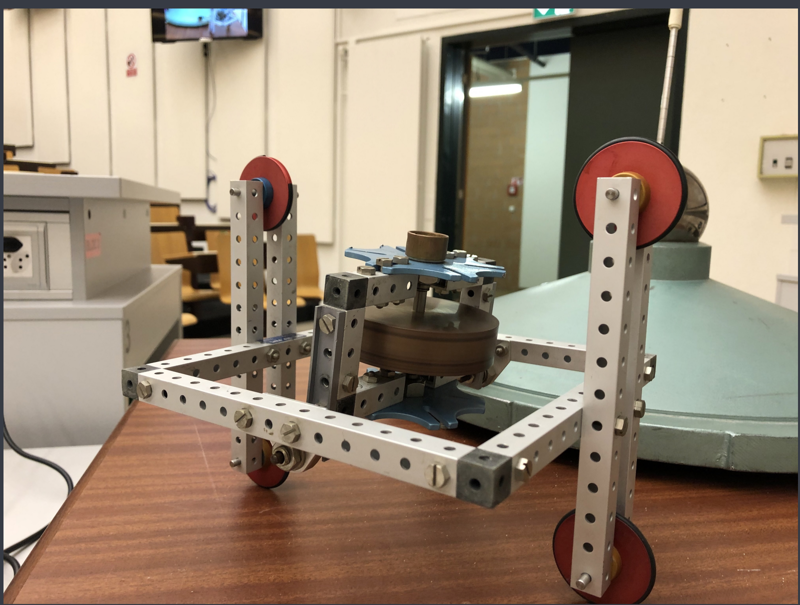
Gyroscope de Mooser – énergies cinétique et potentielle

Exercice Step by step



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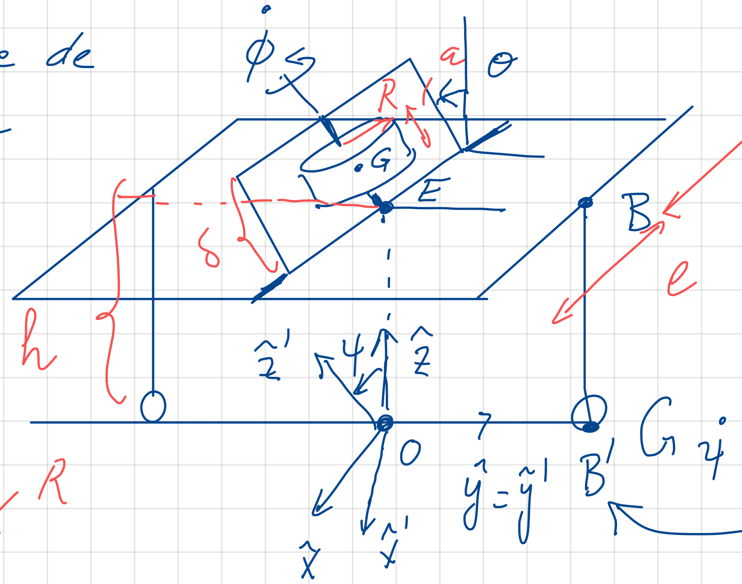
Gyroscope de Moser

$\downarrow \vec{g}$



On suppose la masse du cadre négligeable

l, h et δ, R sont donnés

1° Exprimez la position des points B, E, G en considérant les vecteurs $\vec{OB'}$, $\vec{OB}$, $\vec{OE}$, $\vec{OG}$ et calculez $\dot{\vec{OB}}$ et $\dot{\vec{OE}}$

$$\vec{OB'} = l\,\hat{y} = l\,\hat{y}'$$

$$\vec{OB} = \vec{OB'} + \vec{B'B} = l\,\hat{y}' + h\,\hat{z}'$$

$$\dot{\vec{OB}} = h\,\dot{\hat{z}}' = h\,\dot{\psi}\,\hat{y}' \wedge \hat{z}' = h\,\dot{\psi}\,\hat{x}'$$

$(\dot{\hat{y}}' = 0)$ $\qquad \vec{\omega} = \dot{\psi}\,\hat{y}'$

$$\vec{OE} = h\,\hat{z}' \qquad \dot{\vec{OE}} = h\,\dot{\hat{z}}' = h\,\dot{\psi}\,\hat{y}' \wedge \hat{z}' = h\,\dot{\psi}\,\hat{x}'$$

$$\vec{OG} = \frac{\delta}{2}\,\hat{z}'' + h\,\hat{z}' = \vec{EG} + \vec{OE}$$

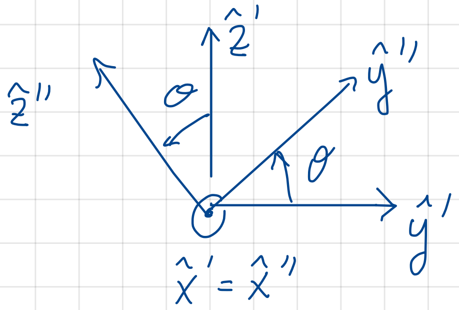
2° Calculez $\dot{\vec{OG}}$ en observant que $\vec{\Omega} = \dot{\psi}\,\hat{y} + \dot{\theta}\,\hat{x}' + \Omega\,\hat{z}''$ et exprimez tout en ''.

$$\dot{\vec{OG}} = \frac{\delta}{2}\,\dot{\hat{z}}'' + h\,\dot{\hat{z}}' = \frac{\delta}{2}\,\vec{\Omega} \wedge \hat{z}'' + h\,\dot{\psi}\,\hat{x}'$$

$$\vec{\Omega} = \Omega\,\hat{z}'' + \dot{\psi}\,\hat{y} + \dot{\theta}\,\hat{x}' = \dot{\psi}\,\hat{y}' + \dot{\theta}\,\hat{x}'' + \Omega\,\hat{z}''$$

car $\hat{x}' = \hat{x}''$ et $\hat{y}' = \hat{y}$

ensuite



$$\hat y'' = \cos\theta\, \hat y' + \sin\theta\, \hat z' = \cos\theta\, \hat y' + \sin\theta\, \hat z'$$

$$\hat z'' = \cos\theta\, \hat z' - \sin\theta\, \hat y' = -\sin\theta\, \hat y' + \cos\theta\, \hat z'$$

$$\hat y' = \cos\theta\, \hat y'' - \sin\theta\, \hat z''$$

$$\hat z' = \sin\theta\, \hat y'' + \cos\theta\, \hat z''$$

pensez à la matrice $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ et à son inverse $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\vec\Omega = (\cos\theta\, \hat y'' - \sin\theta\, \hat z'')\dot\psi + \dot\theta\, \hat x'' + \Omega\, \hat z''$$

$$\boxed{\dot{\vec{OG}} = \frac{\delta}{2}\left(\cos\theta\, \hat y''\dot\psi + \dot\theta\, \hat x''\right) \wedge \hat z'' + h\dot\psi\, \hat z'' = \frac{\delta}{2}\cos\theta\,\dot\psi\, \underline{\underline{\hat x''}} - \frac{\delta}{2}\dot\theta\, \hat y'' + h\dot\psi\, \underline{\underline{\hat x''}}}$$

$$\dot{\vec{OG}} = \left(h\dot\psi + \frac{\delta}{2}\cos\theta\dot\psi\right)\hat x'' - \frac{\delta}{2}\dot\theta\, \hat y''$$

Energie cinétique du solide

$$T = \frac{1}{2} m \vec{V_G}\cdot\vec{V_G} + \frac{1}{2}\vec\Omega \cdot I_G \vec\Omega$$

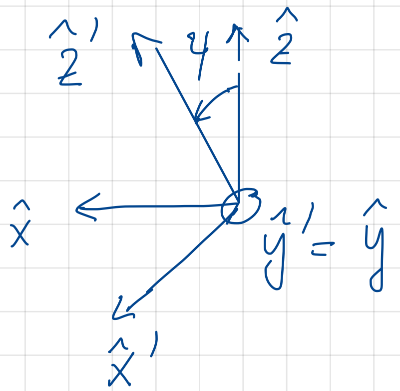
$$\boxed{T = \frac{1}{2} m \left(h\dot\psi + \frac{\delta}{2}\cos\theta\dot\psi\right)^2 + \frac{1}{2} m \frac{\delta^2}{4}\dot\theta^2 + \frac{1}{2} I_{G_x}\dot\theta^2 + \frac{1}{2} I_{G_y}\cos^2\theta\,\dot\psi^2 + \frac{1}{2} I_{G_z}\left(\Omega - \sin\theta\,\dot\psi\right)^2}$$

<u>Remarque:</u> ψ est cyclique en absence de gravité et ainsi $\frac{\partial L}{\partial\dot\psi}$ est conservé en absence de gravité.

Energie potentielle : Utiliser $\vec{OG}$ et $\hat{z}$!

$$V = E_{pot} = mgh = mg\,\vec{OG}\cdot\hat{z}$$

$$\vec{OG} = \frac{\delta}{2}\hat{z}'' + h\hat{z}'$$



$$\hat{x}' = \hat{x}\cos\psi - \hat{z}\sin\psi = \hat{x}\cos\psi - \hat{z}\sin\psi$$

$$\hat{z}' = \hat{z}\cos\psi + \hat{x}\sin\psi = \hat{x}\sin\psi + \hat{z}\cos\psi$$

$$\vec{OG} = \frac{\delta}{2}\hat{z}'' + h\hat{z}'$$

$$= \frac{\delta}{2}\left(-\sin\theta\,\hat{y}' + \cos\theta\,\hat{z}'\right) + h\hat{z}'$$

$$= -\frac{\delta}{2}\sin\theta\,\hat{y} + \left(\frac{\delta}{2}\cos\theta + h\right)\left(\hat{x}\sin\psi + \hat{z}\cos\psi\right)$$

$$\boxed{V = mg\,\vec{OG}\cdot\hat{z} = \cos\psi\left(\frac{\delta}{2}\cos\theta + h\right)mg}$$

(Lorsque $\psi = \theta = 0$ on a $\left(\frac{\delta}{2} + h\right)mg$ ce qui suit l'intuition.)

→ Pour déterminer l'énergie cinétique de rotation du solide, "on grimpe", → non prime → prime → prime, prime.

→ Pour déterminer l'énergie potentielle on exprime $\vec{OG}$, "en descendant" prime, prime → prime → non prime.