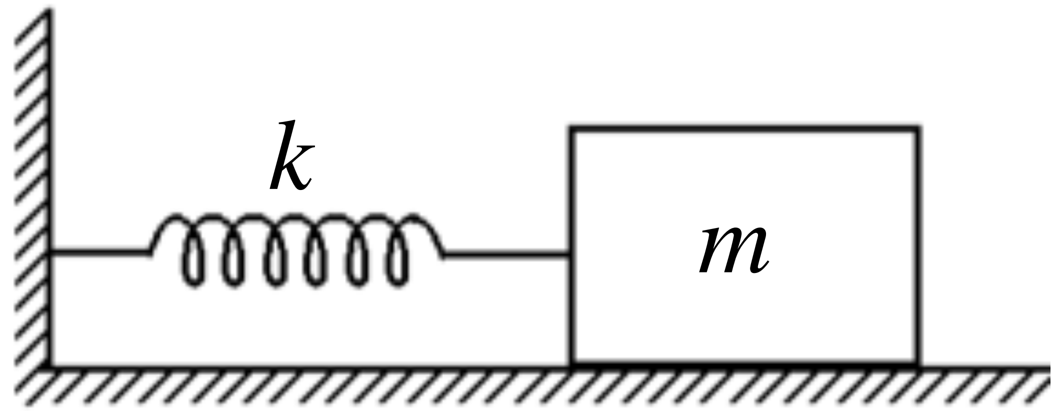
“Simple” harmonic oscillation in 1D



- Consider a frictionless mass-spring system
- The spring has equilibrium position $x_0 = 0$

$$F = -k\Delta x = -k(x - x_0) = -kx$$

“Simple” harmonic oscillation in 1D



- Consider a frictionless mass-spring system
- The spring has equilibrium position $x_0 = 0$, so

$$F = -kx$$

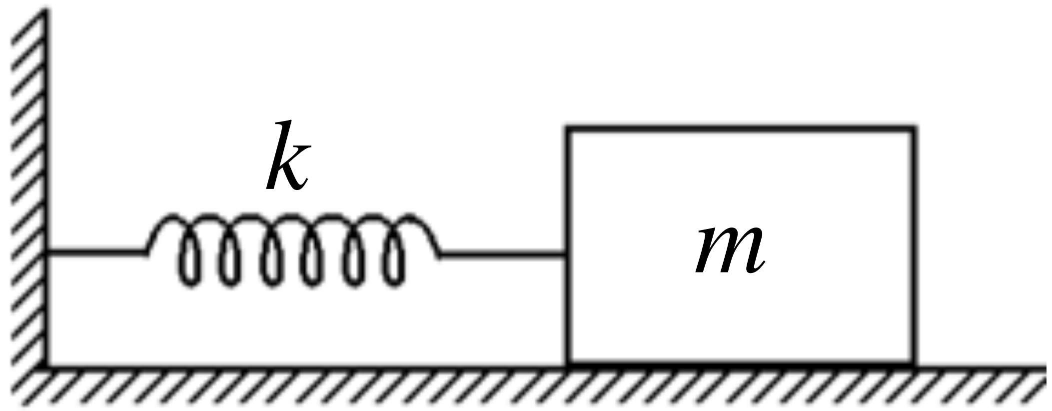
- Newtons 2nd law, $F = ma$, then yields

$$-kx = ma = m\frac{d^2x}{dt^2} \qquad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

“Simple” harmonic oscillation in 1D



- Consider a frictionless mass-spring system
- The spring has equilibrium position $x_0 = 0$, so

$$F = -kx$$

- Newtons 2nd law, $F = ma$, then yields

$$-kx = ma$$

which leads to the differential equation

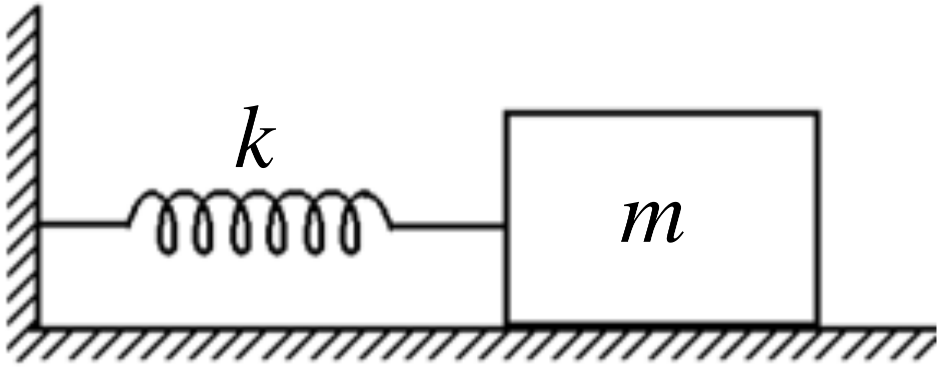
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

"Simple" harmonic oscillation in 1D

- The solution of

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

is $x(t) = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)$



“Simple” harmonic oscillation in 1D

- The solution of

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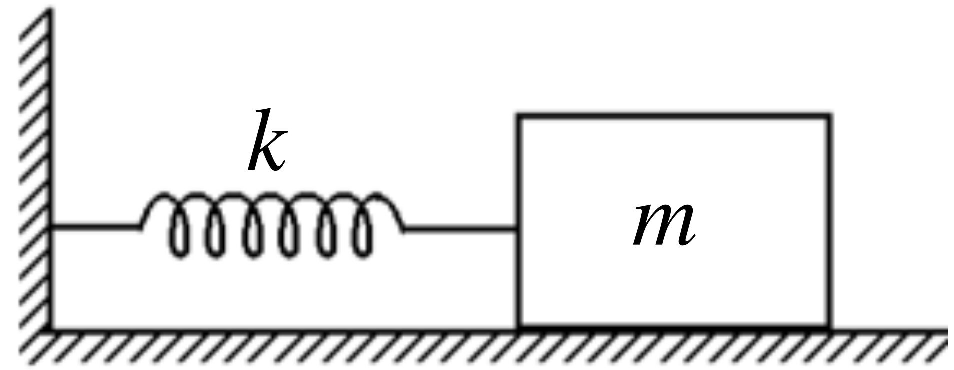


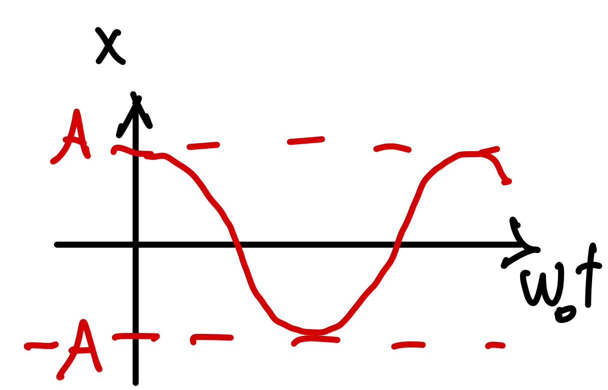
is $x(t) = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)$

- Using trigonometric identities, this is equivalent to

$$x(t) = A \cos(\omega_0 t + \varphi)$$

where

  - A is the amplitude of the oscillation



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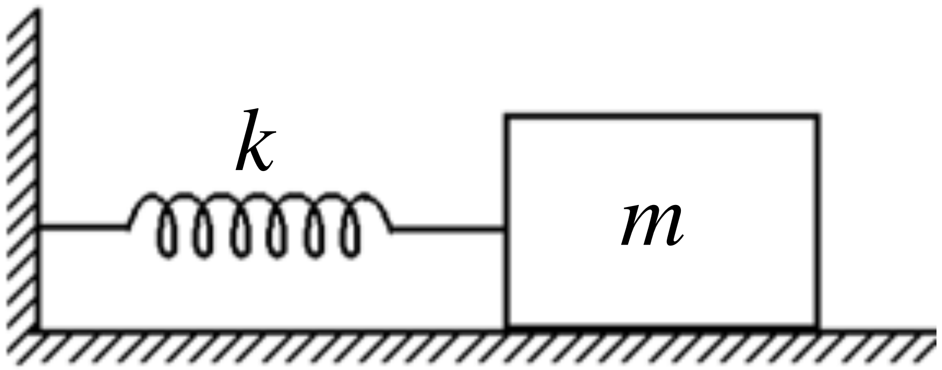

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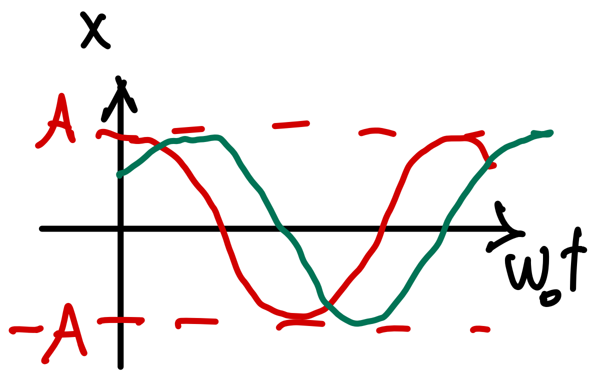
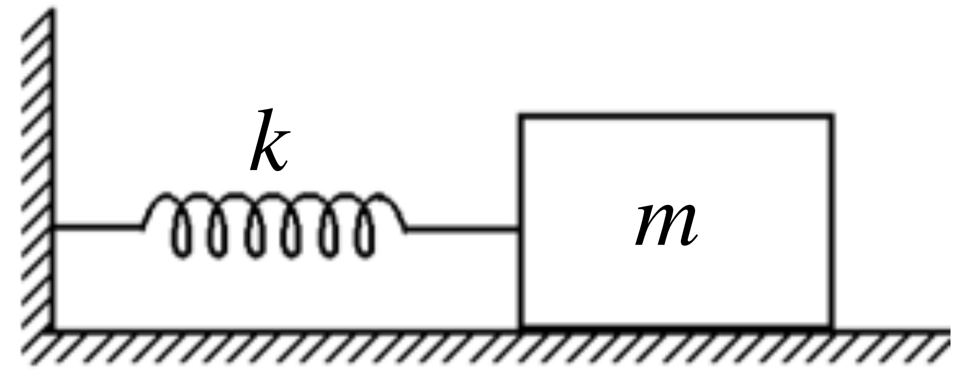
where
  - A is the amplitude of the oscillation
  - φ is the initial phase



“Simple” harmonic oscillation in 1D



- The solution of

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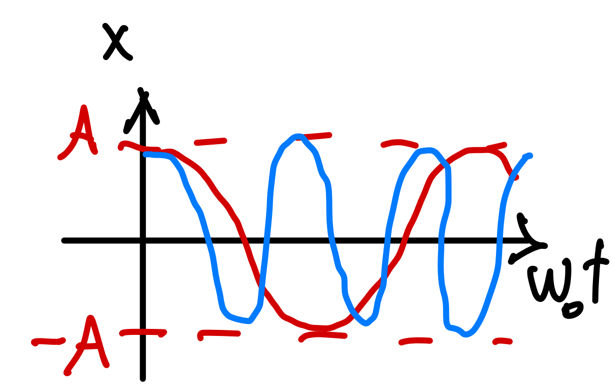

where

  - A is the amplitude of the oscillation
  - φ is the initial phase
  - $\omega_0 = \sqrt{k/m} = 2\pi/T_0$ where T_0 is the period of the oscillation. ω_0 is called the *angular frequency.*

“Simple” harmonic oscillation in 1D

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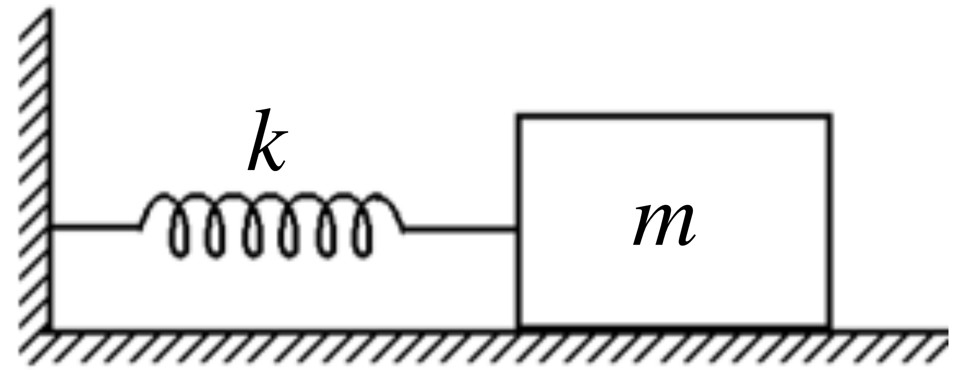
- The solution of

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$



is $x(t) = A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t)$

- Using trigonometric identities, this is equivalent to

$$x(t) = A\cos(\omega_0 t + \varphi)$$

$$v(t) = \dot{x}(t) = \frac{dx}{dt} = \frac{d}{dt}\left[A\cos(\omega_0 t + \varphi)\right] = A\frac{d}{dt}\left[\cos(\omega_0 t + \varphi)\right] = A\left[-\sin(\omega_0 t + \varphi)\right]\underbrace{\frac{d}{dt}\left[\omega_0 t + \varphi\right]}_{\omega_0}$$

$$= -A\omega_0\sin(\omega_0 t + \varphi)$$

$$a(t) = \ddot{x}(t) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left[-A\omega_0\sin(\omega_0 t + \varphi)\right] = -A\omega_0\cos(\omega_0 t + \varphi)\,\omega_0$$

$$= -\omega_0^2 A\cos(\omega_0 t + \varphi) = -\omega_0^2 x(t)$$

$$\Rightarrow\ 0 = \frac{d^2x}{dt^2} + \frac{k}{m}x = -\omega_0^2 x + \frac{k}{m}x = \left(-\omega_0^2 + \frac{k}{m}\right)x \ \Rightarrow\ \omega_0^2 = \frac{k}{m} \ \Rightarrow\ \omega_0 = \sqrt{\frac{k}{m}}$$

Kinetic energy of oscillation

- Kinetic energy is still $K = \frac{m}{2} v^2$

$$K = \frac{1}{2} m \left[-A\omega_0 \sin(\omega_0 t + \varphi) \right]^2 = \frac{1}{2} m A^2 \omega_0^2 \sin^2(\omega_0 t + \varphi)$$

$$= \frac{1}{2} m A^2 \frac{k}{m} \sin^2(\omega_0 t + \varphi) = \frac{1}{2} k A^2 \sin^2(\omega_0 t + \varphi)$$

$$= \frac{1}{2} k A^2 \left[1 - \cos^2(\omega_0 t + \varphi) \right]$$

$$= \frac{1}{2} k A^2 - \frac{1}{2} k A^2 \cos^2(\omega_0 t + \varphi) \quad (\to x^2)$$

$$= \boxed{\frac{1}{2} k A^2 - \frac{1}{2} k x^2}$$

Total energy of oscillation

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- Potential energy is still

$$U = \frac{k}{2} x^2$$

Total energy of oscillation

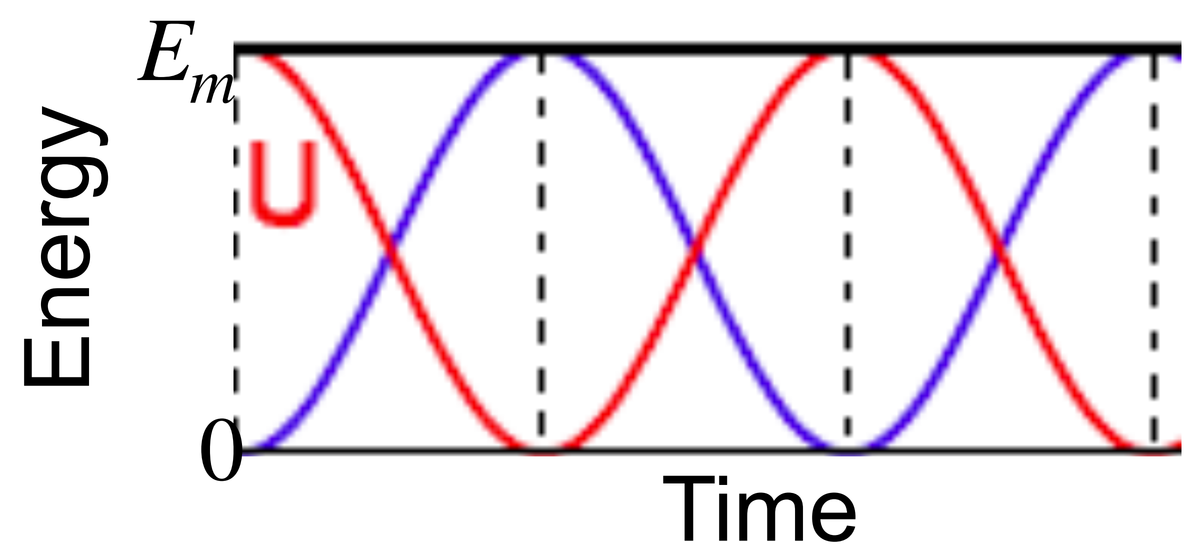
- Potential energy is still

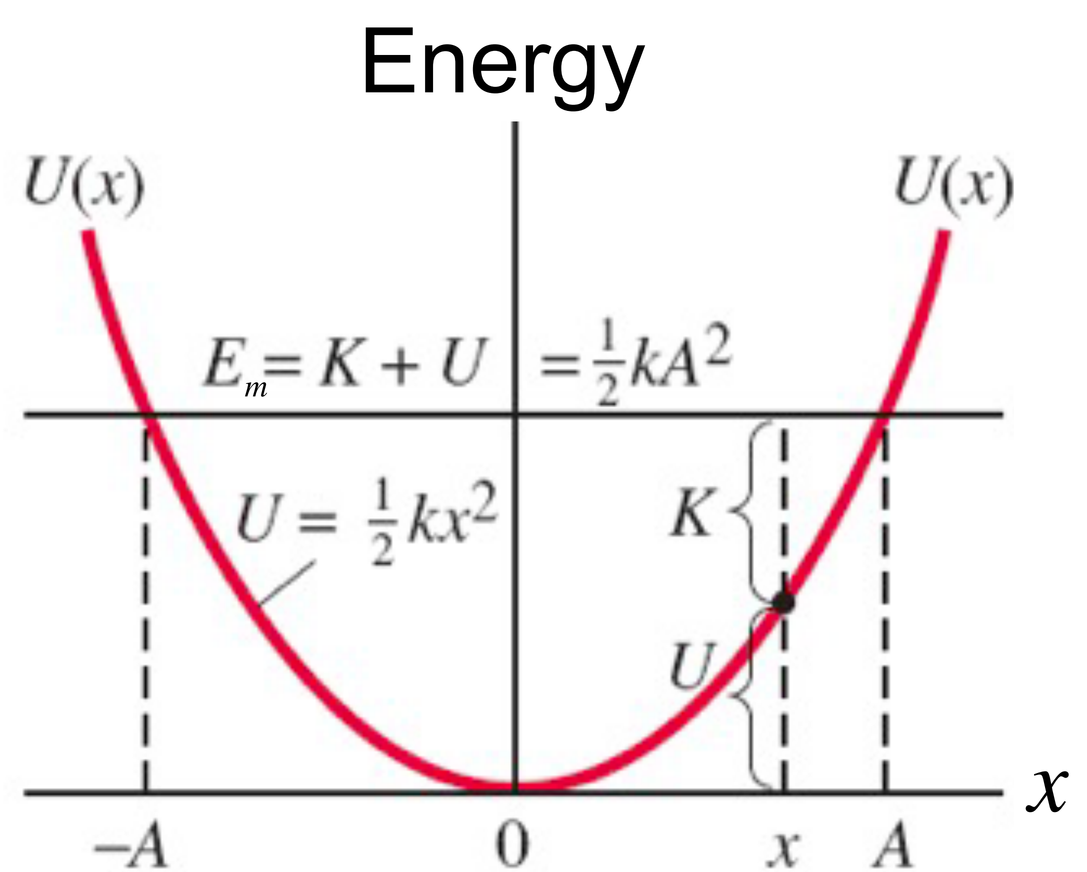
$$U = \frac{k}{2} x^2$$

- Total energy is $E_m = K + U$

$E_m = K + U = \left(\frac{1}{2}kA^2 - \frac{1}{2}kx^2\right) + \frac{1}{2}kx^2$

$= \boxed{\frac{1}{2}kA^2}$





Aside: Oscillation of a simple pendulum

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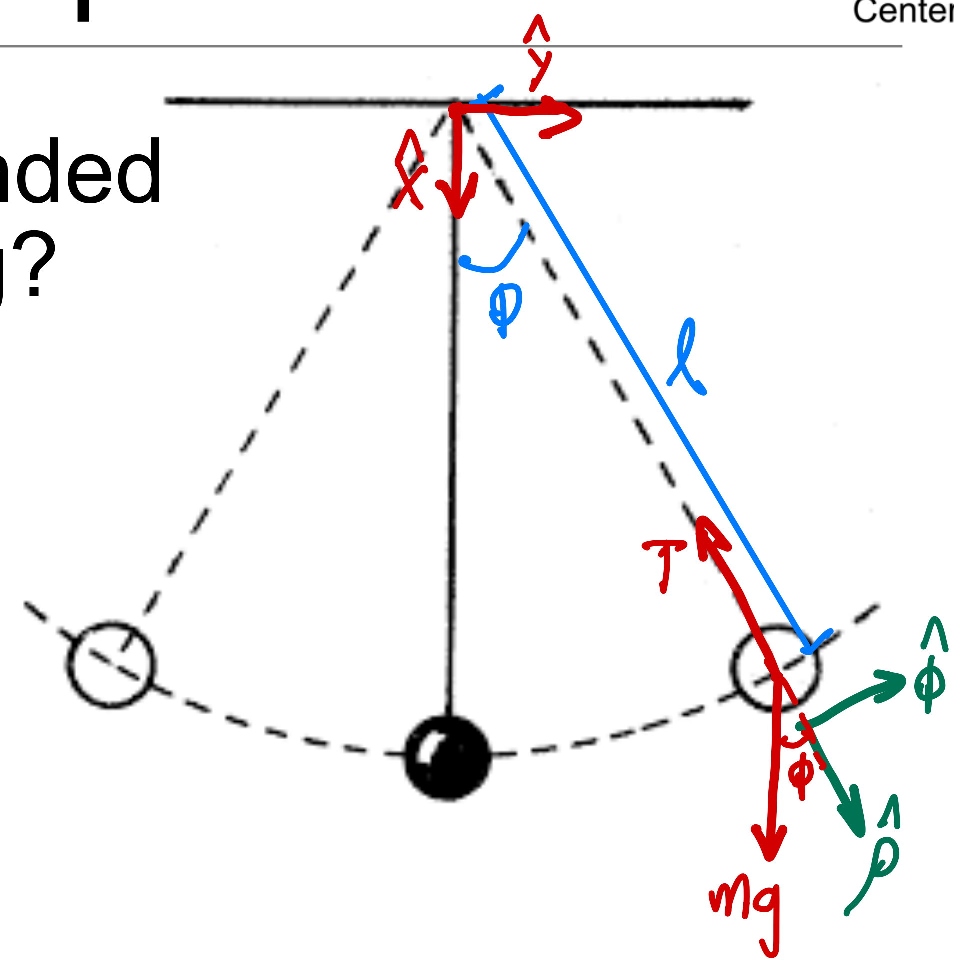
- What is the motion of a mass suspended from a weightless, inextensible string?



$\sum F_\phi : -mg\sin(\phi) = m a_\phi \qquad a_\phi = \ell\alpha = \ell \frac{d^2\phi}{dt^2}$

$\Rightarrow -g\sin(\phi) = \ell \frac{d^2\phi}{dt^2}$

$\Rightarrow \boxed{\frac{d^2\phi}{dt^2} + \frac{g}{\ell}\sin(\phi) = 0}$

If $\phi \ll 1$ then $\sin(\phi) \approx \phi$ (actually $\sin(\phi) = \phi - \frac{1}{3!}\phi^3 + \ldots$)

In that case,

$$\boxed{\frac{d^2\phi}{dt^2} + \frac{g}{\ell}\phi = 0}$$

Simple harmonic oscillator with $\omega_0^2 = \frac{g}{\ell} \Rightarrow \omega_0 = \sqrt{\frac{g}{\ell}}$

DEMO (286)

Mass and frequency of a pendulum

Conceptual question

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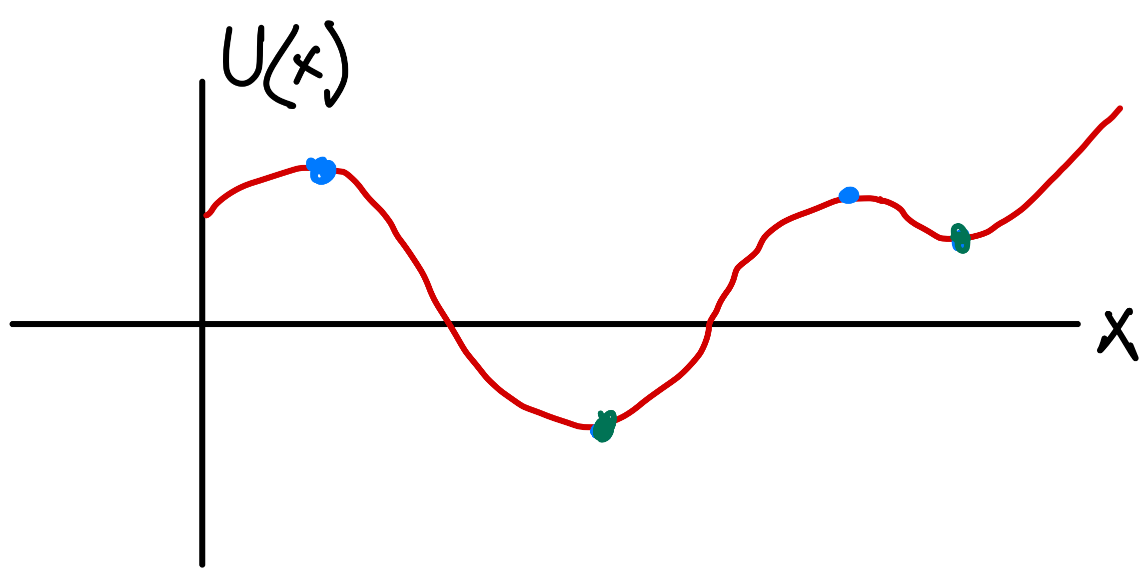
An object can execute harmonic motion (i.e. oscillate) about…

A. any point.
B. any equilibrium point.
C. any stable equilibrium point.
D. any point, provided the forces exerted on the object obey Hooke's law.



See you tomorrow!

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