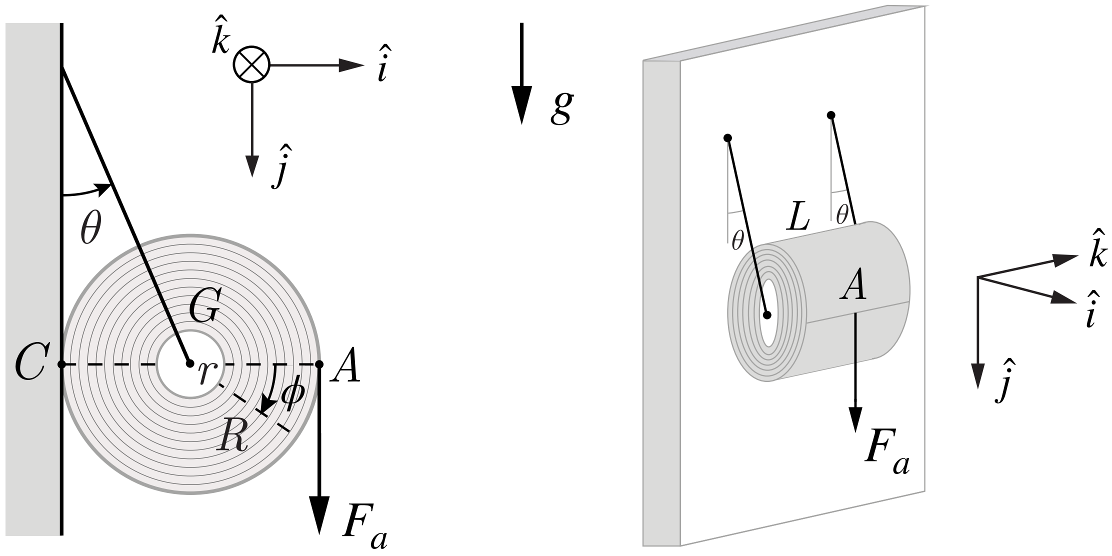
Final exam
PHYS-101(en)
20 January 2023

Problem booklet

Problems

1. Toilet paper (10 points)



We consider the rotation of a roll of toilet paper by modeling it as a uniform, hollow, rigid cylinder with a mass M, length L, inner radius r, outer radius R, and moment of inertia I_G (around its horizontal axis of symmetry, which passes through its center of mass G). The roll of paper is free to rotate around a massless internal plastic cylinder of radius r. This plastic cylinder is suspended from the wall by two metal rods, which both make an angle θ with the wall.

While you may ignore the friction between the internal plastic cylinder and the paper roll, the friction at the point of contact C between the roll and the wall is *not* negligible. It is characterized by a coefficient of static friction μ_s and a coefficient of kinetic friction μ_c, which satisfy

$$\frac{1}{2\tan\theta} > \mu_s > \mu_c.$$

We will model someone pulling on the toilet paper as applying a known constant vertical force $\vec{F}_a = F_a\hat{j}$ at the outer edge of the roll (i.e. point A). The roll starts off at rest. We will assume that the paper is sufficiently thin that the radius R, mass M, and moment of inertia I_G remain constant as it is unrolled vertically to dispense several squares of toilet paper. Thus, you can simultaneously consider θ to be constant and have the roll remain in contact with the wall at point C.

To describe the rotational dynamics of the roll, we will employ the Cartesian coordinate system shown in the figure above, where the horizontal unit vector $\hat{i}$ is perpendicular to the wall, while the vertical $\hat{j}$ and horizontal $\hat{k}$ unit vectors are along the wall. Additionally, we will use the angle ϕ to quantify the angular velocity and angular acceleration of the roll according to

$$\vec{\omega} = \omega\hat{k} = \frac{d\phi}{dt}\hat{k} \qquad \text{and} \qquad \vec{\alpha} = \alpha\hat{k} = \frac{d^2\phi}{dt^2}\hat{k}$$

respectively.

Answers should be expressed in terms of the scalar quantities given above and their time derivatives, the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ of the Cartesian coordinate system, the gravitational acceleration g, and the scalar quantities specified in the statement of each question.

a. Draw the free body diagram for the toilet paper roll, including an indication of the point of application of each force (in order to calculate the torques in part b). This should include the gravitational force $\vec{F}_g$, the normal force $\vec{N}$ from the wall, the friction force $\vec{F}_f$, and the total tension $\vec{T}$ exerted by the two metal rods combined.

b. Find the torques about G that arise from the gravitational force $\vec{\tau}_g$, the normal force $\vec{\tau}_N$, the frictional force $\vec{\tau}_f$, the total tension force $\vec{\tau}_T$, and the applied force $\vec{\tau}_a$. These should be expressed as *vectors* in the Cartesian coordinate system. For this part of the problem, you may use the magnitudes of the normal N and the friction F_f forces in your answers, in addition to other known quantities.

c. What is the largest value of the force F_a that you can apply without the roll rotating (i.e. staying in static equilibrium)? Note that T, N, and F_f should *not* be used in your final answer.

d. Now consider the case when the roll is rotating. Find the total tension force $\vec{T}$ exerted by the rods on the roll and the normal force $\vec{N}$ exerted by the wall on the roll, expressed as *vectors* in Cartesian coordinates.

e. For the case when the roll of paper is rotating, determine its constant angular acceleration α.

f. As the roll of paper rotates, determine the vertical position of the end of the paper $y(t)$ as a function of time. The end of the paper is initially at rest (i.e. $dy/dt = 0$ at $t = 0$) and a length of ℓ is initially hanging off the roll from point A (i.e. $y(0) = \ell$). You may assume that the sheets of paper are inextensible and that the angular acceleration α of the roll is known and constant.

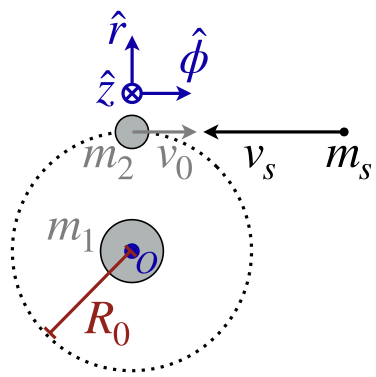
2. DART mission (15 points)

In September 2022, the DART mission tested the ability of humans to alter the trajectory of an asteroid by crashing a spacecraft into it. The purpose was to evaluate this as a method of planetary defense (i.e. to prevent future asteroids from hitting Earth). We are going to study some aspects of this mission.

The system chosen for the DART mission is a double asteroid. It is composed of a main object Didymos (called D1) and a second smaller object Dimorphos (called D2). D1 has a spherical shape, a radius of r_1, and an unknown mass m_1. D2 has a spherical shape, a radius of r_2, and an unknown mass m_2. D2 orbits around D1 in a circular trajectory with a radius of R_0 and a period of T_0. The quantities r_1, r_2, R_0, and T_0 are all known values (as they have been measured from Earth by radar), but m_1 and m_2 are initially unknown. We know the value of the universal gravitational constant G and will perform most of the calculations in the reference frame of D1 (i.e. the origin O is at the center of D1) given the cylindrical coordinate system shown in the figure below. You may assume that all of the objects in this problem have only translational motion (i.e. they do not rotate about their center of mass).

a. Determine $\vec{v}_0$, the orbital *velocity* of D2 in the reference frame of D1.
b. Determine the mass m_1 from the known data.
c. Determine the mass m_2 in terms of m_1, r_1, and r_2. You may assume that the densities of the two objects are identical.

A small spacecraft, sent from Earth with a known mass m_s, collides with D2. It impacts with a known speed of v_s in the reference frame of D1, traveling tangent to the trajectory of D2 but in the opposite direction (see figure). We assume the impact is perfectly head-on (i.e. with an impact parameter of zero) and occurs in a very short time. Note that, for the rest of this problem, the masses m_1 and m_2 as well as the orbital speed v_0 can be considered to be known quantities.



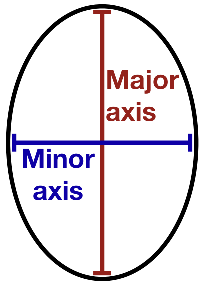
d. If the collision were perfectly inelastic (i.e. the two objects stick together), calculate $\vec{v}_a$, the new velocity of D2 immediately after the collision in the reference frame of D1.

e. Draw a rough sketch that shows, on the same drawing, the orbit of D2 before the collision and the orbit after.

Rather than a perfectly inelastic collision, a more accurate assumption is to think that the objects still stick together, but a certain amount of material is ejected from D2 due to the collision. Let us imagine that a mass m_e is ejected with a speed v_e (in the reference frame of D1) in the direction exactly *opposite* to $\vec{v}_s$. The collision still occurs in a very short amount of time.

f. Given the values of m_e and v_e, calculate $\vec{v}_b$, the new velocity of D2 immediately after the collision in the reference frame of D1.

Our calculations for parts d and f show that the velocity of D2 after the collision will change depending on the exact details of the collision. Thus, rather than assuming we know how the collision will work, a vital task of the DART mission is to determine the velocity from measurements. This will allow us to better understand the nature of the collision. From Earth, it is possible to use radar to measure the new orbital period T_2 of the orbit, which has now become elliptical (see figure).



g. Determine R_2, the minimum distance between D2 and D1 in the orbit, in terms of R_0, T_0, and T_2. Hint: consider a version of Kepler's third law — the square of the orbital period is proportional to the cube of the major axis of the ellipse (i.e. the longest diameter).

h. Determine $\vec{v}_c$, the new velocity of D2 immediately after the collision in the reference frame of D1, in terms of R_0, R_2, m_1, and G.

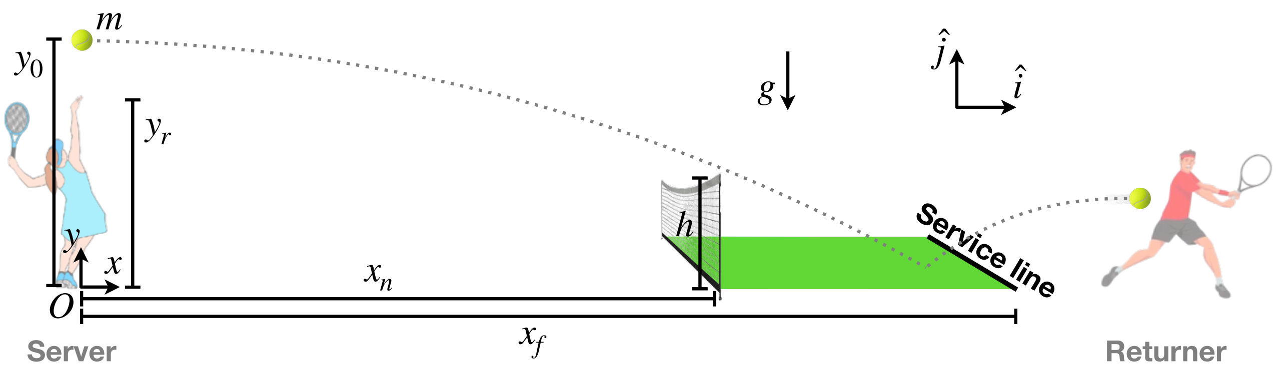
Lastly, we will consider the fuel needed by the spacecraft in order to travel to the asteroid and achieve the collision described above. It was planned that the spacecraft would begin from Earth orbit with an initial velocity of $\vec{v}_i$ *in the reference frame of the Earth* and use its engine to accelerate to the impact velocity $\vec{v}_s$ *in the reference frame of D1.*

i. Given that the asteroid D1 is moving at a velocity $\vec{v}_{D1}$ relative to the Earth, what is the change in velocity $\Delta\vec{v}_s$ that the spacecraft must experience? Express your answer in terms of $\vec{v}_i$, $\vec{v}_s$, and $\vec{v}_{D1}$.

j. Given that the spacecraft impacts the asteroid with a mass of m_s, how much fuel m_f must it have burnt during its journey from Earth to the asteroid? Assume that, as the fuel was burnt it was continuously ejected backwards at a known constant speed of u relative to the spacecraft. Additionally, you may assume that the journey was in a straight line and neglect the effect of gravitational interactions with all objects (e.g. the Earth, D1, and D2). Express your answer in terms of Δv_s (i.e. the magnitude of $\Delta\vec{v}_s$) and other known quantities.

3. Tennis serve (15 points)

Each point in the game of tennis starts with a serve. The "server" stands at the back of their side of the court, tosses a tennis ball vertically into the air, and hits it with their racket towards the "returner". The ball must pass above a net separating the two players, but still land on the near side of the "service line" (i.e. a line on the returner's side of the net). The ball then bounces and the returner tries to hit the ball back over the net.



To understand the dynamics of a tennis serve, we will employ the two-dimensional Cartesian coordinate system shown in the figure above, where the $\hat{j}$ direction points upwards, while the $\hat{i}$ direction is horizontal and defined such that the trajectory of the ball is entirely contained in the x-y plane. The origin is defined to be the point on the ground directly underneath the location where the server's racket hits the ball. You may ignore air drag (except in part g).

Answers should be expressed in terms of the unit vectors $\hat{i}$ and $\hat{j}$ of the Cartesian coordinate system, the mass of the tennis ball m, the gravitational acceleration g, and the scalar quantities specified in the statement of each question.

a. The server starts by tossing the tennis ball, releasing it from their hand at a height of y_r. They wish to toss the ball such that it rises and comes to rest at the height y_0 where they hit it with their racket towards the returner. If the toss is exactly vertical, at what speed v_r should the server release the ball with?

For the serve to be valid, it must land in the green region shown in the figure above. This means it must pass above the net, which has a height of h and is located a distance x_n away from the server, and land on the close side of the service line, which is located a distance x_f away from the server. The ball, immediately after it is hit by the server's racket, has an initial velocity with a horizontal component v_{x0} and a vertical component v_{y0}.

b. Find the trajectory of the ball $y(x)$ as it travels towards the returner, but before it bounces. You may express your answer in terms of y_0, x_n, x_f, v_{x0}, v_{y0}, h, m, and g. Note that you do *not* need to calculate the location at which the ball bounces.

c. What condition must be satisfied in order for the ball to pass above the net? You may express your answer in terms of y_0, x_n, x_f, v_{x0}, v_{y0}, h, m, and g. You may also assume that you know the function $y(x)$ from part b.

d. What condition must be satisfied in order for the ball to land before reaching the service line? You may express your answer in terms of y_0, x_n, x_f, v_{x0}, v_{y0}, h, m, and g. You may also assume that you know the function $y(x)$ from part b.

e. What is the minimum height y_0 that the server can hit the ball at and still hit *down* on the ball (i.e. hit the ball such that it has a negative value of v_{y0} and lands in the green region)? Hint: your final answer should only depend on x_n, x_f, and h.

f. If a tennis court has $x_n = 12$ m, $x_f = 18$ m, and $h = 1$ m, what is the numerical value for the minimum height y_0 from part e?

We will now investigate the effect of air drag on the trajectory of the ball. We assume that drag force is in the laminar regime and has the form $\vec{F}_d = -\beta\vec{v}$. Here $\vec{v}$ is the velocity of the ball, while β is the drag coefficient between the ball and the air (which is a known quantity).

g. Starting from Newton's second law, perform an epic derivation to find the trajectory of the ball $y(x)$ as it travels towards the returner, but before it bounces. You may express your answer in terms of β, y_0, x_n, x_f, v_{x0}, v_{y0}, h, m, and g. Note that you do *not* need to calculate the location at which the ball bounces.

THIS PAGE IS INTENTIONALLY LEFT BLANK