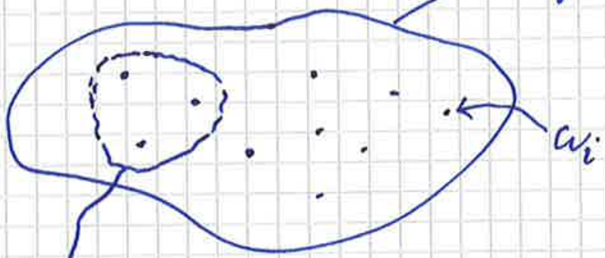
MSE-213 L1 Basics of Probability

Book reference: Ross Ch. 2.1–2.5

1.1 Set Theory – definitions

Sample Space $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$

Element / outcome



size $|\Omega| = N$

Event / Subset
e.g. $A = \{\omega_1, \omega_2, \omega_3\}$

$\boxed{\text{EX}}$ Sample Spaces:

discrete:
- $\Omega = \{heads, tails\}$ (1 coin toss)
- $\Omega = \{hh, ht, th, tt\}$ (2 ordered coin tosses)
- $\Omega = \{2{*}h, 2{*}t, 1{*}h\ 1{*}t\}$ (2 unordered coin tosses)
- Ω = {⚀, ⚁, ⚂, ⚃, ⚄, ⚅} (die throw)
- Ω = {Emilie, Jean, Sam} (pick a person)

continuous: $\Omega = \{t : 0 \le t \le \infty\}$ (decay time of a nucleus)

Elementary events: {heads}, {hh}

Events:

E = {⚀, ⚂, ⚄} "odd" die result

$A = \{hh, ht\}$ "first one heads" coin result.

$B = \{hh, tt\}$ "two of the same"

$C = \{tt\}$ "two tails" (elementary)

1.2. Operations on Sets

- "All elements except A" / "Not-A" : $\bar{A}$ (also: A^c)

<u>EX</u>: $\bar{A} = \{tl, tt\}$

- Union / OR : $A \cup B$

(Venn diagram)

<u>EX</u>: $A \cup B = \{ll, lt, tt\}$

- Intersection / AND : $A \cap B$

<u>EX</u>: $A \cap B = \{ll\}$, $A \cap C = \{\} = \emptyset$ "null event"

A and C "mutually exclusive"

- A without B : $A \setminus B$

<u>EX</u>: $\{lt\}$

- C is contained in B : $C \subset B$ if $B \cap C = C$

<u>EX</u>: $B \cap C = \{tt\} = C$

Note: $A \cap \bar{A} = \{\}$ always

1.3. Axioms of probability

1.3.1 $0 \leq P(A) \leq 1$

1.3.2 $P(\Omega) = 1$

1.3.3 if $A \cap B = \{\}$ then $P(A \cup B) = P(A) + P(B)$

and if $A \cap B = A \cap C = B \cap C = \{\}$ then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

etc...

1.4. Laplace Probabilities

Generally: independent experiments.

Laplace: All elementary events have equal probability.

$$P(\{\omega_i\}) = P(\{\omega_j\}) \text{ for all } i,j$$

as $$\sum_{i=1}^{N} P(\{\omega_i\}) = P(\Omega) \Rightarrow P(\{\omega_i\}) = \frac{1}{|\Omega|} = \frac{1}{N}$$

EX: $P(\{\text{heads}\}) = \frac{1}{2}$

$$\Rightarrow P(A) = \frac{|A|}{|\Omega|} = \frac{(\text{no of elements in } A)}{(\text{total elements})}$$

EX: $P(\text{"odd"}) = \frac{3}{6} = \frac{1}{2}$

1.5. General Results

- $P(\bar{A}) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

don't count 2x

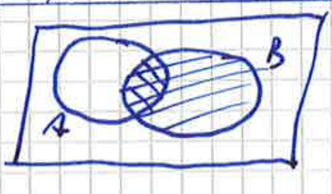
EX: $P(\text{"odd"} \cup \text{"bigger than 4"})$

$$= P(\{1,3,5,6\}) = P(\{1,3,5\}) + P(\{5,6\}) - P(\{5\})$$

$$\frac{4}{6} = \frac{3}{6} + \frac{2}{6} - \frac{1}{6}$$

1.6. Conditional Probabilities / Independent Events

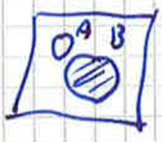
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



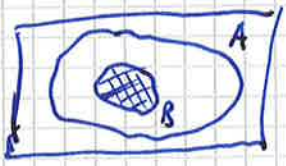
Laplace: $P(A|B) = \frac{|A \cap B|}{|B|}$

"P of A if B is the case"

$P(A|B) = 0$ if $A \cap B = \{\}$



$P(A|B) = 1$ if $B \subset A$ (i.e. $A \cap B = B$)



if $P(A|B) = P(A)$, <u>$P(A)$ is independent of $P(B)$</u>

$$\Leftrightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$$

<u>EX</u>: $P(\text{"odd"} \cap \text{">4"}) = P(\{5\}) = \frac{1}{6}$

$= P(\{1,3,5\}) \cdot P(\{5,6\}) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$\rightarrow$ independent

$P(\text{"odd"} \cap \text{">3"}) = P(\{3,5\}) = \frac{2}{6} = \frac{1}{3}$

$\neq P(\{1,3,5\}) \cdot P(\{4,5,6\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\rightarrow$ dependent