Pendule d'Ansermet

En. cinétique et potentielle.
dérivation des équations par la
méthode de Lagrange

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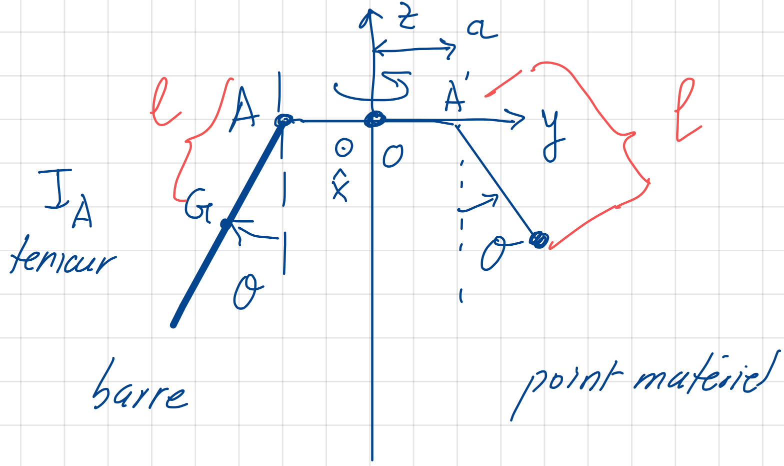
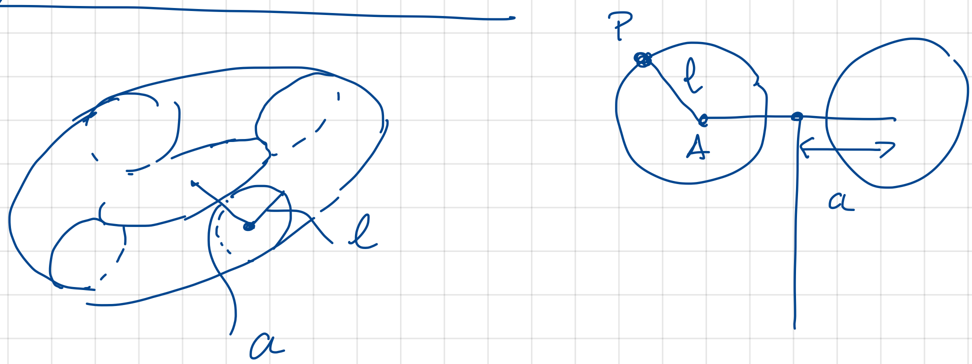


A n'est pas un point fixe

$\|\vec{OG}\| = \ell$

Quel est le type de contrainte si on utilise le repère (O, x, y, z) pour la position de G et de P.

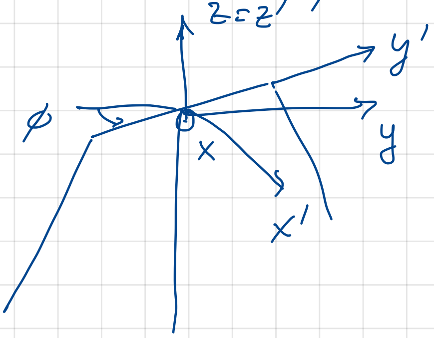
<u>Réponse: c'est un tore :</u>



⚠ aux coordonnées sphériques, elles sont cette fois contraintent pour demeurer sur le tore.

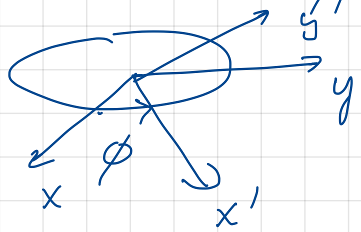
$\Longrightarrow$ utilisons les coordonnées généralisées

θ et ϕ (ce ne sont pas les coordonnées θ et ϕ des sphériques!



Energie cinétique du point matériel

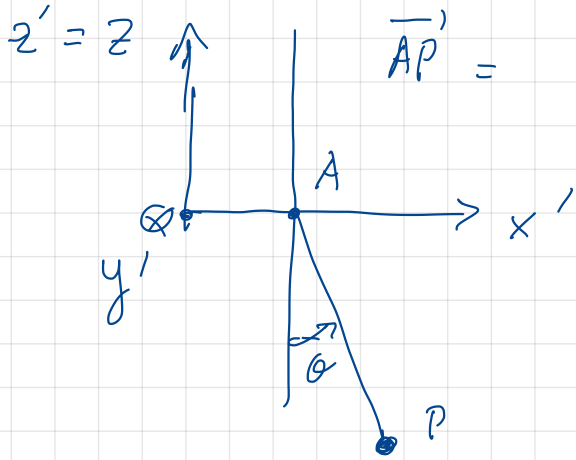
Ecrivons sa position



position du point A

$$x_A = a\cos(\phi) = a\cos(\Omega t)$$
$$y_A = a\sin(\phi) = a\sin(\Omega t)$$
$$z_A = 0.$$

$$\vec{OA} = a(\cos\Omega t\,\hat{x} + \sin\Omega t\,\hat{y}) = a\hat{x}'$$



$\vec{AP} =$

$$\vec{AP} = -l\cos\theta\,\hat{z}' + l\sin\theta\,\hat{x}'$$

$$\vec{OP} = \vec{OA} + \vec{AP}$$
$$= a\hat{x}' + l\sin\theta\,\hat{x}' - l\cos\theta\,\hat{z}'$$
$$= (a + l\sin\theta)\hat{x}' - l\cos\theta\,\hat{z}'$$

$$T = E_{cin} = \frac{1}{2} m \vec{V_p}\cdot\vec{V_p}$$

$$\vec{V_p} = \dot{\vec{OP}} = \dot{\vec{OA}} + \dot{\vec{AP}}$$
$$= a\dot{\hat{x}}' + l\dot{\theta}\cos\theta\,\hat{x}' + l\sin\theta\,\dot{\hat{x}}'$$
$$+ l\sin\theta\,\dot{\theta}\,\hat{z}'$$

$$\dot{\hat{x}}' = \vec{\omega}\wedge\hat{x}' = \Omega\,\hat{z}\wedge\hat{x}' = \Omega\,\hat{y}'$$
$$\vec{\omega} = \Omega\,\hat{z}$$

$$\vec{V_p} = a\Omega\,\hat{y}' + l\dot{\theta}\cos\theta\,\hat{x}' + \Omega(l\sin\theta)\hat{y}'$$
$$+ l\sin\theta\,\dot{\theta}\,\hat{z}'$$

$$\vec{V_p} = \Omega(a + l\sin\theta)\hat{y}' + l\cos\theta\,\dot{\theta}\,\hat{x}' + l\sin\theta\,\dot{\theta}\,\hat{z}'$$

$$\vec{V_p}\cdot\vec{V_p} = \Omega^2(a + l\sin\theta)^2 + l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2$$
$$= \Omega^2(a + l\sin\theta)^2 + l^2\dot{\theta}^2$$

$$E_{cin} = \frac{1}{2} m \Omega^2 (a + l \sin\theta)^2 + \frac{1}{2} m l^2 \dot{\theta}^2$$

$$E_{pot} = mgh = mg(l - l\cos\theta)$$

$L = E_{cin} - E_{pot}$ 1 d° de liberté θ

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m\Omega^2 (a + l\sin\theta) \, l\cos\theta - mgl\sin\theta.$$

!

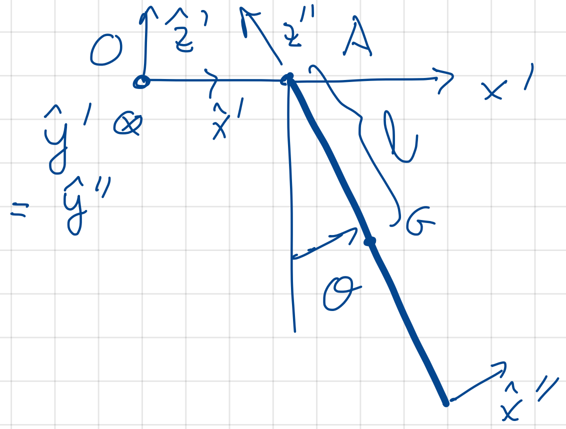
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta}$$

Équations différentielles de la dynamique :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m l^2 \ddot{\theta} - m\Omega^2 l^2 \sin\theta\cos\theta + ma\Omega^2 l\cos\theta + mgl\sin\theta = 0$$

$$\boxed{\theta'' = \Omega^2 \left(\frac{a}{l} + \sin\theta\right)\cos\theta - \frac{g}{l}\sin\theta}$$

Le système se comporte comme la glissière hémisphérique que lorsque $a = 0$, sinon c'est différent.



Energie cinétique du solide
point arbitraire A

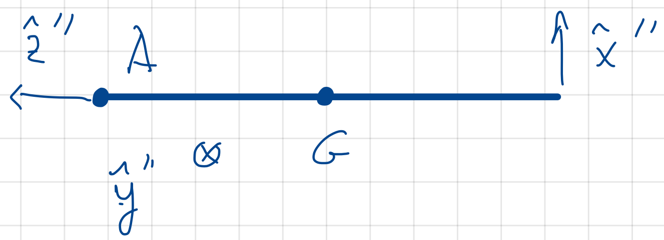
$$E_{cin} = \frac{1}{2} m \vec{V_A} \cdot \vec{V_A} + M(\vec{\omega} \wedge \vec{AG}) \cdot \vec{V_A} + \frac{1}{2} \vec{\omega} \cdot \underset{A}{I} \vec{\omega}$$

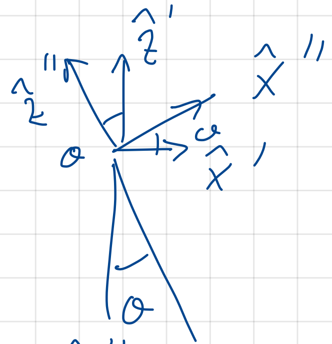
scalaire

$\rightarrow$ $I_A = I_G + m\ell^2$

(Steiner) $= \frac{1}{12} m (2\ell)^2 + m\ell^2 = \frac{1}{3} m\ell^2 + m\ell^2 = \frac{4}{3} m\ell^2$

$\vec{\omega} = -\dot{\theta}\, \hat{y}' + \Omega\, \hat{z}$ $\leftarrow$ du solide





$$\hat{z}' = \hat{z}'' \cos\theta + \hat{x}'' \sin\theta$$

$$\vec{\omega} = -\dot{\theta}\, \hat{y}'' + \Omega \cos\theta\, \hat{z}'' + \Omega \sin\theta\, \hat{x}''$$

$$E_{cin} = \frac{1}{2} m \vec{V_A} \cdot \vec{V_A} + M(\vec{\omega} \wedge \vec{AG}) \cdot \vec{V_A} + \frac{1}{2} \vec{\omega} \cdot \underline{\underline{I}}_A \vec{\omega}$$

$$\underline{\underline{I}}_A \vec{\omega} = -I_A \dot{\theta}\, \hat{y}'' + I_A \Omega \sin\theta\, \hat{x}''$$

$$\frac{1}{2} \vec{\omega} \cdot I_A \vec{\omega} = \frac{1}{2} I_A \dot{\theta}^2 + \frac{1}{2} I_A \Omega^2 \sin^2\theta$$

$\vec{V_A} = \Omega a\, \hat{y}''$ $\qquad \frac{1}{2} m \vec{V_A} \cdot \vec{V_A} = \frac{1}{2} m \Omega^2 a^2$

$\vec{AG} = -\ell\, \hat{z}''$

$\vec{\omega} \wedge AG$

$$\vec{\omega} = -\dot{\theta}\,\hat{y}'' + \Omega\cos\theta\,\hat{z}'' + \Omega\sin\theta\,\hat{x}''$$

$$\vec{\omega}\wedge\vec{AG} = \left(-\dot{\theta}\,\hat{y}'' + \Omega\sin\theta\,\hat{x}''\right)\wedge\left(-\ell\,\hat{z}''\right)$$

$$= +\ell\,\dot{\theta}\,\hat{x}'' + \ell\,\Omega\sin\theta\,\hat{y}''$$

$$m\left(\vec{\omega}\wedge\vec{AG}\right)\cdot\vec{V_A} = m\,\ell\,\Omega^2 a\sin\theta$$

$$E_{cin} = \frac{1}{2}m\,\vec{V_A}\cdot\vec{V_A} + m\left(\vec{\omega}\wedge\vec{AG}\right)\cdot\vec{V_A} + \frac{1}{2}\vec{\omega}\cdot I_A\vec{\omega}$$

$$\frac{1}{2}m\,\Omega^2 a^2 + m\,\ell\,\Omega^2 a\sin\theta$$

$$+\frac{1}{2}I_A\dot{\theta}^2 + \frac{1}{2}I_A\sin^2\theta\,\Omega^2$$

$$E_{pot} = mg\left(\ell - \ell\cos\theta\right)$$

$$\mathcal{L} = \frac{1}{2}m\Omega^2 a^2 + m\,\ell\,\Omega^2 a\sin\theta$$

$$+\frac{1}{2}I_A\left(\dot{\theta}^2 + \sin\theta\,\Omega^2\right) - mg\left(\ell - \ell\cos\theta\right)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = I_A\dot{\theta}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = m\,\ell\,\Omega^2 a\cos\theta + I_A\sin\theta\cos\theta\,\Omega^2 - mg\ell\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \qquad I_A\ddot{\theta} - I_A\sin\theta\cos\theta\,\Omega^2 - m\ell a\,\Omega^2\cos\theta + mg\ell\sin\theta = 0$$

$$\boxed{\theta'' = \Omega^2\left(\frac{m\ell a}{I_A} + \sin\theta\right)\cos\theta - \frac{mg\ell}{I_A}\sin\theta = 0}$$

à comparer avec

Si $I_A = m\ell^2$ (point matériel)

$$\theta'' = \Omega^2\left(\frac{a}{\ell} + \sin\theta\right)\cos\theta - \frac{g}{\ell}\sin\theta$$

 du

<u>Application numérique</u> $\ell = 16$ [cm] $a = 5$ [cm]

$$\Omega = 75 \ [t/min]$$

$$= \frac{75 \cdot 2\pi}{60} \ [rad/s]$$

Angle d'équilibre (dynamique) du pendule simple.

$$\tan\theta = + \frac{\Omega^2}{g}\left(a + \ell\sin\theta\right)$$

$$\tan\theta = \left(\frac{75\cdot 2\pi}{60}\right)^2 \frac{1}{9.81}\left(0.05 + 0.16\cdot\sin\theta\right)$$

$$\theta = 46,1072$$

$$\boxed{\Omega^2\left(\frac{m\ell a}{I_A} + \sin\theta\right)\cos\theta - \frac{mg\ell}{I_A}\sin\theta = 0}$$

A l'équilibre dynamique

$$I_A = \frac{4}{3} m\ell^2$$

$$\Omega^2\left(\frac{3}{4}\frac{a}{\ell} + \sin\theta\right)\cos\theta - \frac{3}{4}\frac{g}{\ell}\sin\theta = 0$$

$a = 0.05$ $\quad g = 9.81$ $\quad \Omega = \left(\frac{75\cdot 2\pi}{60}\right)$

$\ell = 0,16$

$$\Rightarrow \theta = 54,61826025°$$