

Solutions of homework # 2

Exercise 1

1. Doing the derivative with $\Phi_n(x, t) = e^{-\frac{i}{\hbar}Et}\psi_n(x)$:

$$e^{-\frac{i}{\hbar}Et}E\psi_n(x) = e^{-\frac{i}{\hbar}Et}\hat{H}\psi_n(x) \quad (1)$$

where we have used that \hat{H} operates only on space components (note that even if V is just a multiplicative operator which depends on time, E would depend on time and that is not allowed by the way we took the derivative of the LHS). Since the $e^z \neq 0$, we can simplify $e^{-\frac{i}{\hbar}Et}$ and read the time-independent Schrödinger equation:

$$E\psi_n(x) = \hat{H}\psi_n(x). \quad (2)$$

From the physical point of view E represent the energy, which is the eigenvalue of the Hamiltonian operator from the mathematical point of view.

2. In order to compute C_n we need to impose that $\langle \psi_n | \psi_n \rangle = 1$. Considering the hint given in the Exercise sheet we write:

$$1 = \langle \psi_n | \psi_n \rangle = C_n^2 \int_0^a dx \sin^2\left(\frac{n\pi x}{a}\right) = C_n^2 \frac{a}{n\pi} \int_0^{n\pi} dt \sin^2(t) = C_n^2 \frac{a}{2} \quad (3)$$

from which we easily obtain $C_n = \sqrt{\frac{2}{a}}$, which is independent from n . Since $|\psi_n(x)|^2$ represents the probability of finding the particle in a generic point x , setting $\langle \psi_n | \psi_n \rangle = 1$ is equivalent to imposing that the particle must be found inside the box. The $n = 0$ solution $\psi_0 = 0$, which is a trivial solution of the Schrödinger equation, is not physically acceptable because it corresponds to a zero probability of finding the particle everywhere in space.

3. The expectation value of the Hamiltonian is given by the expression $\langle H \rangle_n = \langle \psi_n | H | \psi_n \rangle$. So:

$$\begin{aligned} \langle H \rangle_n &= \int_0^a dx \psi_n^*(x) \left(-\frac{\hbar^2}{2m} \right) \frac{d^2}{dx^2} \psi_n(x) \\ &= \frac{2}{a} \int_0^a dx \sin\left(\frac{n\pi x}{a}\right) \left(-\frac{\hbar^2}{2m} \right) \frac{d^2}{dx^2} \sin\left(\frac{n\pi x}{a}\right) \\ &= \frac{2}{a} \frac{\hbar^2}{2m} \left(\frac{n\pi}{a}\right)^2 \int_0^a dx \sin^2\left(\frac{n\pi x}{a}\right) \\ &= \frac{\hbar^2 \pi^2 n^2}{2ma^2} = E_n \end{aligned} \quad (4)$$

The minimum-energy level is $\frac{\hbar^2 \pi^2}{2ma^2}$, which corresponds to $n = 1$. Therefore, no state with zero energy is accessible to the particle. In contrast, for an analogous classical system a zero-energy state (i.e. a still particle in the box) is an acceptable solution for this problem.

Exercise 2

1. Time-independent Schrödinger equation for this problem:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V_0 \theta(x)\psi(x) = E\psi(x) \quad (5)$$

2. If $E < V_0$, K becomes imaginary $K = i\tilde{K} = i\sqrt{\frac{2m|E-V_0|}{\hbar^2}}$, so the eigensolution becomes:

$$\psi(x) = \left(Ae^{ikx} + Be^{-ikx} \right) \theta(-x) + \left(Ce^{-\tilde{K}x} \right) \theta(x), \quad (6)$$

thus the plane-wave on the right becomes a decaying exponential.

3. The probability is:

$$P = \int_{\frac{a_b}{2}}^{a_b} dx \left| \left(Ae^{ikx} + Be^{-ikx} \right) \theta(-x) + \left(Ce^{-\tilde{K}x} \right) \theta(x) \right|^2 = |C|^2 \int_{\frac{a_b}{2}}^{a_b} dx e^{-2\tilde{K}x} > 0. \quad (7)$$

We pass from the first to the second equality thanks to the theta function and the fact that we are integrating on the positive axis. P is different from 0 and positive since a sum of strictly positive numbers and $C \neq 0$ as it is written in the text. This is an example of quantum tunneling and is strictly a non-classical effect. In classical mechanics a particle coming from $-\infty$ when approaches $x \rightarrow 0^-$ would “rebound back”, i.e. would not surpass the barrier if its energy E is lower than the V_0 .