

# Solution to the Final Exam

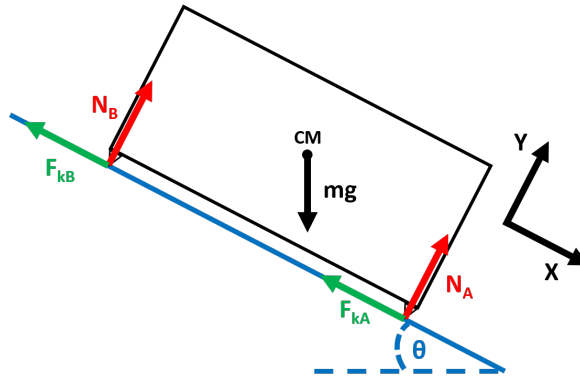
17 January 2025

PHYS-101(en)

## 1. Block on an inclined surface

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a. (2.0 points) The diagram may look as follows, where CM indicates the center of mass of the block:



b. (4.0 points) First, apply Newton's second law to solve for the acceleration of the block:

$$\sum F_y : N_A + N_B - mg \cos(\theta) = 0 \Rightarrow N_A + N_B = mg \cos(\theta) \quad (1)$$

$$\sum F_x : mg \sin(\theta) - F_{kA} - F_{kB} = ma_x \quad (2)$$

$$ma_x = mg \sin(\theta) - \mu_k N_A - \mu_k N_B = mg \sin(\theta) - \mu_k (N_A + N_B) \quad (3)$$

Substituting in the result from equation 1 to equation 3, we find:

$$ma_x = mg \sin(\theta) - \mu_k mg \cos(\theta) \quad (4)$$

$$a_x = g [\sin(\theta) - \mu_k \cos(\theta)] \quad (5)$$

The block therefore experiences constant acceleration in the x-direction, so the following equations apply:

$$v_x(t) = v_0 + a_x t \quad (6)$$

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_x t^2 \quad (7)$$

After traveling some distance  $d$ , we can express equations 6 and 7 as:

$$d = x(t_d) - x_0 = v_0 t_d + \frac{1}{2} a_x t_d^2 \quad (8)$$

$$v_f = v_x(t_d) = v_0 + a_x t_d \Rightarrow t_d = \frac{1}{a_x} (v_f - v_0) \quad (9)$$

Substituting the result for  $t_d$  into equation 8, we find:

$$d = v_0 \frac{1}{a_x} (v_f - v_0) + \frac{1}{2} a_x \frac{1}{a_x^2} (v_f - v_0)^2 \quad (10)$$

$$d = \frac{v_0}{a_x} (v_f - v_0) + \frac{1}{2a_x} (v_f - v_0)^2 = \frac{1}{2a_x} (v_f^2 - v_0^2) \quad (11)$$

Rearranging and introducing the value for  $a_x$  found in equation 5, we arrive at an expression for  $v_f$ :

$$v_f^2 = v_0^2 + 2d a_x \quad (12)$$

$$|\vec{v}_f| = \sqrt{v_0^2 + 2dg [\sin(\theta) - \mu_k \cos(\theta)]} \quad (13)$$

- c. **(3.0 points)** For  $|\vec{v}_f|$  to be less than  $|\vec{v}_0|$ , the second term inside the radical ( $2dg [\sin(\theta) - \mu_k \cos(\theta)]$ ) must be negative, as  $d$  and  $g$  are both positive. This gives us the condition:

$$2dg [\sin(\theta) - \mu_k \cos(\theta)] < 0 \quad \Rightarrow \quad \sin(\theta) - \mu_k \cos(\theta) < 0 \quad (14)$$

$$\mu_k > \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \quad (15)$$

A full stop occurs when the terms inside the radical cancel out. To avoid that, it must be the case that:

$$v_0^2 + 2dg [\sin(\theta) - \mu_k \cos(\theta)] > 0 \quad \Rightarrow \quad v_0^2 > -2dg [\sin(\theta) - \mu_k \cos(\theta)] \quad (16)$$

$$\frac{v_0^2}{2dg} > \mu_k \cos(\theta) - \sin(\theta) = \cos(\theta) [\mu_k - \tan(\theta)] \quad (17)$$

This gives us the following condition for  $\mu_k$  to ensure that the block does not come to a stop:

$$\mu_k < \tan(\theta) + \frac{v_0^2}{2dg \cos(\theta)} \quad (18)$$

The two conditions for  $\mu_k$  from equations 15 and 18 can then be written as a single expression in the form provided in the problem statement:

$$\tan(\theta) < \mu_k < \tan(\theta) + \frac{v_0^2}{2dg \cos(\theta)} \quad (19)$$

- d. **(4.0 points)** To solve for the normal force acting on point B ( $\vec{N}_B$ ), you can use the fact that the rotational state of the block does not change around any choice of pivot point, i.e.,  $\vec{\tau}_{\text{net}} = 0$ . By choosing the CM as the pivot point,  $\tau \neq 0$  for frictional forces. Note that, due to its uniform density, the CM of the block is at the center of the rectangle. We can then write the following, where the subscripts  $FkA$  and  $FkB$  correspond to the kinetic friction at points  $A$  and  $B$ :

$$\vec{\tau}_{\text{net,CM}} = \vec{\tau}_{N_A} + \vec{\tau}_{FkA} + \vec{\tau}_{N_B} + \vec{\tau}_{FkB} + \vec{\tau}_{mg} \quad (20)$$

We can then write expressions for the components of  $\vec{\tau}_{\text{net,CM}}$ . As the weight acts on the pivot,  $\vec{\tau}_{mg} = 0$ . For the rest, we have:

$$\vec{\tau}_{N_A} = \vec{r}_{CM,A} \times \vec{N}_A = (l \hat{x} - h \hat{y}) \times (N_A \hat{y}) = l N_A \hat{z} \quad (21)$$

$$\vec{\tau}_{FkA} = \vec{r}_{CM,A} \times \vec{F}_{kA} = (l \hat{x} - h \hat{y}) \times (-F_{kA} \hat{x}) = h F_{kA} (\hat{y} \times \hat{x}) = -h \mu_k N_A \hat{z} \quad (22)$$

$$\vec{\tau}_{N_B} = \vec{r}_{CM,B} \times \vec{N}_B = (-l \hat{x} - h \hat{y}) \times (N_B \hat{y}) = -l N_B \hat{z} \quad (23)$$

$$\vec{\tau}_{FkB} = \vec{r}_{CM,B} \times \vec{F}_{kB} = (-l \hat{x} - h \hat{y}) \times (-F_{kB} \hat{x}) = h F_{kB} (\hat{y} \times \hat{x}) = -h \mu_k N_B \hat{z} \quad (24)$$

Substituting into  $\vec{\tau}_{\text{net,CM}}$ :

$$\vec{\tau}_{\text{net,CM}} = l N_A \hat{z} - h \mu_k N_A \hat{z} - l N_B \hat{z} - h \mu_k N_B \hat{z} = [N_A (l - h \mu_k) - N_B (l + h \mu_k)] \hat{z} \quad (25)$$

Since  $\vec{\tau}_{\text{net,CM}} = 0$ , we can set equation 25 equal to zero:

$$0 = N_A (l - h \mu_k) - N_B (l + h \mu_k) \quad (26)$$

We can then use equation 1 to find that  $N_A = mg \cos(\theta) - N_B$ . Plugging this into equation 26 and solving for  $N_B$ :

$$0 = (l - h \mu_k) [mg \cos(\theta) - N_B] - (l + h \mu_k) N_B = (l - h \mu_k) mg \cos(\theta) - 2l N_B \quad (27)$$

$$N_B = \frac{1}{2} \left(1 - \frac{h}{l} \mu_k\right) mg \cos(\theta) \quad \Rightarrow \quad |\vec{N}_B| = \frac{1}{2} mg \cos(\theta) \left|1 - \frac{h}{l} \mu_k\right| \quad (28)$$

- e. **(2.0 points)** For the block to always remain in contact with the surface,  $N_B > 0$ . Since  $\cos(\theta) \geq 0$  for  $0 \leq \theta \leq \frac{\pi}{2}$ , the requirement is satisfied if  $1 - \frac{h}{l} \mu_k > 0$ . Solving for  $\mu_k$ , the block remains in contact with the surface if:

$$\mu_k < \frac{l}{h} \quad (29)$$

- f. **(2.0 points)** If the block does not slide, friction is static. The forces acting on the block are then as follows, where  $F_{sA}$  and  $F_{sB}$  are the forces of static friction acting on the block at points  $A$  and  $B$ , respectively:

$$\sum F_y : N_A + N_B - mg \cos(\theta) = 0 \Rightarrow N_A + N_B = mg \cos(\theta) \quad (30)$$

$$\sum F_x : mg \sin(\theta) - F_{sA} - F_{sB} = ma_x \quad (31)$$

We know that  $F_{sA} + F_{sB} \leq \mu_s N_A + \mu_s N_B$ , and from equation 30 that  $\mu_s (N_A + N_B) = \mu_s mg \cos(\theta)$ . The maximum value of the sum of friction forces is then:

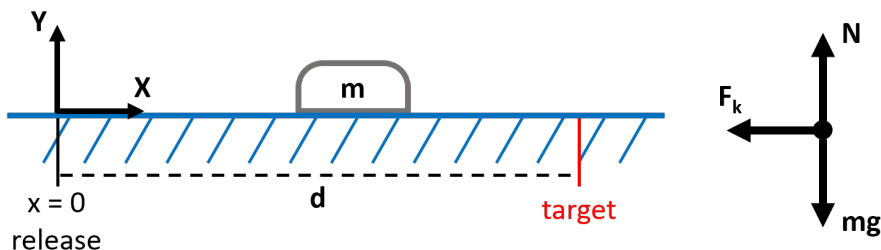
$$(F_{sA} + F_{sB})_{\text{max}} = \mu_s mg \cos(\theta) \quad (32)$$

According to equation 30, the block will start sliding down the inclined surface (i.e.,  $a_x > 0$ ), only when:

$$mg \sin(\theta) - (F_{sA} + F_{sB})_{\text{max}} > 0 \quad \Rightarrow \quad mg \sin(\theta) > \mu_s mg \cos(\theta) \quad \Rightarrow \quad \tan(\theta) > \mu_s \quad (33)$$

## 2. Curling

- a. (1.0 points) The diagram may look something like what is shown below. The forces acting on the stone after it is released are also indicated at right:



- b. (3.0 points) Applying Newton's second law to solve for the acceleration of the stone:

$$\sum F_y : N - mg = 0 \Rightarrow N = mg \quad (1)$$

$$\sum F_x : -F_k = ma_x \Rightarrow ma_x = -\mu_k N = -\mu_k mg \Rightarrow a_x = -\mu_k g \quad (2)$$

This tells us that the stone has a constant acceleration up to the moment it stops,  $t_s$ . We can then describe the velocity of the stone before it stops as:

$$v_x(t) = v_0 + a_x t = v_0 - \mu_k g t \quad (3)$$

$$\vec{v}(t) = (v_0 - \mu_k g t) \hat{x} \quad (4)$$

Note that, if  $t > t_s$ , then  $\vec{v}(t) = 0$ . To find  $t_s$ , we solve equation 4 for  $t = t_s$ :

$$v_x(t_s) = 0 = v_0 - \mu_k g t_s \Rightarrow t_s = \frac{v_0}{\mu_k g} \quad (5)$$

- c. (3.0 points) Since acceleration is constant for  $0 \leq t \leq t_s$ , we can describe the position of the stone in the  $x$ -direction as a function of time using:

$$x(t) = x_0 + v_0 t - \frac{1}{2} \mu_k g t^2 \quad (6)$$

In our given choice of reference frame,  $x_0 = 0$ . To stop on the target, we need  $x(t_s) = d$ . Substituting this condition into equation 6:

$$d = x(t_s) = v_0 t_s - \frac{1}{2} \mu_k g t_s^2 = v_0 \left( \frac{v_0}{\mu_k g} \right) - \frac{1}{2} \mu_k g \left( \frac{v_0}{\mu_k g} \right)^2 \quad (7)$$

$$d = \frac{1}{\mu_k g} (v_0^2 - \frac{1}{2} v_0^2) = \frac{1}{2\mu_k g} v_0^2 \quad (8)$$

Finally, solving for the required initial velocity  $v_0$  of the stone:

$$v_0 = \sqrt{2\mu_k g d} \quad (9)$$

- d. **(3.0 points)** To find the work done on the stone by friction, we first recognize that the friction force ( $\vec{F}_k$ ) is constant over the entire trajectory of the stone. Therefore, we can apply the expression of work done by a constant force:

$$W_f = \vec{F}_k \cdot \vec{l} = (-F_k \hat{x}) \cdot (d \hat{x}) = (-F_k d) (\hat{x} \cdot \hat{x}) = -F_k d = -\mu_k m g d \quad (10)$$

Here, it is important to recognize that  $W_f < 0$ .

To now solve for the change in kinetic energy, we use:

$$\Delta K = K_f - K_i = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2 \quad (11)$$

Since  $v_f = 0$ , we find:

$$\Delta K = -\frac{1}{2} m v_0^2 \quad (12)$$

Substituting in the expression found for  $v_0$  in equation 9:

$$\Delta K = -\frac{1}{2} m (\sqrt{2\mu_k g d})^2 = -m \mu_k g d \quad (13)$$

This is the same expression found for  $W_f$  in equation 10, so  $\Delta K = W_f$ . This result should be expected from the "Work-Kinetic energy" theorem, as friction is the only force doing work on the stone.

- e. **(3.0 points)** The work done by friction can be separated into the part of the stone's trajectory with sweeping and the part without, denoted by subscripts  $s$  and  $ns$ , respectively:

$$W_f = \int_d \vec{F}_k \cdot d\vec{l} = \int_s \vec{F}_k \cdot d\vec{l} + \int_{ns} \vec{F}_k \cdot d\vec{l} \quad (14)$$

The force is constant in each part, so:

$$\int_s \vec{F}_k \cdot d\vec{l} = \vec{F}_{k,s} \cdot \int_s d\vec{l} = \vec{F}_{k,s} \cdot \vec{l}_s \quad (15)$$

$$\int_{ns} \vec{F}_k \cdot d\vec{l} = \vec{F}_{k,ns} \cdot \int_{ns} d\vec{l} = \vec{F}_{k,ns} \cdot \vec{l}_{ns} \quad (16)$$

From the problem statement, we know that:

$$\vec{F}_{k,ns} = -\mu_k m g \hat{x} \quad \vec{F}_{k,s} = -\frac{1}{2} \mu_k m g \hat{x} \quad (17)$$

$$\vec{l}_{ns} = l_{ns} \hat{x} \quad \vec{l}_s = \frac{d}{3} \hat{x} \quad (18)$$

where  $l_{ns}$  is the total distance without sweeping. We can then find an expression for the work done by friction:

$$W_f = \vec{F}_{k,s} \cdot \vec{l}_s + \vec{F}_{k,ns} \cdot \vec{l}_{ns} = -\left(\frac{1}{2} \mu_k m g\right) \left(\frac{d}{3}\right) - (\mu_k m g) (l_{ns}) \quad (19)$$

$$W_f = -\left(l_{ns} + \frac{d}{6}\right) \mu_k m g \quad (20)$$

From the "Work-Kinetic energy" theorem, we then have:

$$\Delta K = \frac{1}{2}m v_f^2 - \frac{1}{2}m v_0^2 = -\frac{1}{2}m v_0^2 = -m \mu_k g d = W_f \quad (21)$$

Equating the result from equation 20 and solving for  $l_{ns}$ :

$$\Delta K = W_f \quad \Rightarrow \quad -m \mu_k g d = -\left(l_{ns} + \frac{d}{6}\right) \mu_k m g \quad (22)$$

$$d = l_{ns} + \frac{d}{6} \quad \Rightarrow \quad l_{ns} = d - \frac{d}{6} = \frac{5d}{6} \quad (23)$$

The total distance then traveled by the stone is:

$$l_{tot} = l_s + l_{ns} = \frac{d}{3} + \frac{5d}{6} = \frac{7d}{6} \quad (24)$$

- f. **(3.0 points)** Using the impulse approximation, we can say that the total momentum is conserved between right before and right after the collision. The use of the impulse approximation is necessary because the presence of an external force during the collision (friction) tells us that momentum is not conserved during the collision. The impulse approximation, however, allows us to simplify the calculations and proceed as if momentum is conserved.

To show that the velocity of the sticking stones after the collision is in the same direction and has half the speed as your stone's velocity right before the collision, we use conservation of momentum, as implied by the impulse approximation. The subscripts 1 and 2 correspond to the moving stone (your stone) and the target stone (your opponent's stone), respectively. The subscript  $c$  denotes the combined stones immediately following the collision.

$$\vec{p}_i = m \vec{v}_{1,i} + m \vec{v}_{2,i} \quad (25)$$

$$\vec{p}_f = 2m \vec{v}_{c,f} \quad (26)$$

Since  $\vec{v}_{2,i} = 0$ , we have:

$$\vec{p}_i = \vec{p}_f \quad \Rightarrow \quad m \vec{v}_{1,i} = 2m \vec{v}_c \quad \Rightarrow \quad \vec{v}_c = \frac{1}{2} \vec{v}_{1,i} \quad (27)$$

- g. **(2.0 points)** To stop at the target, which is  $\frac{1}{10}d$  away from the location of the collision, the object of mass  $2m$  must undergo a change in kinetic energy of:

$$\Delta K = 0 - \frac{1}{2}(2m) v_c^2 \quad (28)$$

As friction is the only force doing work on the stones, this change in kinetic energy must be equal to the work done by friction. This allows us to solve for  $v_c$ :

$$W_f = \vec{F}_k \cdot \vec{l} = -\mu_k (2m) g \left(\frac{1}{10}d\right) = \Delta K = -\frac{1}{2}(2m) v_c^2 \quad (29)$$

$$v_c^2 = \frac{1}{5} \mu_k g d \quad \Rightarrow \quad v_c = \sqrt{\frac{1}{5} \mu_k g d} \quad (30)$$

- h. **(2.0 points)** From part (f), the speed of your stone immediately before the collision must be twice  $v_c$ . To find  $v_n$ , we can then use the change in kinetic energy from the moment you release the stone to right before the collision:

$$\Delta K = \frac{1}{2}m (2v_c^2) - \frac{1}{2}m v_n^2 = \frac{1}{2}m \left[ \frac{4}{5} \mu_k g d - v_n^2 \right] \quad (31)$$

As before, this change in kinetic energy must be equal to the work done by friction over the path of the stone:

$$W_f = \vec{F}_k \cdot \vec{l} = -\mu_k mg \left( \frac{9}{10}d \right) \quad (32)$$

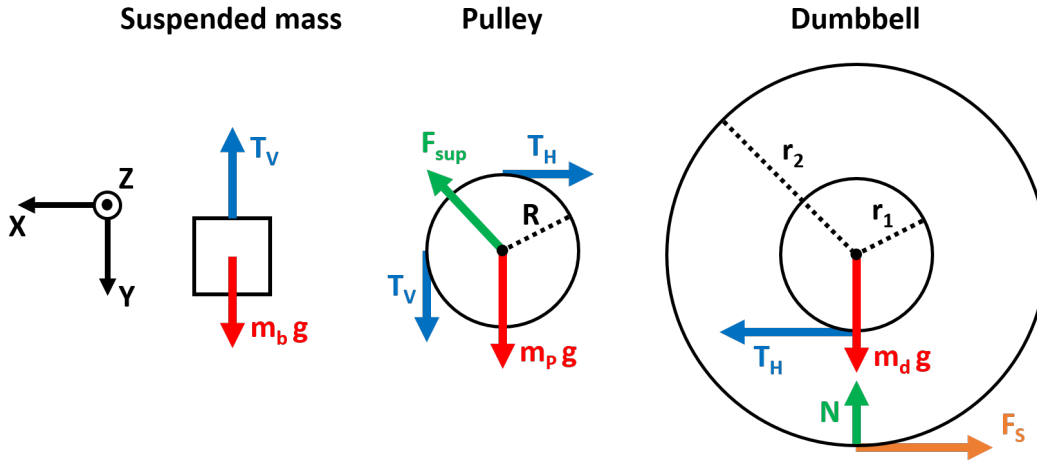
Equating  $W_f$  and  $\Delta K$ , we can solve for  $v_n$ :

$$\Delta K = W_f \quad \Rightarrow \quad \frac{1}{2}m \left[ \frac{4}{5}\mu_k g d - v_n^2 \right] = -\mu_k mg \left( \frac{9}{10}d \right) \quad (33)$$

$$v_n^2 - \frac{4}{5}\mu_k g d = \frac{g}{5}\mu_k g d \quad \Rightarrow \quad v_n = \sqrt{\frac{13}{5}\mu_k g d} \quad (34)$$

### 3. Dumbbell, pulley and block

- a. (2.0 points) The diagrams may look as follows for the suspended mass, pulley, and dumbbell:



The "no slipping" condition suggests that static friction ( $F_s$ ) is acting on point  $C_2$ . This force acts in the negative  $x$  direction, opposing the tension force of the rope ( $T_H$ ).

- b. (5.0 points) To find the acceleration of the dumbbell's CM, first apply Newton's second law to the dumbbell;

$$\sum F_y : m_d g - N = 0 \Rightarrow N = m_d g \quad (1)$$

$$\sum F_x : T_H - F_s = m_d a_{CM} \Rightarrow F_s = T_H - m_d a_{CM} \quad (2)$$

We also need to consider the torques about the axis of the dumbbell, as the dumbbell can rotate about its axis. The net torque can be described as:

$$\vec{\tau}_{\text{net}} = \vec{\tau}_{m_d g} + \vec{\tau}_N + \vec{\tau}_{F_s} + \vec{\tau}_{T_H} \quad (3)$$

Since the weight is applied at the pivot,  $\vec{\tau}_{m_d g} = 0$ . Breaking out the other individual components of  $\vec{\tau}_{\text{net}}$ , we have:

$$\vec{\tau}_N = (r_2 \hat{y}) \times (N \hat{y}) = r_2 N (\hat{y} \times \hat{y}) = 0 \quad (4)$$

$$\vec{\tau}_{F_s} = (r_2 \hat{y}) \times (-F_s \hat{x}) = -r_2 F_s (\hat{y} \times \hat{x}) = r_2 F_s \hat{z} \quad (5)$$

$$\vec{\tau}_{T_H} = (r_1 \hat{y}) \times (T_H \hat{x}) = r_1 T_H (\hat{y} \times \hat{x}) = -r_1 T_H \hat{z} \quad (6)$$

Substituting these values into equation 3, we find:

$$\vec{\tau}_{\text{net}} = (r_2 F_s - r_1 T_H) \hat{z} \quad (7)$$

A net torque will cause an angular acceleration  $\vec{\alpha}_d$  in the  $\hat{z}$  direction:

$$\vec{\tau}_{\text{net}} = I_d \vec{\alpha}_d \Rightarrow r_2 F_s - r_1 T_H = I_d \alpha_d \quad (8)$$

The "no-slipping" condition at  $C_2$  implies that  $\alpha_d = \frac{a_{CM}}{r_2}$ . Replacing this and  $F_s$  (from equation 2) in equation 8, gives:

$$r_2(T_H - m_d a_{CM}) - r_1 T_H = I_d \frac{a_{CM}}{r_2} \Rightarrow (r_2 - r_1) T_H = \frac{1}{r_2} I_d a_{CM} + m_d r_2 a_{CM} \quad (9)$$

$$a_{CM} = T_H (r_2 - r_1) \frac{r_2}{I_d + m_d r_2^2} \quad (10)$$

Since  $T_H > 0$  and  $r_2 > r_1$ ,  $a_{CM} > 0$ . This tells us that the dumbbell will accelerate to the left (in the positive  $x$  direction).

- c. **(4.0 points)** As seen from the dumbbell axis (i.e., from a reference frame whose origin coincides with the dumbbell axis but does not rotate), for the rope to not slip at  $C_1$ , it must have the same tangential velocity as the dumbbell at that point. The tangential speed (with respect to the axis) is simply  $|v_{ax,C_1}| = r_1 |\omega|$ , where  $\omega$  is the angular speed. Keep in mind that  $|v_{ax,C_2}| = r_2 |\omega| = |v_{CM}|$  and  $|v_{ax,C_1}| = r_1 \left| \frac{v_{CM}}{r_2} \right| = \frac{r_1}{r_2} |v_{CM}|$  from the no-slip condition on the table.

To find the direction, we notice that since  $a_{CM} > 0$  (the dumbbell moves to the left), the dumbbell must rotate counterclockwise. Therefore, because  $v_{CM} > 0$ :

$$\vec{v}_{ax,C_1} = \frac{r_1}{r_2} |v_{CM}| (-\hat{x}) = -\frac{r_1}{r_2} v_{CM} \hat{x} \quad (11)$$

In the original reference frame, the dumbbell axis has a translational motion with velocity  $\vec{v}_{ax} = v_{CM} \hat{x}$  because the axis contains the CM. If  $\vec{r}_{ax}$  is the position of the axis in the original reference frame and  $\vec{r}_{ax,C_1}$  is the position of  $C_1$  with respect to the axis, then the position of  $C_1$  in the original reference frame is:

$$\vec{r}_{C_1} = \vec{r}_{ax} + \vec{r}_{ax,C_1} \quad (12)$$

After differentiating equation 12, we can express the velocity at  $C_1$  in the original frame as:

$$\vec{v}_{C_1} = \vec{v}_{ax} + \vec{v}_{ax,C_1} = v_{CM} \hat{x} - \frac{r_1}{r_2} v_{CM} \hat{x} \quad (13)$$

$$\vec{v}_{C_1} = \left(1 - \frac{r_1}{r_2}\right) v_{CM} \hat{x} \quad (14)$$

The rope must then have this same velocity:

$$\vec{v}_{\text{rope}} = \vec{v}_{C_1} = \left(1 - \frac{r_1}{r_2}\right) v_{CM} \hat{x} \quad (15)$$

Since  $v_{\text{rope}} < v_{CM}$ , the axis of the dumbbell overtakes the rope. This implies that the rope winds up around the handle.

- d. **(2.0 points)** Looking at our diagram in part (a) and applying Newton's second law to the block, we find:

$$\sum F_y : m_b g - T_V = m_b a_{by} \Rightarrow a_{by} = g - \frac{1}{m_b} T_V \quad (16)$$

- e. **(4.0 points)** From part (a), we can compute the net torque around the axis of the pulley as follows:

$$\vec{\tau}_{\text{net}} = \vec{\tau}_{m_p g} + \vec{\tau}_{\text{sup}} + \vec{\tau}_{T_V} + \vec{\tau}_{T_H} \quad (17)$$

Here, both  $\vec{\tau}_{m_p g} = 0$  and  $\vec{\tau}_{\text{sup}} = 0$  because the forces act on the pivot point. The two remaining non-zero components of the net torque are then:

$$\vec{\tau}_{T_V} = (R\hat{x}) \times (T_V\hat{y}) = RT_V\hat{z} \quad (18)$$

$$\vec{\tau}_{T_H} = (-R\hat{y}) \times (-T_H\hat{x}) = -RT_H\hat{z} \quad (19)$$

Substituting these into equation 17, we find that the net torque is:

$$\vec{\tau}_{\text{net}} = R(T_V - T_H)\hat{z} \quad (20)$$

Since  $\vec{\tau}_{\text{net}}$  can also be expressed in terms of the angular acceleration ( $\vec{\alpha}_P$ ) and moment of inertia  $I_P$  of the pulley ( $\vec{\tau}_{\text{net}} = I_P\vec{\alpha}_P$ ), we can then use the result from equation 20 to solve for the angular acceleration:

$$I_P\vec{\alpha}_P = R(T_V - T_H)\hat{z} \quad \Rightarrow \quad \vec{\alpha}_P = \frac{R}{I_P}(T_V - T_H)\hat{z} \quad (21)$$

- f. **(3.0 points)** Since the rope does not slip on the pulley, it must accelerate with the tangential acceleration:

$$\alpha_P = \frac{a_{\text{rope}}}{R} \quad \Rightarrow \quad a_{\text{rope}} = R\alpha_P \quad (22)$$

Then, because the rope is attached to  $m_b$  and inextensible:

$$a_{by} = a_{\text{rope}} = R\alpha_P \quad (23)$$

Plugging the expression for  $\alpha_P$  into equation 23, we can write an expression for  $T_H$ :

$$a_{by} = \frac{R^2}{I_P}(T_V - T_H) \quad \Rightarrow \quad T_H = T_V - \frac{I_P}{R^2}a_{by} \quad (24)$$

Then, rearranging the result from equation 16 to find an expression for  $T_V$ , we can solve for  $T_H$  with the requested terms:

$$T_H = m_b g - m_b a_{by} - \frac{I_P}{R^2}a_{by} \quad \Rightarrow \quad T_H = m_b g - a_{by} \left( m_b + \frac{I_P}{R^2} \right) \quad (25)$$

- g. **(3.0 points)** To find the relation between the acceleration of the dumbbell's CM and the acceleration of the block, we can start with equation 15, which describes the relationship between the velocity of the dumbbell's CM and the velocity of the rope. We can then differentiate both sides of equation 15 with respect to time, to get an expression that relates the accelerations of the rope and dumbbell's CM:

$$\frac{d}{dt}(v_{\text{rope}}) = \left( 1 - \frac{r_1}{r_2} \right) \frac{d}{dt}(v_{CM}) \quad \Rightarrow \quad a_{\text{rope}} = \left( 1 - \frac{r_1}{r_2} \right) a_{CM} \quad (26)$$

Since  $a_{\text{rope}} = a_{by}$  as argued in part (e) (see equation 23), we can relate the vertical acceleration of the block and the horizontal acceleration of the dumbbell's CM:

$$a_{by} = a_{CM} \left( 1 - \frac{r_1}{r_2} \right) \quad (27)$$

Next, to solve for  $a_{CM}$  in terms of the parameters of the problem, we can first revisit equation 25. By substituting the result of equation 27, we can find an expression for  $T_H$  that includes  $a_{CM}$ :

$$T_H = m_b g - a_{by} \left( m_b + \frac{I_P}{R^2} \right) \Rightarrow T_H = m_b g - a_{CM} \left( 1 - \frac{r_1}{r_2} \right) \left( m_b + \frac{I_P}{R^2} \right) \quad (28)$$

$$T_H = m_b g - a_{CM} \frac{(r_2 - r_1)}{r_2 R^2} (I_P + m_b R^2) \quad (29)$$

We can then rewrite equation 10 as an expression for  $T_H$  that includes  $a_{CM}$ :

$$a_{CM} = T_H (r_2 - r_1) \frac{r_2}{I_d + m_d r_2^2} \Rightarrow T_H = a_{CM} \frac{I_d + m_d r_2^2}{r_2 (r_2 - r_1)} \quad (30)$$

This allows us to equate equations 29 and 30 and find an expression for  $a_{CM}$  that includes only parameters given in the problem statement:

$$a_{CM} \frac{I_d + m_d r_2^2}{r_2 (r_2 - r_1)} = m_b g - a_{CM} \frac{(r_2 - r_1)}{r_2 R^2} (I_P + m_b R^2) \quad (31)$$

$$a_{CM} \left[ \frac{I_d + m_d r_2^2}{r_2 (r_2 - r_1)} + \frac{(r_2 - r_1)}{r_2 R^2} (I_P + m_b R^2) \right] = m_b g \quad (32)$$

$$a_{CM} = m_b g \left[ \frac{R^2 r_2 (r_2 - r_1)}{R^2 (I_d + m_d r_2^2) + (r_2 - r_1)^2 (I_P + m_b R^2)} \right] \quad (33)$$