RECAP L2

Random Variable: $\underbrace{X}_{\text{function}}(\underbrace{\omega_i}_{\text{element}})$ = some (real) number, written as x_i

$\to P(X)$: probability $\underbrace{\text{mass}}_{\text{discrete}}$ / $\underbrace{\text{density}}_{\text{continuous}}$ function

Can have different random variables for the same element, e.g. if our experiment is "pick a random person" then the random variable X_1 could be "the weight in kg" and X_2 could be "the height in cm" so $X_1(\text{"Jean"}) = 78$ and $X_2(\text{"Jean"}) = 171$

Cumulative distribution function:

"CDF" $F(X) = \sum_{i=-\infty}^{X} P(X=i)$ for discrete

$= \int_{-\infty}^{X} P(x')\,dx'$ for continuous.

Expectation Value: $\mathbb{E}(X) = \sum_{\text{all } i} P(\omega_i)\, X(\omega_i) = \sum_i p_i x_i$

$= \int_{-\infty}^{\infty} P(X)\, X\, dX$

$\mathbb{E}$ is a linear function so $\mathbb{E}(\underbrace{a + bX}_{\text{numbers}}) = a + b\,\mathbb{E}(X)$

Useful for discrete uniform probabilities: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

As $p_i = \frac{1}{n}$, $\mathbb{E}(X) = \sum_i p_i x_i = \frac{1}{n} \sum_{i=1}^{n} i = \frac{n+1}{2} = \mu$

(for $x_i = i$ (e.g. dice))

Variance: $\mathbb{V}(X) = \mathbb{E}((X-\mu)^2) = \sigma^2 = SD(X)^2$

standard deviation

$= \mathbb{E}(X^2) - E(X)^2$

Not a linear function

$$\mathbb{V}(a + b X) = b^2\, \mathbb{V}(X)$$

numbers

useful for uniform discrete distributions:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \Rightarrow \mathbb{V}(X) = \frac{(n+1)(n-1)}{12}$$

prove this as an exercise!

MSE-213 L3

3.1 Sums of Expectation Values:

$$\underline{\mathbb{E}(X_1 \pm X_2)} = \sum_{\text{all } i} P(\omega_i)\left(X_1(\omega_i) \pm X_2(\omega_i)\right)$$

$$= \sum_i p_i \left(x_{1,i} \pm x_{2,i}\right)$$

[Ex: for ω_1 = Jean $X_1(\text{Jean}) = 75$ $X_2(\text{Jean}) = 180$]
weight in kg height in cm

$$= \sum_i p_i x_{1,i} \pm \sum_i p_i x_{2,i}$$

$$= \underline{\mathbb{E}(X_1) \pm \mathbb{E}(X_2) = \mu_1 \pm \mu_2}$$

→ The expectation value of a sum (or difference) of two random variables is the sum (or difference) of the individual expectation values.

3.2 Sums of Variances

$$\underline{\mathbb{V}(X_1 \pm X_2)} = \mathbb{E}\left((X_1 \pm X_2)^2\right) - \left(\mathbb{E}(X_1 \pm X_2)\right)^2$$

$$= \mathbb{E}\left(X_1^2 + X_2^2 \pm 2X_1X_2\right) - \left(\mathbb{E}(X_1) \pm \mathbb{E}(X_2)\right)^2$$

$$= \mathbb{E}(X_1^2) + \mathbb{E}(X_2^2) \pm 2\mathbb{E}(X_1X_2) - \mathbb{E}(X_1)^2 - \mathbb{E}(X_2)^2 \mp 2\mathbb{E}(X_1)\mathbb{E}(X_2)$$

$$= \underline{\mathbb{V}(X_1) + \mathbb{V}(X_2)} \pm 2\underbrace{\left(\mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)\right)}_{\mathrm{Cov}(X_1, X_2)}$$

note: only +.

$\mathrm{Cov}(X_1, X_2) = 0$ iff X_1 and X_2 are independent.
else: can be > or < 0.

For <u>independent</u> random variables, the variances add up.

Note: height and weight are for example usually <u>not</u> independent.

3.3 More on Binomial Distributions

- n independent "trials"
- in each trial "win" or "succeed with proba. "p", fail with proba $(1-p) = q$
- X is the "total number of wins"
- The probability to win k times is: (Ex: k heads from n throws)

$$P(X=k) = p^k (1-p)^{n-k} \cdot \binom{n}{k}$$

binomial coefficient.

for $p = 1/2$:

$$= \frac{1}{2}^n \binom{n}{k}$$

What are $\mathbb{E}(X)$ and $\mathbb{V}(X)$? Can calculate by "brute force" (see exercise/solution 3) or we realize

$$X = X_1 + X_2 + \dots X_n$$

first trial, nth trial

and all X_j are <u>independent</u>.

And X_j is Bernoulli distributed.

So:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n) = p + \dots + p$$

$$= n \cdot p = \mu$$

$$\mathbb{V}(X) = \mathbb{V}(X_1 + \dots + X_n) = \mathbb{V}(X_1) + \dots + \mathbb{V}(X_n) = p(1-p) + \dots + p(1-p)$$

because X_j all independent.

$$= n\, p(1-p) = \sigma^2$$

So $\sigma = \sqrt{\mathbb{V}(X)} = \sqrt{n\, p(1-p)}$

<u>Relative width:</u>

$$\frac{\sigma}{\mu} = \frac{1}{\sqrt{n}} \frac{\sqrt{p(1-p)}}{p} \quad \text{for } p = 1/2: \ \frac{1}{\sqrt{n}}$$

$\rightarrow$ decreases with n! (see slides...)

In here $X = X_1 + X_2 + X_3 \dots + X_n$

$\uparrow$
$=1$ for win in 1st trial
(ex: coin #1 in heads)
$=0$ for fail in 1st trial → direct as full-set solution in follow exercise

$E(X) = E(X_1 + \dots X_n)$
$= E(X_1) + \dots E(X_n)$ → Bernoulli $\sum 1p + 0(1-p)$
$= p$

$\underline{E(X) = n \cdot p = \mu}$

$V(X) = V(X_1 + \dots X_n)$ } independent $X_1, X_2 \dots$
$= V(X_1) + \dots V(X_n)$

$\underline{= n \cdot p(1-p)}$ → Bernoulli $V = p(1-p)$
$= p - p^2$

$\sigma = \sqrt{np(1-p)}$

$$\frac{\sigma}{\mu} = \frac{\sqrt{np(1-p)}}{n \cdot p} = \frac{1}{\sqrt{n}} \frac{\sqrt{p(1-p)}}{p} \quad \text{for } p = \tfrac{1}{2}: \ \frac{1}{\sqrt{n}}$$

→ $\boxed{\text{slides}}$

3.4 ~~CDF? $P(k \geq 10)$?~~

Can we get an approximate/smooth fit for $P(X=k)$? to do derivatives, integrals, ...

Use Stirling's / de Moivre's approximation

$n! \approx n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}$ (correct in the limit $n \to \infty$)

$$P(k) \approx \frac{1}{\sqrt{2\pi np(1-p)}} e^{-\frac{(k-np)^2}{2np(1-p)}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} = \mathcal{N}(\mu, \sigma^2)$$

"Gaussian"/"Normal" probability distribution → more general! $\rightarrow \int$? $\rightarrow x < 0$?

3.4 FROM BINOMIAL TO GAUSSIAN/NORMAL

Can we get an appropriate smooth/continuous function for $P(X=k)$?

Use "Stirling's / deMoivre's" approximation

$$n! \simeq \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}$$ (correct in $n \to \infty$ limit)

then we get:

$$P(X=k) = \frac{1}{\sqrt{2\pi np(1-p)}}\, e^{-\frac{(k-np)^2}{2np(1-p)}}$$

$$= \boxed{\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(k-\mu)^2}{2\sigma^2}} = \mathcal{N}(\mu, \sigma^2)}$$

the "normal" or "Gaussian" distribution

→ Appears in many contexts

3.5 THE "STANDARD NORMAL" Distribution

Also known as the "scaled and centred" or "reduced and centered"

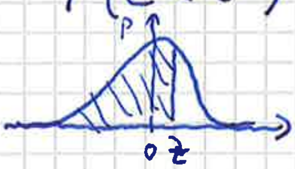
→ we shift and scale all values as: $\boxed{z = \frac{X-\mu}{\sigma}}$ (dimensionless!)

So z means: how many standard deviations away from the mean is a value?

$$\underset{Z=z}{P()} = \mathcal{N}(\underset{\text{mean}}{0}, \underset{\text{std.}}{1}) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$$

Cumulative distribution fct:

$$P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, e^{-z^2}\, dz = \Phi(z)$$ ← found in a "z-table"



$$= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right)$$

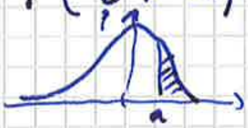
← "error function"

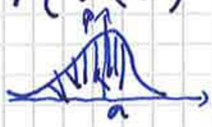
We have:

$$P(X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) = \Phi(z)$$

$$P(X \leq \mu) = \Phi(0) = 0.5$$

$$P(z > a) = 1 - P(z \leq a) = P(z \leq -a)$$





by symmetry of the Gaussian.

$$P(a \leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$