Solutions to the Final Exam

16 January 2026

PHYS-101(en)

Multiple Choice Questions (2 points each)

Note: Explanations to the multiple choice questions are just to clarify the answers and are not needed on the exam.

Q1. **Correct answer: B. It is smaller than in the case of the massless pulley.**

Explanation: The acceleration is smaller because the massive pulley has a nonzero moment of inertia, so part of the net force is required to accelerate the pulley. This reduces the force available to accelerate the mass, resulting in a smaller acceleration than in the case of the massless pulley.

Q2. **Correct answer: C. Vector 3.**

Explanation: In the absence of the centripetal force provided by the string, the mass will continue in the direction of the tangential velocity vector, orthogonal to the position vector at the moment the string breaks.

Q3. **Correct answer: B. It is deflected toward the west.**

Explanation: At the moment of launch, both the ground-based observer and the rocket have the same tangential velocity with respect to the earth's axis due to the rotation of the earth. As the rocket travels straight up, its distance from the rotational axis increases, but its tangential velocity remains constant. As a result, the angular velocity ($\omega = v_\phi/r$) of the rocket decreases. To the ground-based observer, this appears as if the rocket is deflected to the west. Another way to see this is by realizing that the ground-based observer is in a non-inertial reference frame rotating with angular velocity Ω_T. Then, the motion of the rocket as seen by the observer will be affected by a Coriolis term $-2\, m_b\, \Omega_T \times \vec{v}$, which points to the West.

Q4. **Correct answer: C. The equilibrium point corresponds to a minimum of the sum of the spring's elastic potential energy and gravitational potential energy.**

Explanation: Since the system is frictionless and subject only to conservative forces, equilibrium occurs at a minimum of the total potential energy. In this case, the total potential energy is the sum of the gravitational potential energy and the spring's elastic potential energy.

Q5. **Correct answer: A. The wheel undergoes a precession motion about axis $\hat{z}$ in the counterclockwise direction (top view), with angular velocity $\vec{\Omega}_p = \Omega\,\hat{z}$**

Explanation: Gravity exerts a torque about the support point given by $\vec{\tau} = \vec{r} \times m\vec{g}$. In cylindrical coordinates, the position vector of the wheel's center of mass points along $+\hat{\rho}$ and the gravitational force points along $-\hat{z}$. The direction of the torque is then $\vec{\tau} = (+\hat{\rho}) \times (-\hat{z}) = +\hat{\phi}$. The wheel has angular momentum $\vec{L} = I\vec{\omega} = I\omega\,\hat{\rho}$. The total external torque satisfies $\vec{\tau} = \frac{d\vec{L}_{\text{tot}}}{dt}$. Since $\vec{\tau}$ is along $+\hat{\phi}$ and $\vec{L}$ is along $+\hat{\rho}$, there must be some change in angular momentum that is in the direction of the torque, $\hat{\phi}$. As seen in class, this happens if the wheel rotates around the z-axis in the counterclockwise direction with a constant angular speed. This rotation is called "precession".

Q6. **Correct answer: E. Cylinder B.**

Explanation: As the cylinders roll without slipping, gravitational potential energy is converted into both translational and rotational kinetic energy. Since both cylinders have the same mass and outer radius, the distribution of kinetic energy depends on their moments of inertia. The hollow cylinder, cylinder A, has a larger moment of inertia, so a greater fraction of the available energy goes into rotational motion, leaving less energy for translation. As a result, its acceleration is smaller, and the solid cylinder reaches the bottom first.

1. Inclined plane, blocks, and pulley (13 points)

a. (**1.0 points**) Free-body diagrams for blocks A and B are shown in figure 1. Note that, since the pulley is massless, $T_A = T_B = T$.

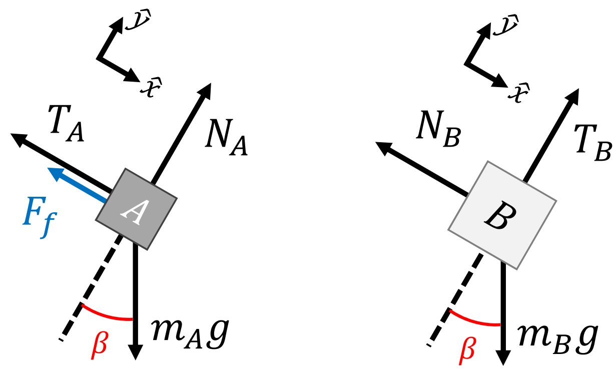

Figure 1: Free-body diagrams for blocks A and B.

b. (**3.0 points**) Using the free-body diagrams, determine the forces acting on blocks A and B:

$$\text{Block B:}\quad \Sigma F_y = T - m_B\, g\cos(\beta) = 0 \quad\Rightarrow\quad T = m_B\, g\cos(\beta) \tag{1}$$

$$\text{Block A:}\quad \Sigma F_y = N_A - m_A\, g\cos(\beta) = 0 \quad\Rightarrow\quad N_A = m_A\, g\cos(\beta) \tag{2}$$

$$\Sigma F_x = m_A\, g\sin(\beta) - T - F_f = 0 \tag{3}$$

In this case, friction is static:

$$F_f = F_s \le \mu_s\, N_A = \mu_s\, m_A\, g\cos(\beta) \tag{4}$$

Therefore, for block A not to slide down the inclined plane:

$$F_f = m_A\, g\sin(\beta) - T = m_A\, g\sin(\beta) - m_B\, g\cos(\beta) \le \mu_s\, m_A\, g\cos(\beta) \tag{5}$$

$$\Rightarrow\; m_A(\sin(\beta) - \mu_s\cos(\beta)) \le m_B\cos(\beta) \tag{6}$$

$$\Rightarrow\; m_A \le \frac{m_B\cos(\beta)}{\sin(\beta) - \mu_s\cos(\beta)} \tag{7}$$

The minimum mass of block A is then:

$$m_A^{\min} = \frac{m_B\cos(\beta)}{\sin(\beta) - \mu_s\cos(\beta)} \tag{8}$$

c. (**2.0 points**) If block A starts sliding down the plane, then for block B:

$$\Sigma F_y = T - m_B\, g\cos(\beta) = m_B\, a_{By} \tag{9}$$

For block A:

$$\Sigma F_y = N_A - m_A\, g\cos(\beta) = 0 \quad\Rightarrow\quad N_A = m_A\, g\cos(\beta) \quad \text{(same as before)} \tag{10}$$
$$\Sigma F_x = m_A\, g\sin(\beta) - T - F_f = m_A\, a_{Ax} \tag{11}$$
$$\Rightarrow m_A\, g\sin(\beta) - [m_B\, g\cos(\beta) + m_B\, a_{By}] - \mu_k\, m_A\, g\cos(\beta) = m_A\, a_{Ax} \tag{12}$$
$$\Rightarrow m_A\, a_{Ax} + m_B\, a_{By} = m_A\, g\,[\sin(\beta) - \mu_k\cos(\beta)] - m_B\, g\cos(\beta) \tag{13}$$

The inextensible rope connecting block A and block B provides the constraint $a_{Ax} = a_{By}$. As a result:

$$\vec{a}_A = \frac{g}{m_A + m_B}\Big[m_A\big(\sin(\beta) - \mu_k\cos(\beta)\big) - m_B\cos(\beta)\Big]\,\hat{x} \tag{14}$$

d. (**1.0 points**) Figure 2 shows the forces acting on the pulley and their point of application. F_{sup} is the force exerted by the support of the pulley.

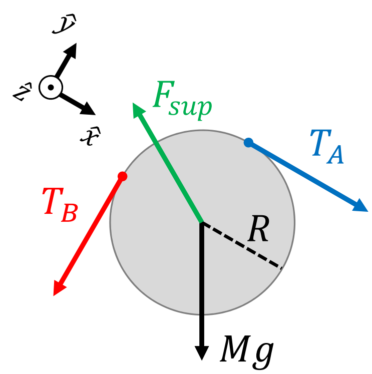

Figure 2: Diagram showing the forces acting on the pulley and their point of application.

e. (**2.0 points**) To find the angular acceleration of the pulley ($\vec{\alpha}$), it is first necessary to determine the net torque acting on the pulley:

$$\Sigma\vec{\tau} = \vec{\tau}_{T_A} + \vec{\tau}_{T_B} + \vec{\tau}_{F_{\text{sup}}} + \vec{\tau}_{M_g} \tag{15}$$

However, $\vec{\tau}_{F_{\text{sup}}} = \vec{\tau}_{M_g} = 0$ because these forces act on the pivot. The tension of the rope acting on the pulley then gives:

$$\Sigma\vec{\tau} = \vec{r}_{T_A} \times (T_A\,\hat{x}) + \vec{r}_{T_B} \times (-T_B\,\hat{y}) \tag{16}$$
$$= (R\,\hat{y}) \times (T_A\,\hat{x}) + (-R\,\hat{x}) \times (-T_B\,\hat{y}) \tag{17}$$
$$= -R\,T_A\,\hat{z} + R\,T_B\,\hat{z} = R(T_B - T_A)\,\hat{z} \tag{18}$$

For the pulley, the net torque is related to the angular acceleration by $\vec{\tau} = I_p\,\vec{\alpha}$. This gives:

$$\vec{\alpha} = \frac{R}{I_p}(T_B - T_A)\,\hat{z} \quad\Rightarrow\quad |\vec{\alpha}| = \frac{R}{I_p}|T_B - T_A| = \frac{R}{I_p}|T_A - T_B| \tag{19}$$

f. (**4.0 points**) Because the rope does not slip on the pulley, $\vec{\alpha} = \frac{1}{R}|a_{\text{rope}}|$. In this reference frame, $\vec{\alpha}$ is positive if the pulley accelerates counterclockwise. This corresponds to a tangential acceleration in $-\hat{x}$ at the point of contact of the rope attached to block A. As a result, $a_\phi = -a_{\text{rope}} = -a_{Ax}$ and

$\alpha = -\frac{1}{R} a_{Ax}$.

With the massive pulley, $T_A \neq T_B$. Therefore, for block B:

$$\Sigma F_y = T_B - m_B\, g\cos(\beta) = m_B\, a_{By} \tag{20}$$

For block A, the sum of forces in the y-direction is the same as before, given by equation 10. In the x-direction, it becomes:

$$\Sigma F_x = m_A\, g\sin(\beta) - T_A - F_f = m_A\, a_{Ax} \tag{21}$$

$$\Rightarrow m_A\, a_{Ax} = m_A\, g\left[\sin(\beta) - \mu_k\cos(\beta)\right] - T_A \tag{22}$$

Adding together equations 20 and 22:

$$m_A\, a_{Ax} + m_B\, a_{By} = m_A\, g\left[\sin(\beta) - \mu_k\cos(\beta)\right] - T_A + T_B - m_B\, g\cos(\beta) \tag{23}$$

From the relationship found in equation 19, we can substitute $T_B - T_A = \frac{I_p}{R}\alpha$ into equation 23:

$$m_A\, a_{Ax} + m_B\, a_{By} = m_A\, g\left[\sin(\beta) - \mu_k\cos(\beta)\right] - m_B\, g\cos(\beta) + \frac{I_p}{R}\alpha \tag{24}$$

$$\Rightarrow m_A\, a_{Ax} + m_B\, a_{By} - \frac{I_p}{R}\alpha = m_A\, g\left[\sin(\beta) - \mu_k\cos(\beta)\right] - m_B\, g\cos(\beta) \tag{25}$$

To solve for a_{Ax}, recall that $a_{By} = a_{Ax}$ and that $\alpha = -\frac{1}{R}a_{Ax}$. Also, for a solid cylinder pulley $I_p = (1/2)\, M_p\, R^2$:

$$m_A\, a_{Ax} + m_B\, a_{By} - \frac{I_p}{R}\alpha = \left(m_A + m_B + \frac{M_p}{2}\right) a_{Ax} \tag{26}$$

Substituting this into the left-hand side of equation 25, gives the expression for a_{Ax}. Since $a_{Ay} = 0$, we have:

$$\vec{a}_A = \frac{g}{m_A + m_B + \frac{M_p}{2}}\Big[m_A\big(\sin(\beta) - \mu_k\cos(\beta)\big) - m_B\cos(\beta)\Big]\,\hat{x} \tag{27}$$

2. Hole-in-one (12 points)

a. (**2.0 points**) To find the velocity of m_A before the collision with the ball, conservation of mechanical energy can be used since gravity is the only force that does work in this system. As a result, for m_A, $E_{m,\theta_0} = E_{m,\beta}$. By considering the potential and kinetic energy of m_A at angles θ_0 and β, we can write:

$$K_{\theta_0} + U_g(\theta_0) = K_\beta + U_g(\beta) \tag{1}$$

Recognizing that $K_{\theta_0} = 0$ and $K_\beta = \frac{1}{2} m_A v_A^2$:

$$U_g(\theta_0) = \frac{1}{2} m_A v_A^2 + U_g(\beta) \quad \Rightarrow \quad U_g(\theta_0) - U_g(\beta) = \frac{1}{2} m_A v_A^2 \tag{2}$$

To solve for v_A, the difference in the height of m_A at angles θ_0 and β must then be found. From figure 3 and the geometry of the system, it can be seen that the height difference is $\Delta h = l(\cos(\beta) - \cos(\theta_0))$:

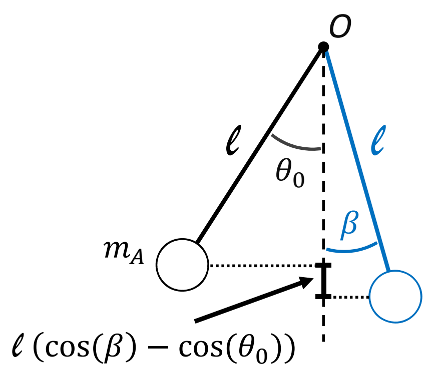

Figure 3: Schematic of table-top golf.

Plugging this value into equation 2, we find v_A:

$$U_g(\theta_0) - U_g(\beta) = m_A\, g\, \Delta h \tag{3}$$

$$m_A\, g\, l\big(\cos(\beta) - \cos(\theta_0)\big) = \frac{1}{2} m_A v_A^2 \tag{4}$$

$$v_A = \sqrt{2\, g\, l\, [\cos(\beta) - \cos(\theta_0)]} \tag{5}$$

b. (**3.0 points**) Since the collision is head on and elastic, the ball, m_B, will move in the same direction that m_A is moving immediately prior to impact. To determine the trajectory of m_A at that moment, we recognize that its motion is circular. As a result, the velocity of m_A is perpendicular to the string. The direction of motion of m_A right before impacting the ball is $\hat{v}_A = \cos(\beta)\,\hat{x} + \sin(\beta)\,\hat{y}$. Therefore, $\hat{v}_B = \cos(\beta)\,\hat{x} + \sin(\beta)\,\hat{y}$.

To find the magnitude of v_B, the formula for 1D elastic collisions can be used. In this case, $v_{B,i} = 0$, $v_{A,i} = v_A$, and $v_{B,f} = v_B$. Since $m_B = m_A/2$, we find:

$$v_B = \frac{2\, m_A}{m_A + m_B} v_A = \frac{2\, m_A}{m_A + m_A/2} v_A = \frac{4}{3} v_A \tag{6}$$

Combining the magnitude and direction, the expression for $\vec{v}_B$ is:

$$\vec{v}_B = v_B\,\hat{v}_B = \frac{4}{3}v_A\,\hat{v}_B \tag{7}$$

$$\vec{v}_B = \frac{4\sqrt{2}}{3}\sqrt{g\,l\,[\cos(\beta) - \cos(\theta_0)]}\,\Big(\cos(\beta)\,\hat{x} + \sin(\beta)\,\hat{y}\Big) \tag{8}$$

The impulse approximation is necessary so that external forces can be ignored when analyzing the collision between m_A and m_B. This allows the application of the formula for 1D elastic collisions.

c. (**3.0 points**) The ball, m_B, experiences projectile motion with an initial velocity of $\vec{v}_B$ and an initial position of $x_0 = y_0 = 0$. The trajectory of m_B is shown in figure 4.

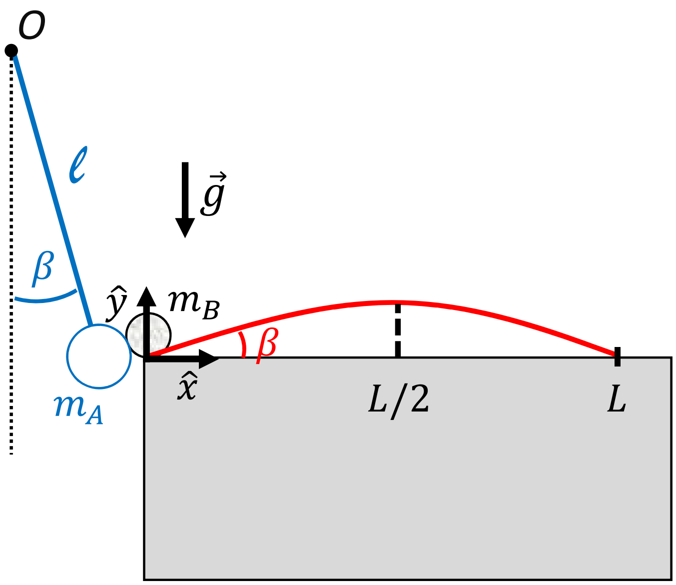

Figure 4: Trajectory of m_B is shown in red, with the launch angle β.

The time it takes to reach the apex of its trajectory, t_a, at $x = L/2$ and where $v_y = 0$ is:

$$v_y = -g\,t_a + v_{0,y} \quad \Rightarrow \quad t_a = \frac{1}{g}v_B\sin(\beta) \tag{9}$$

We can then write an expression for L:

$$\frac{L}{2} = v_{0,x}\cdot t_a = v_B\cos(\beta)\cdot\frac{1}{g}v_B\sin(\beta) \tag{10}$$

$$L = \frac{v_B^2}{g}2\sin(\beta)\cos(\beta) = \frac{v_B^2}{g}\sin(2\beta) \tag{11}$$

Substituting in the magnitude of v_B from equation 8:

$$L = \frac{1}{g}\sin(2\beta)\Big(\frac{4\sqrt{2}}{3}\sqrt{g\,l\,[\cos(\beta)-\cos(\theta_0)]}\Big)^2 \tag{12}$$

$$= \frac{1}{g}\sin(2\beta)\cdot\frac{32}{9}g\,l\,[\cos(\beta)-\cos(\theta_0)] \tag{13}$$

$$= \frac{32}{9}l\,[\cos(\beta)-\cos(\theta_0)]\,\sin(2\beta) \tag{14}$$

It is then clear that L has a maximum value when $\theta_0 = \frac{\pi}{2}$, for which $\cos(\theta_0) = 0$:

$$L_{\max} = L(\theta_0 = \frac{\pi}{2}) = \frac{32}{9}l\cos(\beta)\sin(2\beta) = \frac{64}{9}l\sin(\beta)\cos^2(\beta) \tag{15}$$

d. (**1.0 points**) To find the initial angle, θ_0, required for the ball to land in the hole, we solve for $L = d$ in the expression given in equation 14:

$$d = \frac{32}{9}l\,[\cos(\beta)-\cos(\theta_0)]\,\sin(2\beta) \tag{16}$$

$$\frac{d}{\sin(2\beta)} = \frac{32}{9}l\,[\cos(\beta)-\cos(\theta_0)] \tag{17}$$

$$\frac{d}{\sin(2\beta)}\frac{9}{32\,l} = \cos(\beta)-\cos(\theta_0) \tag{18}$$

Solving for θ_0:

$$\theta_0 = \arccos\left[\cos(\beta) - \frac{9}{32\,l}\frac{d}{\sin(2\beta)}\right] \tag{19}$$

e. (**3.0 points**) In the case where $m_B \ll m_A$, equation 6 becomes:

$$v_B = \frac{2\,m_A}{m_A+m_B} \approx \frac{2\,m_A}{m_A}v_A = 2\,v_A \tag{20}$$

The direction for $\hat{v}_B$ remains the same as before (i.e., $\hat{v}_B = \hat{v}_A$), so the expression for range L from equation 11 still applies. Now, however, $v_B = 2\,v_A$:

$$L\,(m_B \ll m_A) = \frac{1}{g}\sin(2\beta)(2\,v_A)^2 \tag{21}$$

Earlier, when $m_B = m_A/2$, $L = d$ and was given by:

$$L\,(m_B = m_A/2) = d = \frac{1}{g}\sin(2\beta)(\frac{4}{3}v_A)^2 \tag{22}$$

Taking the ratio of the different expressions of L from equations 21 and 22, we find:

$$\frac{L\,(m_B \ll m_A)}{L\,(m_B = m_A/2)} = \frac{\frac{1}{g}\sin(2\beta)(2\,v_A)^2}{\frac{1}{g}\sin(2\beta)(\frac{4}{3}v_A)^2} = \frac{4}{\frac{16}{9}} = \frac{9}{4} \tag{23}$$

Since the ball travels a distance d when $m_B = m_A/2$, it now travels $\frac{9}{4}d$ with $m_B \ll m_A$. As a result, the ball lands at a distance from the hole of:

$$L\,(m_B \ll m_A) - L\,(m_B = m_A/2) = \frac{9}{4}d - d = \frac{5}{4}d \tag{24}$$

3. Rolling sphere (13 points)

a. (**3.0 points**) Here, we are asked to find the relationship between $\dot{\phi}$ and $\dot{\varphi}$. To do so, we first note that the velocity of the point of the sphere "p" in contact with the surface with respect to the fixed frame (subscript F) is:

$$\vec{v}_{F,p} = \vec{v}_{F,CM} + \vec{v}_{CM,p} = (R+r)\dot{\phi}\,\hat{\phi} + r\,\dot{\varphi}\,\hat{\varphi} \tag{1}$$

At "p", it can be seen that $\hat{\varphi} = -\hat{\phi}$. Also, since the sphere does not slip, $\vec{v}_{F,p} = 0$. Substituting these terms into equation 1, we can show the requested relationship:

$$0 = (R+r)\dot{\phi}\,\hat{\phi} + r\,\dot{\varphi}(-\hat{\phi}) = \Big[(R+r)\dot{\phi} - r\,\dot{\varphi}\Big]\,\hat{\phi} \tag{2}$$

$$\Rightarrow (R+r)\dot{\phi} = r\,\dot{\varphi} \tag{3}$$

$$\Rightarrow \dot{\varphi} = (1+\frac{R}{r})\dot{\phi} \tag{4}$$

b. (**2.0 points**) To find the total kinetic energy of the sphere, we must first recognize that both the translational and rotational kinetic energies contribute to the total:

$$K = K_{\text{tr}} + K_{\text{rot}} = \frac{1}{2}m\,v_{CM}^2 + \frac{1}{2}I_s\,\omega_s^2 \tag{5}$$

As noted in equation 1, $v_{F,CM} = (R+r)\dot{\phi}$. In the problem statement, we are given the angular speed of the sphere, $\omega_s = \dot{\varphi}$. Substituting these terms and the expression for $\dot{\varphi}$ from equation 4 into equation 5, we find the requested relationship for the kinetic energy:

$$K = \frac{1}{2}m\,(R+r)^2(\dot{\phi})^2 + \frac{1}{2}I_s\,\dot{\varphi}^2 \tag{6}$$

$$= \frac{1}{2}m\,(R+r)^2(\dot{\phi})^2 + \frac{1}{2}I_s\left(1+\frac{R}{r}\right)^2(\dot{\phi})^2 \tag{7}$$

$$= \frac{1}{2}(R+r)^2\left(m+\frac{I_s}{r^2}\right)(\dot{\phi})^2 \tag{8}$$

c. (**1.0 points**) To find the change in potential energy, ΔU_g, as a function of ϕ, we can define $U_g = 0$ at $x = 0$. Note that $\hat{x}$ is along the vertical axis in the coordinate system defined in the problem. Using this definition:

$$U_g(\phi = 0) = m\,g\,(R+r) \tag{9}$$

$$U_g(\phi) = m\,g\,(R+r)\cos(\phi) \tag{10}$$

We can then find an expression for ΔU_g:

$$\Delta U_g = U_g(\phi = 0) - U_g(\phi) \tag{11}$$

$$\Delta U_g = m\,g\,(R+r)\big(1-\cos(\phi)\big) \tag{12}$$

d. (**2.0 points**) Mechanical energy is conserved in this system because gravity (a conservative force) is the only force that does work. Note that the static friction required for the no-slip condition does no work.

Because mechanical energy is conserved, we can write:

$$E_{m,\phi} = E_{m,0} \quad \Rightarrow \quad K_0 + U_{g,0} = K_\phi + U_{g,\phi} \tag{13}$$

Initially, $K_0 = 0$, so:

$$U_{g,0} = K_\phi + U_{g,\phi} \quad \Rightarrow \quad \Delta U_g = K_\phi \tag{14}$$

Substituting in the expressions for K_ϕ and ΔU_g from equations 8 and 12, respectively, we can find the requested relationship for $(\dot{\phi})^2$:

$$m\,g\,(R+r)\big(1-\cos(\phi)\big) = \frac{1}{2}(R+r)^2\left(m + \frac{I_s}{r^2}\right)(\dot{\phi})^2 \tag{15}$$

$$(\dot{\phi})^2 = \frac{2\,g}{R+r}\left(\frac{1}{1+I_s/(m\,r^2)}\right)\big(1-\cos(\phi)\big) \tag{16}$$

e. (**3.0 points**) The forces acting on the sphere while the sphere is still in contact with the surface are shown in figure 5.

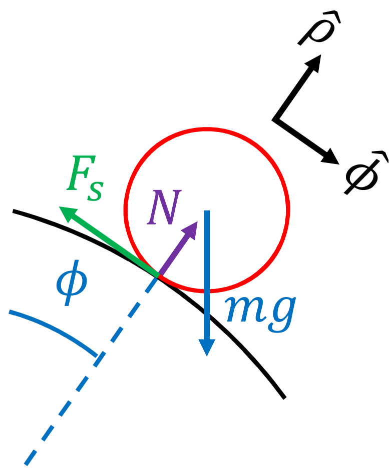

Figure 5: The forces acting on the sphere and the point of their application.

To find the relationship pertaining to ϕ_d, we analyze the forces acting on the sphere in the $\hat{\rho}$-direction:

$$\Sigma F_\rho = N - m\,g\cos(\phi) = m\,a_\rho \tag{17}$$

Because the sphere is undergoing circular motion, a_ρ must be equal to the centripetal acceleration of the sphere, a_{cent}. From the geometry of the problem:

$$a_\rho = a_{\text{cent}} = (R+r)(\dot{\phi})^2 \tag{18}$$

To find an expression involving ϕ_d, we recognize that, at the point the sphere loses contact with the surface, $N = 0$. We can then combine equations 17 and 18 to find the requested relationship:

$$0 - m\,g\cos(\phi_d) = m\,(R+r)(\dot{\phi}_d)^2 \tag{19}$$

$$g\cos(\phi_d) = (R+r)(\dot{\phi}_d)^2 \tag{20}$$

f. (**2.0 points**) To solve for ϕ_d, we can substitute our general expression for $(\dot{\phi})^2$ from equation 16 into the relationship given in equation 20:

$$g\cos(\phi_d) = (R+r)\frac{2\,g}{R+r}\left(\frac{1}{1+I_s/(m\,r^2)}\right)\big(1-\cos(\phi_d)\big) \tag{21}$$

$$\cos(\phi_d) = \frac{2}{3+\frac{I_s}{m\,r^2}} \tag{22}$$

$$\phi_d = \arccos\left[\frac{2}{3+\frac{I_s}{m\,r^2}}\right] \tag{23}$$

For a uniform sphere, $I_s = \frac{2}{5}m\,r^2$. Plugging this into the expression for ϕ_d, we find:

$$\phi_d = \arccos\left[\frac{10}{17}\right] \tag{24}$$

For a thin hoop, $I_s = m\,r^2$. For ϕ_d, this gives:

$$\phi_d = \arccos\left[\frac{1}{2}\right] \tag{25}$$

Since the cosine function decreases from 0 to $\pi/2$, a lower cosine value corresponds to a larger angle. From the geometry of the problem, this tells us the sphere leaves the surface at a larger angle in the case of the thin hoop.