# General Physics: Mechanics

## PHYS-101(en)

### Lecture 12a: Angular momentum

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Swiss Plasma Center

# Announcements

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- We’ll hold another ***mock exam*** tomorrow
  - You can bring a “cheat” sheet containing formulas or all of your notes, as you wish
  - Turn in at the end if you want exam to be graded (optional). Graded exams will be returned on Monday December 15th
  - Solutions will be published on the Moodle
  - Does **not** matter at all for your final grade
  - Exam will contain some Multiple choice questions to practice for the final

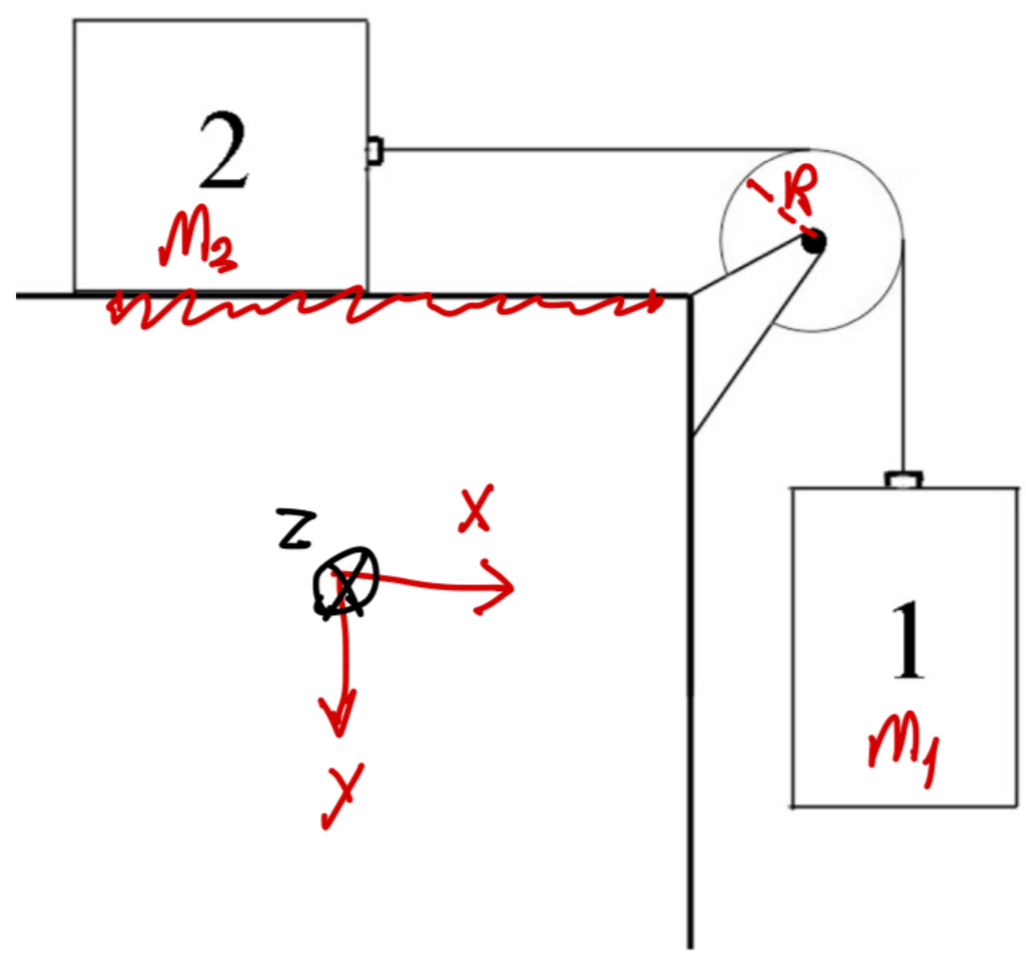
# Today's agenda (Serway 11,13, MIT 19)

Conclusion of rotation of rigid objects about a fixed axis

1. Work and power for rotation
2. Work-kinetic energy theorem for rotation
3. Angular momentum and its conservation

# Example (continued): Massive pulley

A pulley (with radius $R$ and moment of inertia about its center of mass $I$ ) is attached to the edge of a table. A massless string connects two blocks as shown. Block 1 has mass $m_1$ and hangs off the edge of the table. Block 2 has mass $m_2$ and can slide along a table with a coefficient of kinetic friction of $\mu$. Note that $m_1 > \mu m_2$. The blocks are released from rest and the string does not slip around the pulley.




Find the magnitude of the acceleration of each block. Express your answer in terms of $R$, $I$, $m_1$, $m_2$, and $\mu$ as needed.

# Example (continued): Massive pulley

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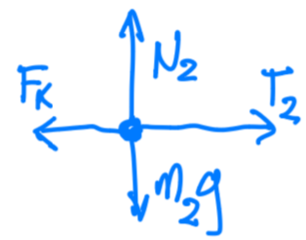
Block 1


$$\Sigma F_y: \; m_1 g - T_1 = m_1 a_{1y}$$

$$\Rightarrow \; T_1 = m_1 g - m_1 a_{1y} \quad (1)$$

Block 2




$$\Sigma F_y: \; N_2 = m_2 g$$

$$\Sigma F_x: \; T_2 - \mu m_2 g = m_2 a_{2x} \quad (2)$$

# Example (continued): Massive pulley

# Summary of rotation and translation

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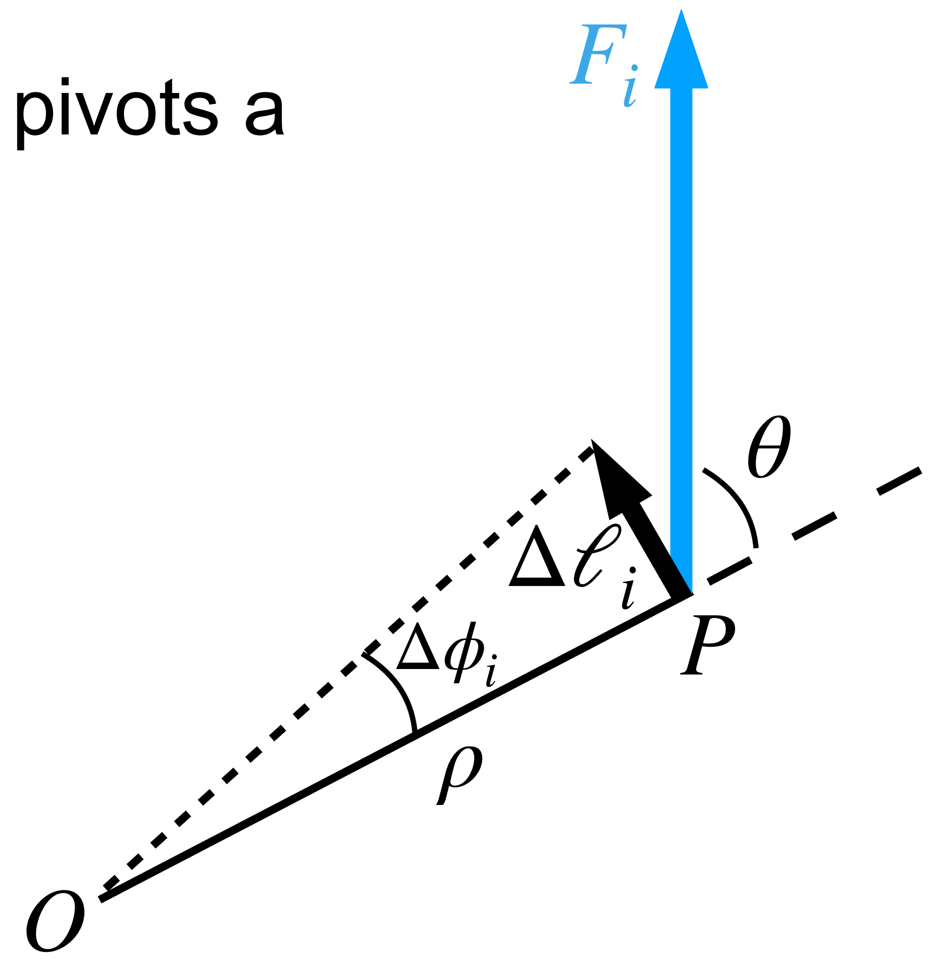
| Rotational motion (about a fixed axis) | | Translational motion (in one dimension) | |
|---|---|---|---|
| Angular position | $\phi$ | Position | $x$ |
| Angular speed | $\omega = d\phi/dt$ | Speed | $v = dx/dt$ |
| Angular acceleration | $\alpha = d\omega/dt$ | Acceleration | $a = dv/dt$ |
| Moment of inertia | $I = \int \rho^2 dm$ | Mass | $m$ |
| Net torque | $\Sigma\tau_{ext} = I\alpha$ | Net force | $\Sigma F_{ext} = ma$ |
| Rotational kinetic energy | $K^{rot} = I\omega^2/2$ | Translational kinetic energy | $K^{trans} = mv^2/2$ |
| Work | ? | Work | $W = \int_{x_a}^{x_b} F dx$ |
| Power | ? | Power | $P = Fv$ |
| Angular momentum | ? | Momentum | $p = mv$ |
| Net torque | ? | Net force | $\Sigma F_{ext} = dp/dt$ |

# Work and power for rotation

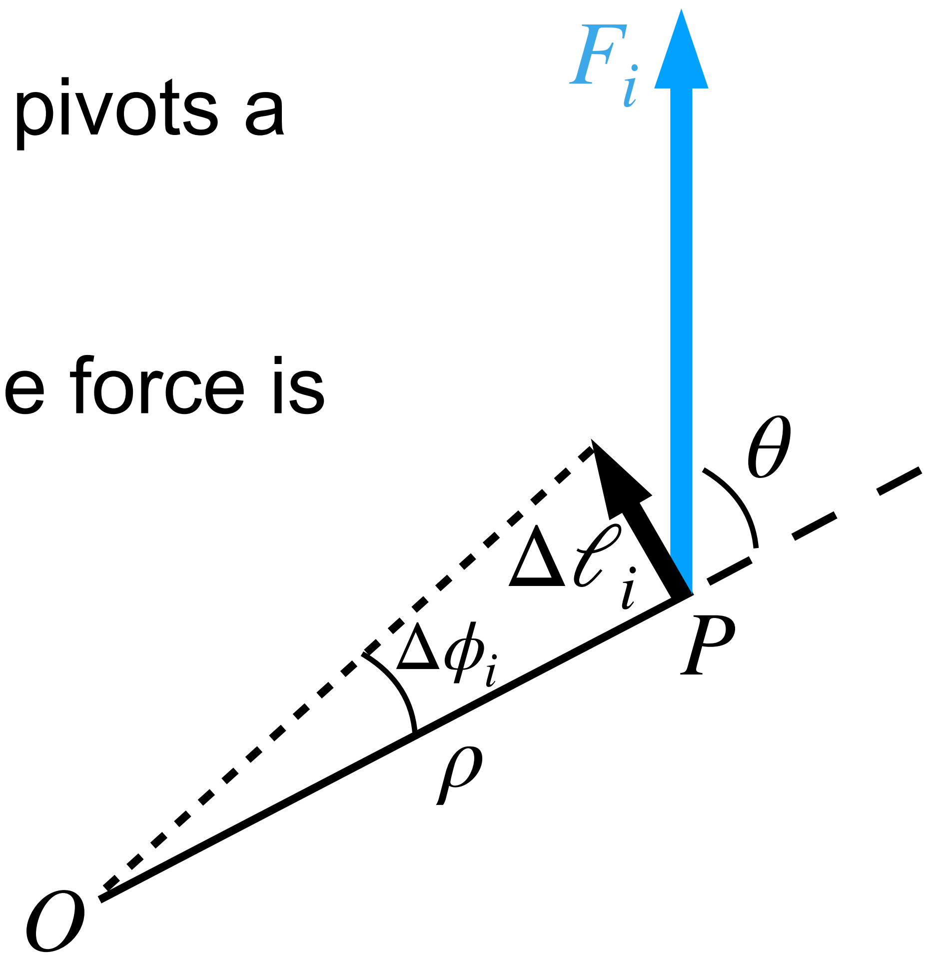
- A force $\overrightarrow{F}_i$ is applied at a point $P$, which pivots a small distance $\Delta\ell_i$ about point $O$



# Work and power for rotation

- A force $\overrightarrow{F}_i$ is applied at a point $P$, which pivots a small distance $\Delta \ell_i$ about point $O$
- As in week 8, the work done by a variable force is

$$\Delta W_i = \overrightarrow{F}_i \cdot \Delta \overrightarrow{\ell}_i$$



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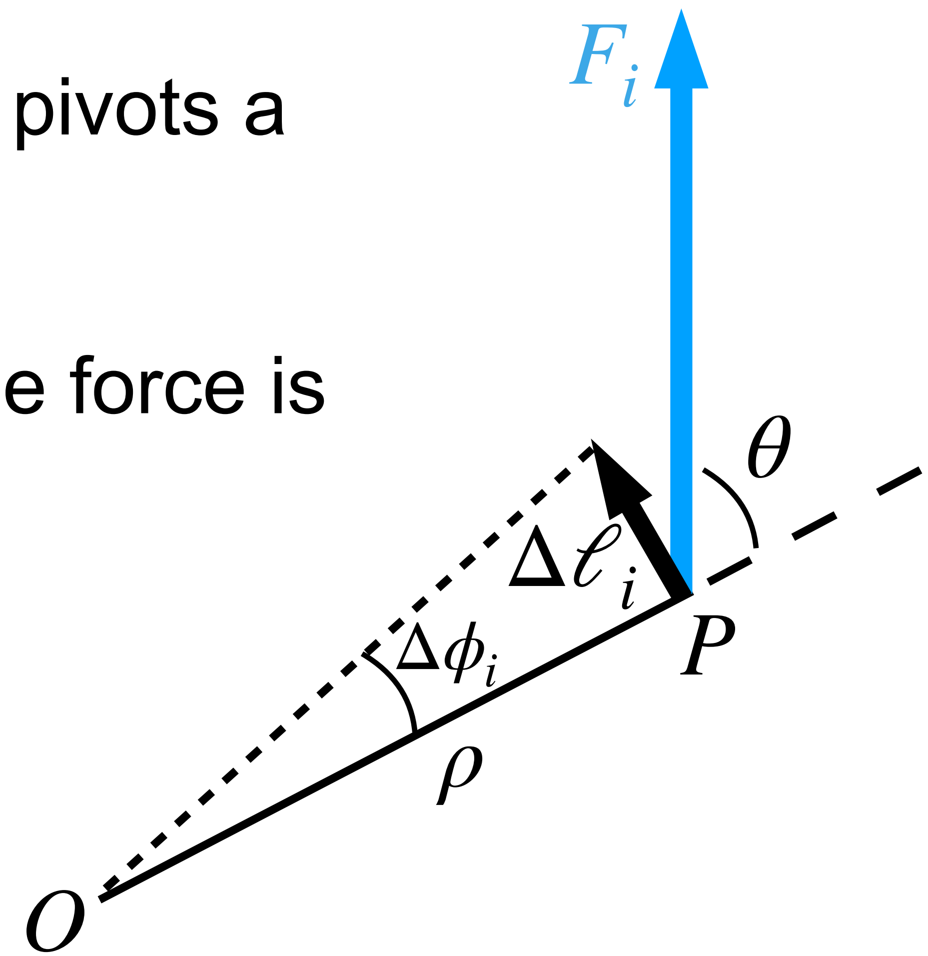
$$\Delta W_i = \overrightarrow{F}_i \cdot \Delta\overrightarrow{\ell}_i = \left(F_i \sin\theta\right)\Delta\ell_i$$

$$= F_i \sin\theta\left(\rho\Delta\phi_i\right) = \tau_i\,\Delta\phi_i$$



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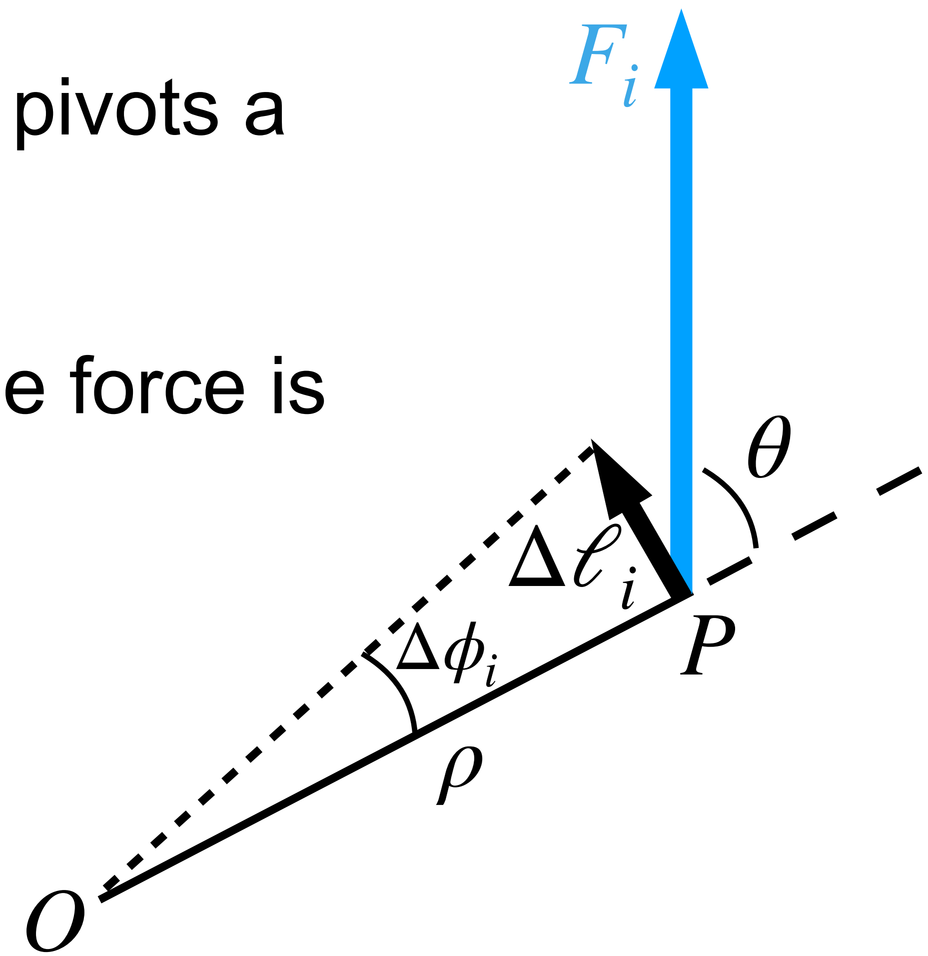
- Total work is the sum over differential changes in angle

$$W = \lim_{\Delta\phi_i \to 0} \sum_i \tau_i \, \Delta\phi_i = \int_{\phi_a}^{\phi_b} \tau \, d\phi$$



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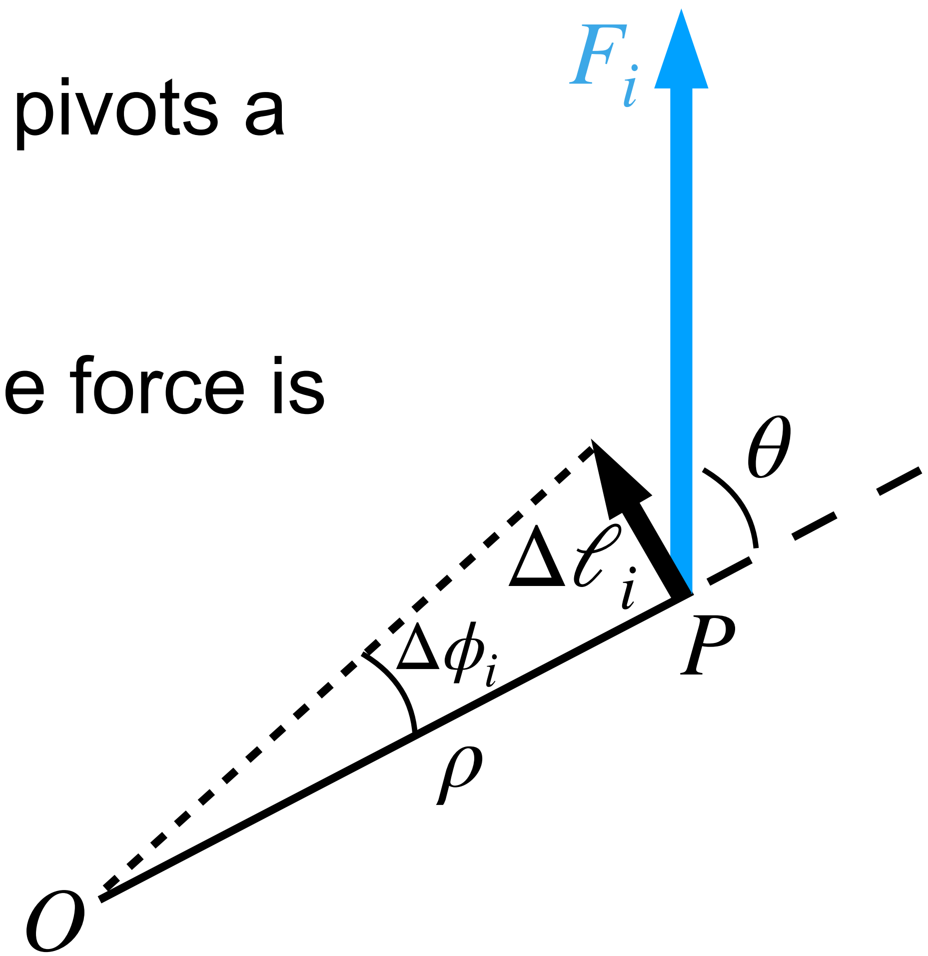


- Total work is the sum over differential changes in angle

$$W = \lim_{\Delta\phi_i \to 0} \sum_i \tau_i \Delta\phi_i = \int_{\phi_a}^{\phi_b} \tau \, d\phi$$

- Thus, the power is $P = \dfrac{dW}{dt} = \tau\,\omega$

# Today's agenda (Serway 11,13, MIT 19)

Conclusion of rotation of rigid objects about a fixed axis

2. Work-kinetic energy theorem for rotation

# Work-kinetic energy theorem for rotation

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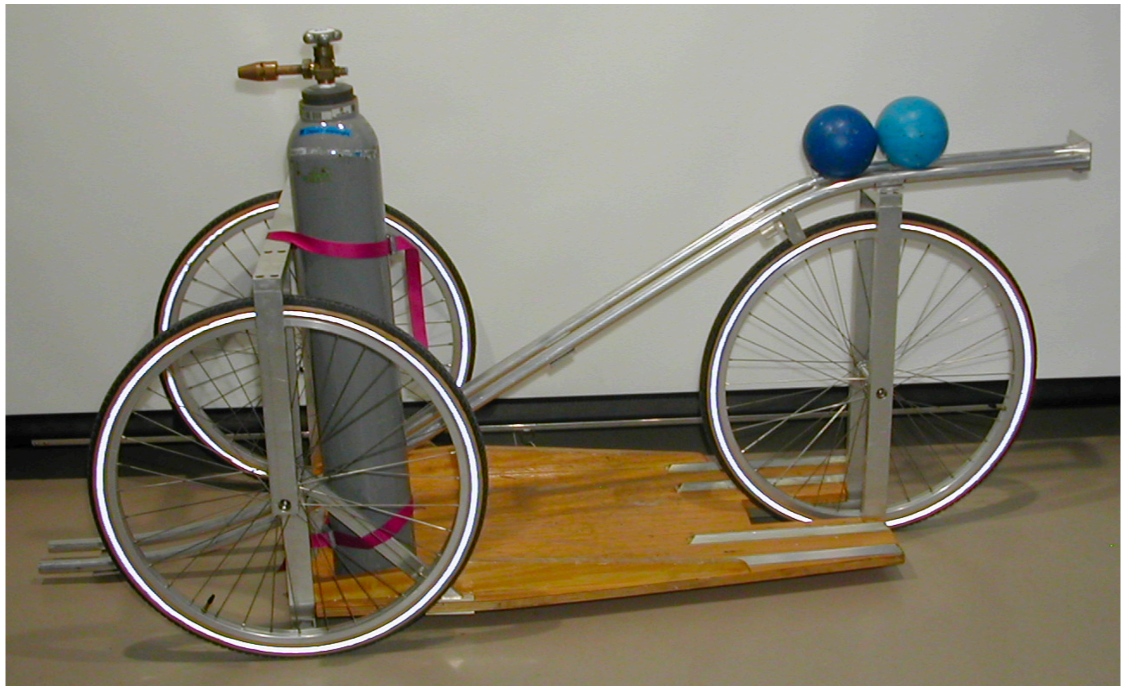
- Reminder: This tells us how the kinetic energy of an object changes due to the work performed on it

# DEMO (15)

Action-reaction **disk**



# Today's agenda (Serway 11,13, MIT 19)

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Conclusion of rotation of rigid objects about a fixed axis

3. Angular momentum and its conservation

# Angular momentum

- As you may suspect, **angular momentum** $\overrightarrow{L}$ is the rotational analogue of *linear* momentum $\vec{p} = m\vec{v}$ (lecture 6)

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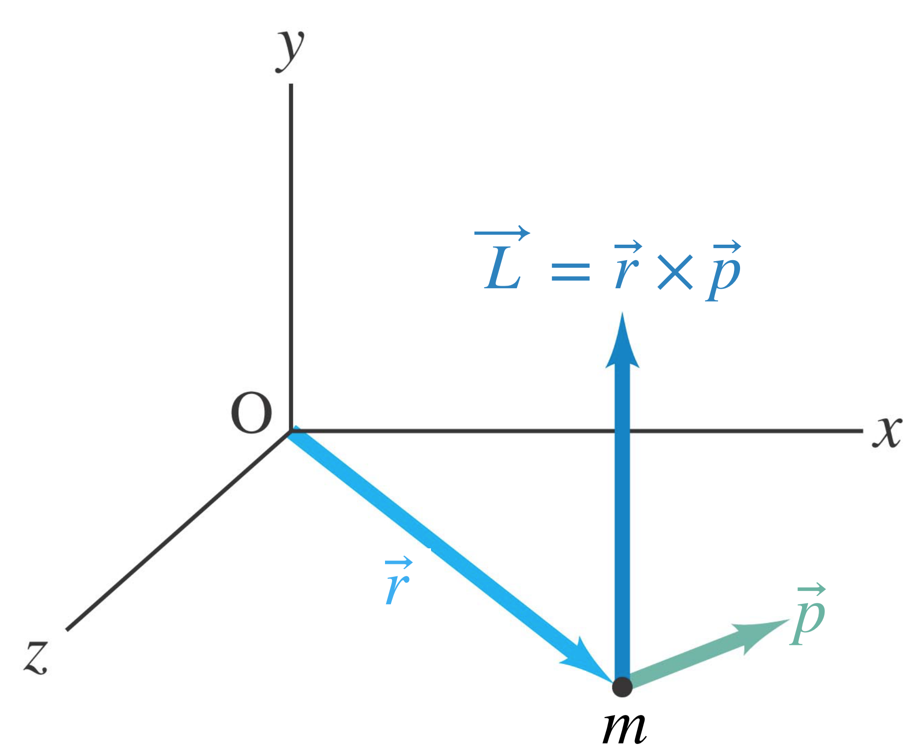
- It is the *moment* of momentum, defined as

$$\overrightarrow{L} = \vec{r} \times \vec{p}$$

where $\vec{r}$ is the position vector from a pivot point to the object



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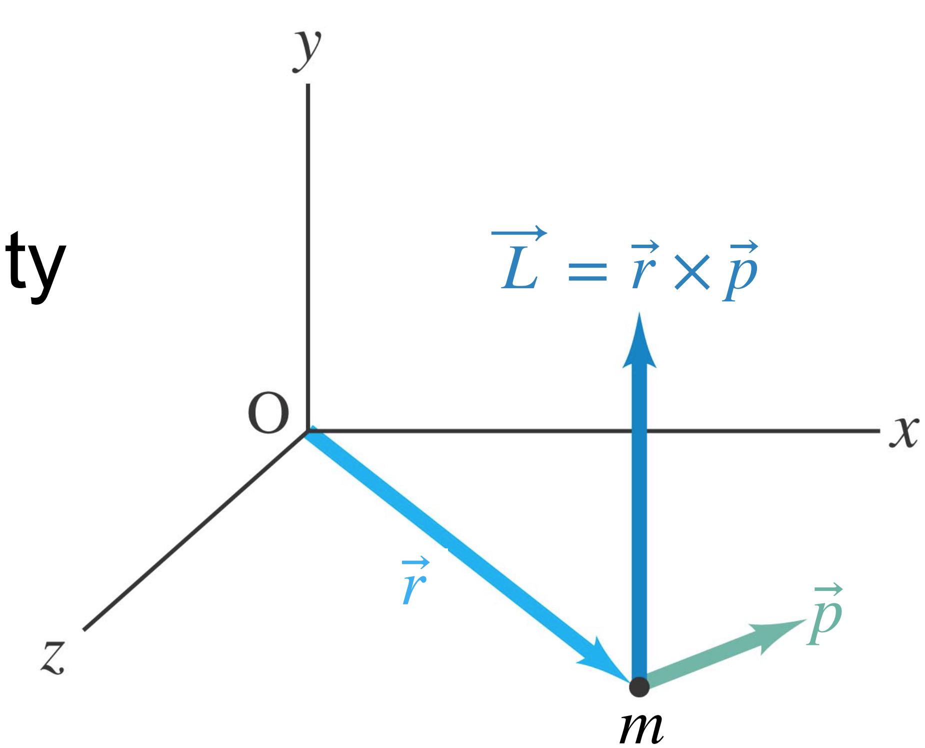
- It has units of [kg·m$^2$/s]
- Like linear momentum, it is a vector quantity and will be conserved under certain conditions



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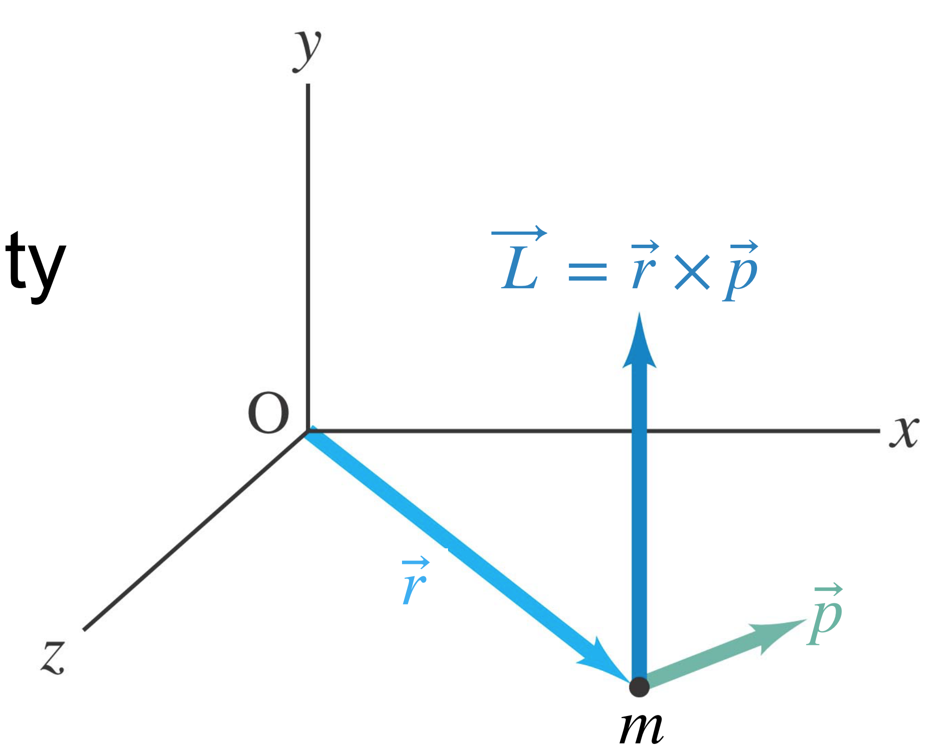
- Unlike linear momentum, it depends on where the pivot is chosen

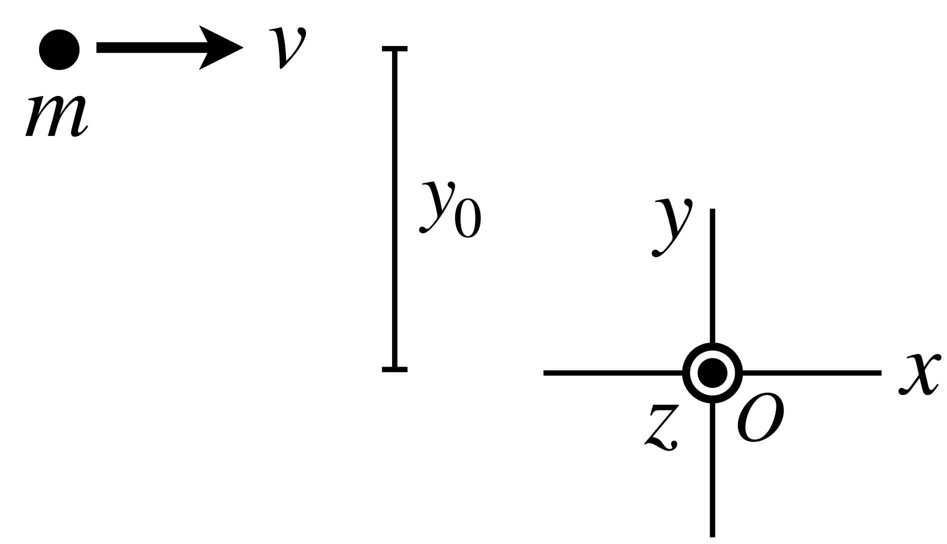


# Conceptual question



A particle is moving in the $x$-$y$ plane with a constant velocity and constant height $y_0$ (as shown below). The magnitude of the angular momentum $L_O$ about the origin…

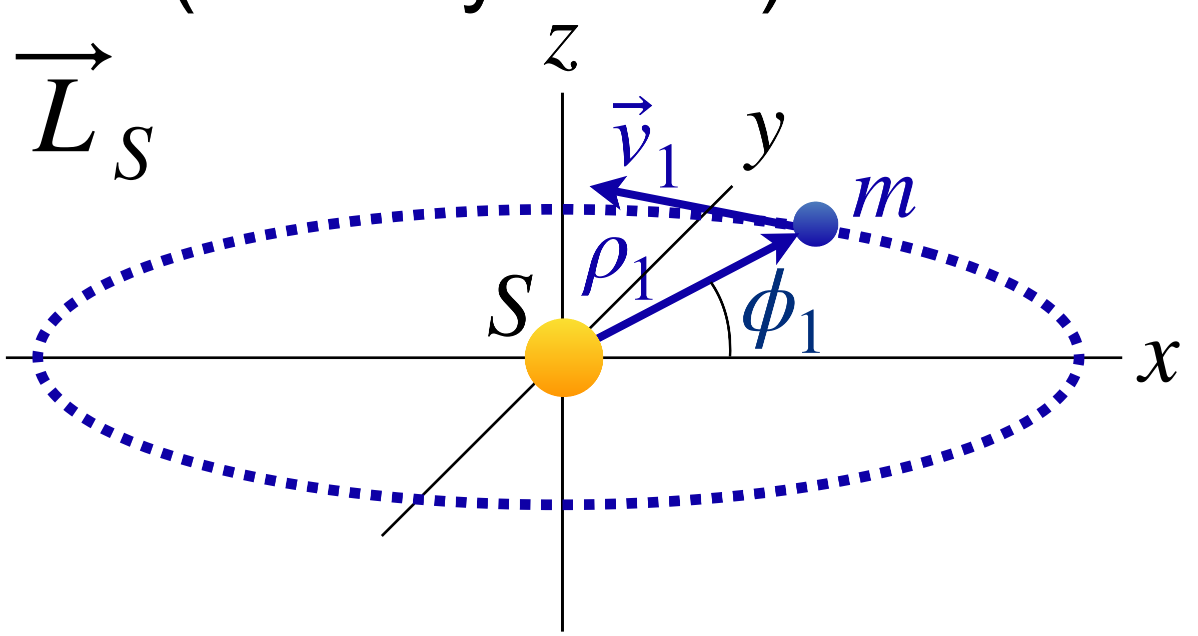
A. is zero because this is not circular motion.
B. decreases, then increases.
C. increases, then decreases.
D. is constant.



# Example: Point particle angular momentum

A point of mass $m$ (lets say a planet) is executing uniform circular motion with $\overrightarrow{\omega}$ around point $S$ (lets say a star).

A. What is its angular momentum $\overrightarrow{L}_S$ about $S$?



# Example: Point particle angular momentum

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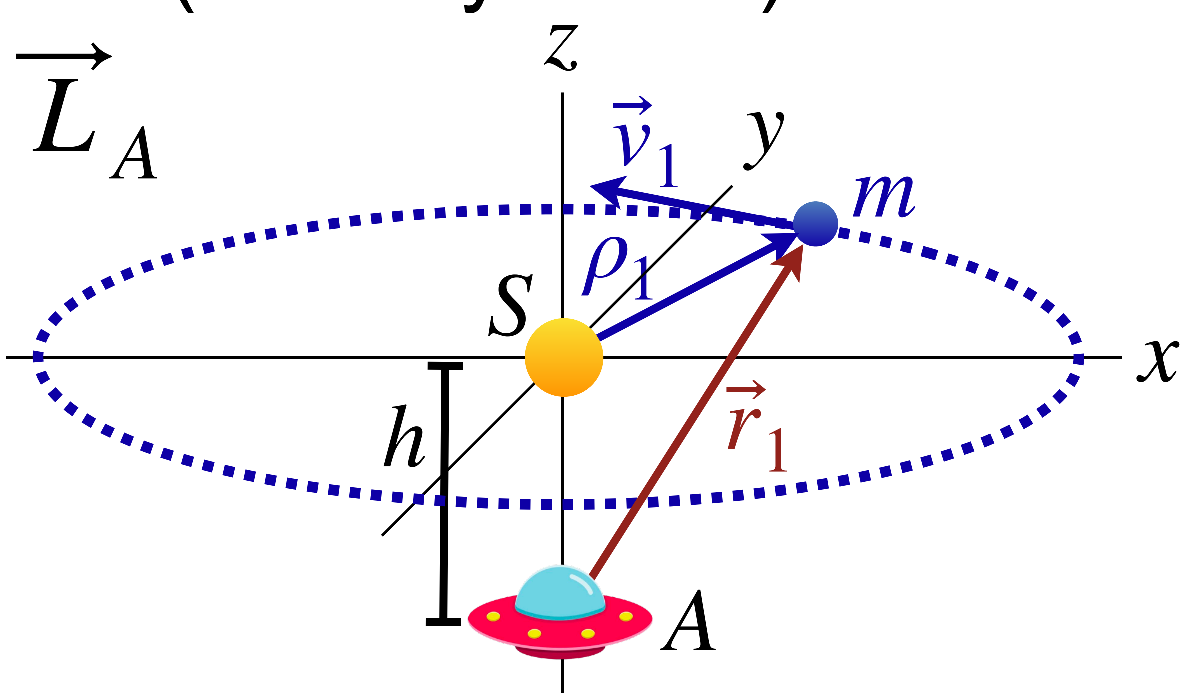
A point of mass $m$ (lets say a planet) is executing uniform circular motion with $\overrightarrow{\omega}$ around point $S$ (lets say a star).

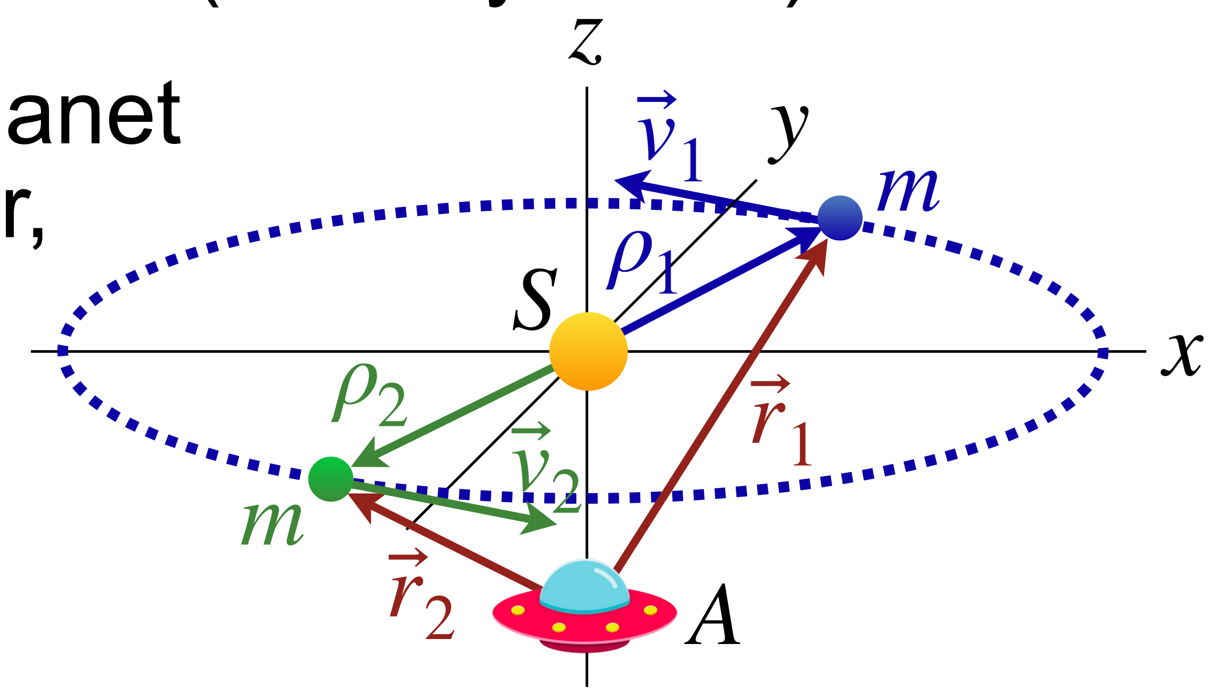
B. What is its angular momentum $\overrightarrow{L}_A$ about a lower point $A$?



# Example: Point particle angular momentum

A point of mass $m$ (lets say a planet) is executing uniform circular motion with $\overrightarrow{\omega}$ around point $S$ (lets say a star).

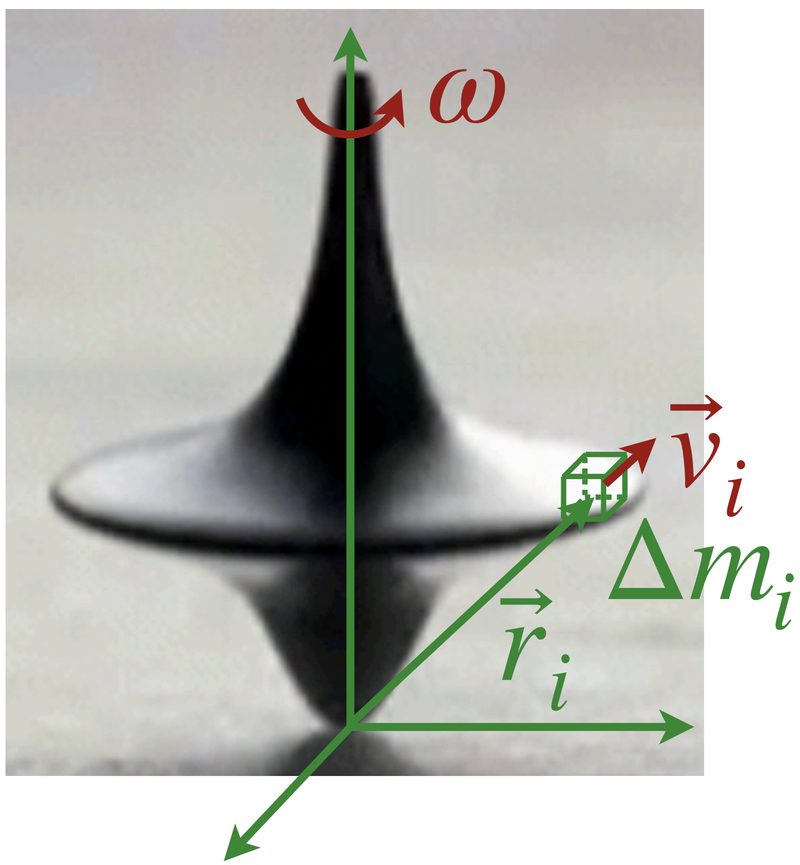
C. If we add a second identical planet on the opposite side of the star, what is $\overrightarrow{L}_{sys}$ of the system?



# Angular momentum of rotating rigid bodies

- Imagine the object is composed of many differential elements, labeled $i = 1, 2, 3, \ldots$, with positions $\vec{r}_i$

$$\vec{L} = \sum_i \vec{L}_i$$



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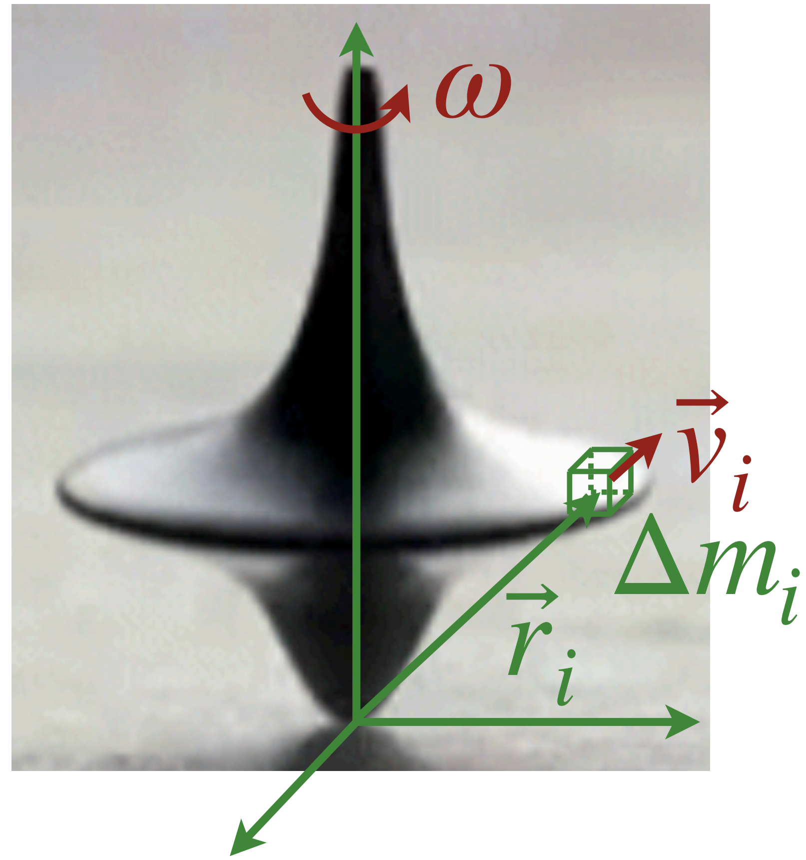
$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \Delta m_i \vec{v}_i$$



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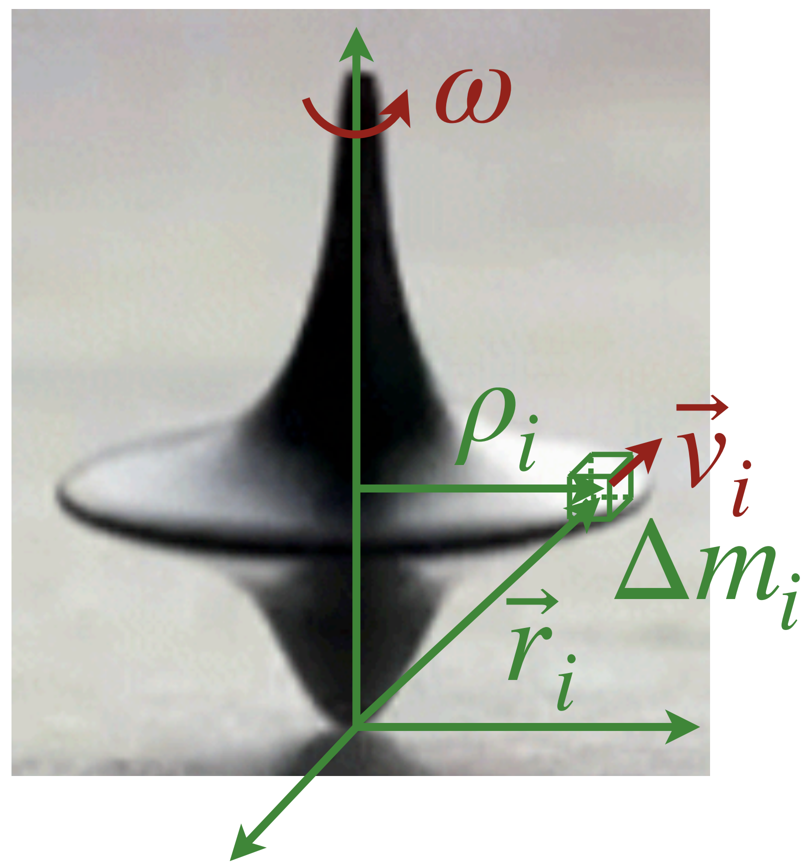
- For pure rotation of a symmetric object, the $\hat{\rho}$ component of $\vec{L}$ cancels



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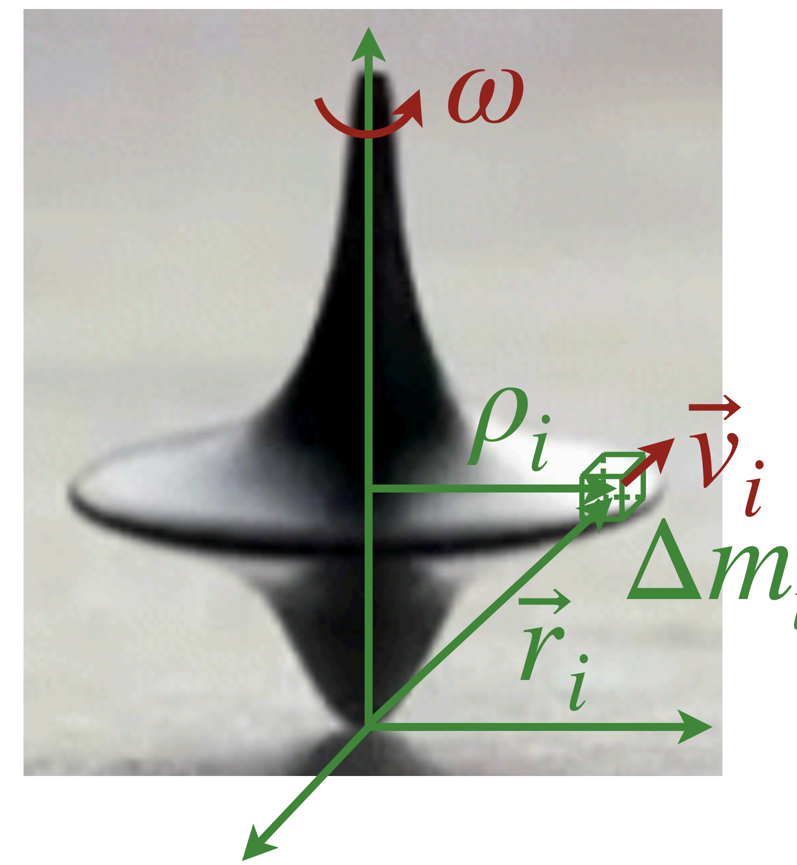


- For pure rotation of a symmetric object, the $\hat{\rho}$ component of $\vec{L}$ cancels

- Since $v_i = \rho_i \omega$, we see that $\vec{L} = \sum_i \rho_i^2 \Delta m_i \, \omega \hat{z}$

# Angular momentum of rotating rigid bodies

- Imagine the object is composed of many differential elements, labeled $i = 1, 2, 3, \ldots$, with positions $\vec{r}_i$

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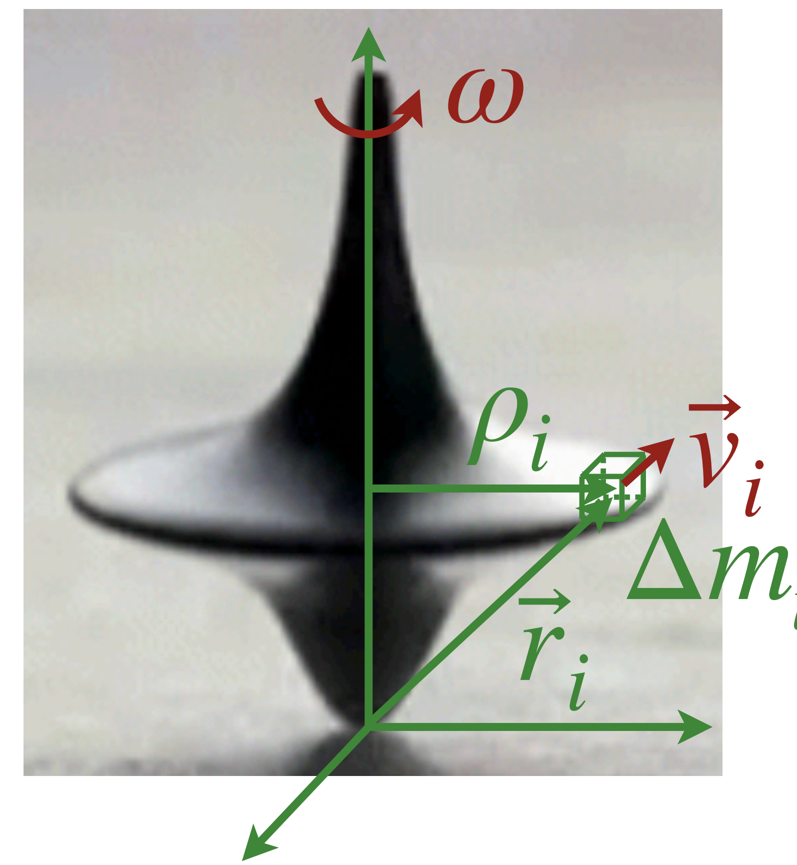
- For pure rotation of a <u>symmetric</u> object, the $\hat{\rho}$ component of $\overrightarrow{L}$ cancels




- Since $v_i = \rho_i \omega$, we see that $\overrightarrow{L} = \sum_i \rho_i^2 \Delta m_i \, \omega \hat{z}$

- In the limit of $\Delta m_i \to 0$, $\overrightarrow{L} = \int_M \rho^2 dm \overrightarrow{\omega} \Rightarrow$ $\boxed{\overrightarrow{L} = I \overrightarrow{\omega}}$

# Angular momentum and torque



- If angular momentum $\overrightarrow{L}$ is analogous to momentum and torque $\vec{\tau}$ is analogous to force, what is their relationship?

# Conservation of angular momentum



$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

# Conservation of angular momentum

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*(and matter is not exchanged),*
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# Conservation of angular momentum

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$$\vec{\tau}_{net} = \frac{d\overrightarrow{L}}{dt}$$

*If the net torque on a system is **zero***
*(and matter is not exchanged),*
*the total angular momentum does not change with time.*

- In other words, if $\vec{\tau}_{net} = 0$ then the angular momentum is conserved:

$$\overrightarrow{L}_i = \overrightarrow{L}_f$$

# DEMO (17)

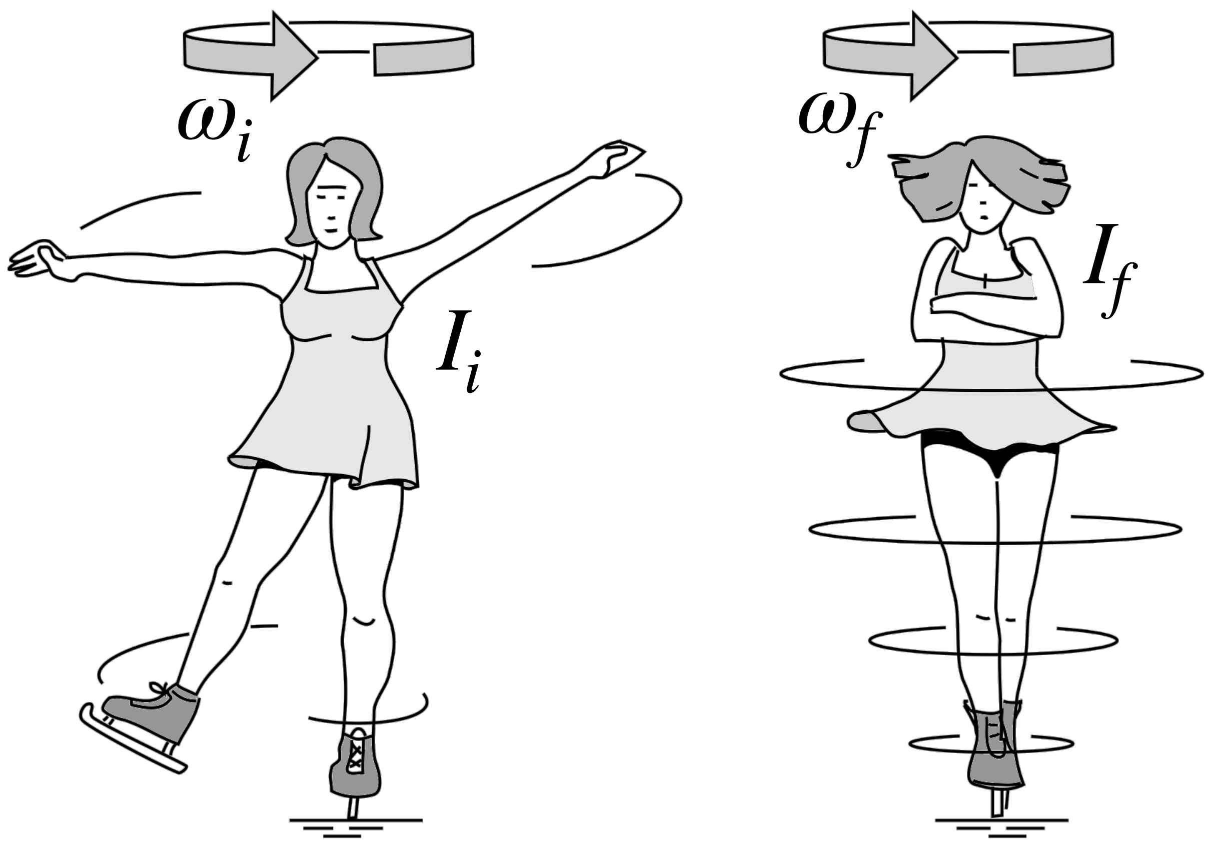
Swivel stool

# Conceptual question

responseware.eu
Session ID: epflphys101en

A figure skater stands on one spot on the ice (assumed frictionless) and spins around with her arms extended. When she pulls her arms in, she reduces her moment of inertia and her angular speed increases. Compared to her initial **rotational kinetic energy**, her **rotational kinetic energy** after she has pulled her arms in must be…

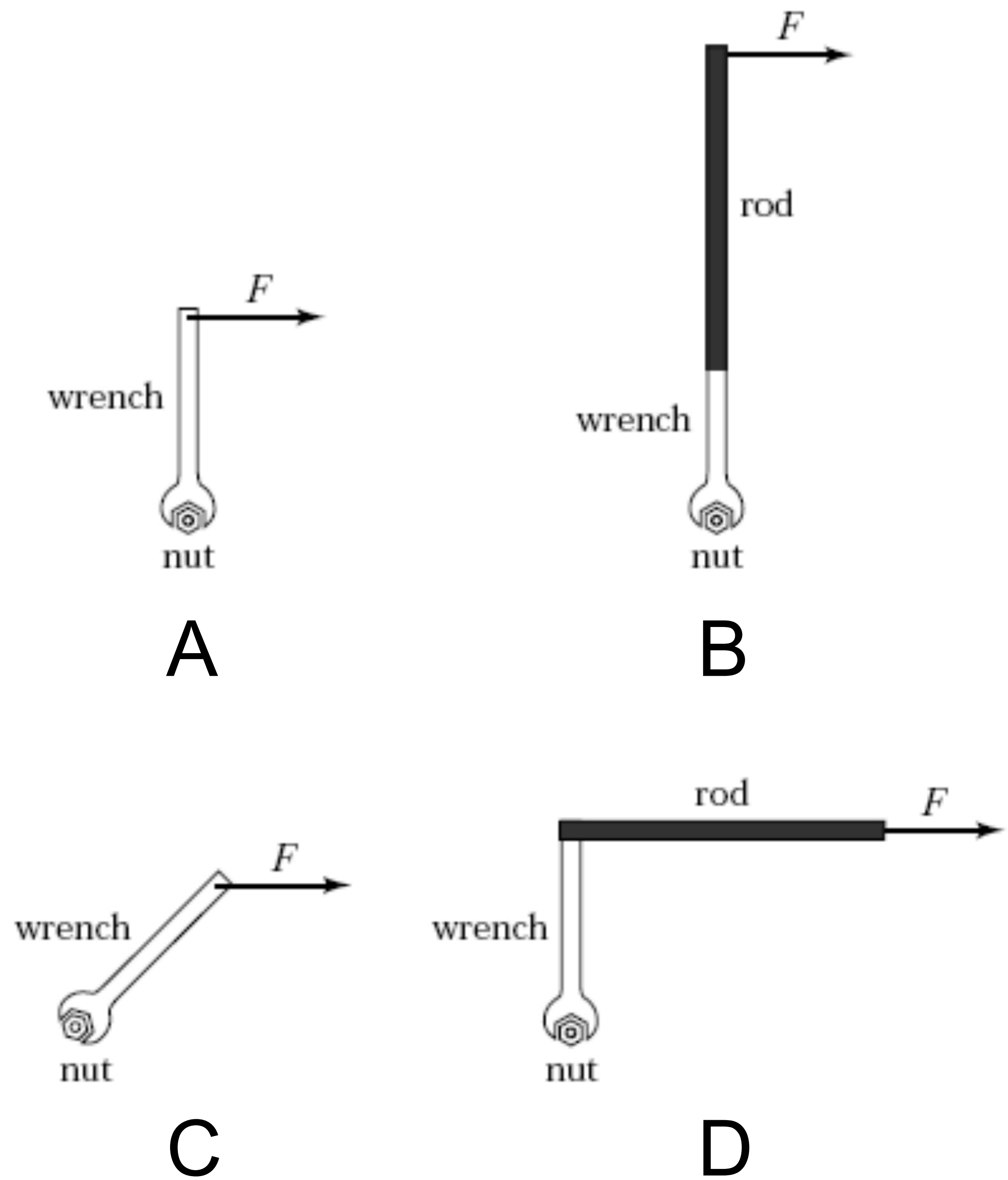
A. the same.
B. larger.
C. smaller.



# Conceptual question



You are using a wrench and trying to loosen a rusty nut. Which of the arrangements shown is most effective in loosening the nut?



# Conceptual question

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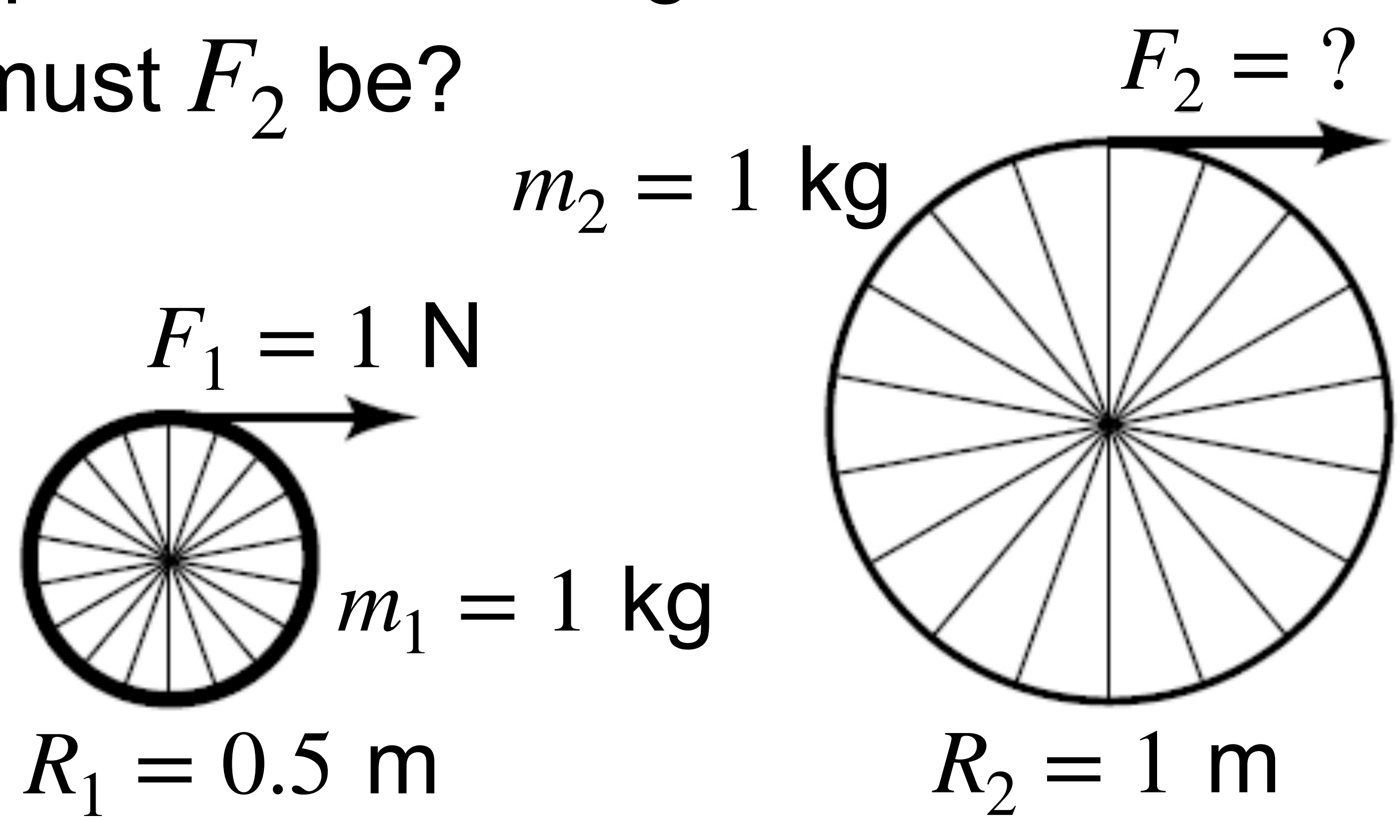
Two wheels with fixed hubs, each having a mass of 1 kg, start from rest, and forces are applied as shown. Assume the hubs and spokes are massless, so that the moment of inertia is $I = mR^2$. In order to impart identical angular accelerations, how large must $F_2$ be?

A. 0.25 N
B. 0.5 N
C. 1.0 N
D. 2.0 N
E. 4.0 N



# Summary of rotation and translation

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| Net torque | $\Sigma\tau_{ext} = I\alpha$ | Net force | $\Sigma F_{ext} = ma$ |
| Rotational kinetic energy | $K^{rot} = I\omega^2/2$ | Translational kinetic energy | $K^{trans} = mv^2/2$ |
| Work | $W = \int_{\phi_a}^{\phi_b} \tau d\phi$ | Work | $W = \int_{x_a}^{x_b} F dx$ |
| Power | $P = \tau\omega$ | Power | $P = Fv$ |
| Angular momentum | $L = I\omega$ | Momentum | $p = mv$ |
| Net torque | $\Sigma\tau_{ext} = dL/dt$ | Net torque | $\Sigma F_{ext} = dp/dt$ |

# Mock exam tomorrow

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Bon courage!