Exercise Set 4

Give your results with 2 significant digits precision e.g. 0.95 or 0.15%, or as a fraction e.g. 1/3

1 Nanoparticles with a normal/Gaussian distribution [normal / essential]

Your lab has developed two methods to create gold nanoparticles (in solution). They absorb light in a way that depends on their size, so good control over the size is important. Method A produces nanoparticles with an average size of 400nm and a standard deviation of 25nm. Method B produces nanoparticles with an average size of 440nm and a standard deviation of 40nm. In both cases, the distributions are normal/Gaussian.

a) For each method, within which interval (centered on the mean), would you expect to find 90% of the particles?

b) For a specific application, it is very important to have as few as possible nanoparticles which have a size of 350nm or smaller. Which method is better suited?

c) In another application, it is very important to have as few as possible nanoparticles which have a size of 300nm or smaller. Which method is better suited?

d) You produce 1 million nanoparticles with each method, and then measure the *mean size* of the particles in each sample. Within which interval would you expect the *measured mean* to lie with 90% probability.

e) You have created an experimental new method to produce nanoparticles, method C. You attempt 10 production runs with this method, each containing 10'000 nanoparticles. In each production run you measure the mean nanoparticle size. The mean of the measured means over all 10 production runs is 390nm, and the standard deviation over all 10 production runs is 0.1nm.

   - Which mean and standard deviation (for individual nanoparticles) do you expect for nanoparticles produced with this method?
   - Would it be better suited for the applications in b and c than the methods A or B? You can assume that method C also produces particles with a normal/Gaussian distribution.

2 The influence of replicates on the precision of a measure: the average chemical composition of the interstellar medium [basic-normal, essential]

According to the Big Bang theory, the universe should be composed of about 75% of hydrogen. Two competing Big Bang models predict two slightly different compositions, the first one 74% and the second one 77%.

You can do a measurement which gives you a value for the hydrogen content. This measurement has a measurement error of $\sigma = 1\%$, i.e. you expect measurement outcomes to be described by a probability density distribution with this standard deviation around the true mean.

a) Let X be this random experiment modelled by a normal law. If the universe follows theory A, the measurement results should be distributed as $\mathcal{N}(74, 1)$. Within that model, what is the probability of measuring 77% or higher?

b) Inversely, if nature is following $\mathcal{N}(77, 1)$ what is the probability of measuring 74% or lower?

c) Compute for both cases, the corresponding 99.99% prediction confidence intervals for the measurement outcome.

d) If the law is $\mathcal{N}(74, 1)$ for one single measurement, what is the law for an average of ten measurements, $\frac{1}{10}\sum_{i=1}^{10} X_i$?

e) Let us assume we measure one single measurement at 75%. For which range of means would this result be in the 99.99% prediction confidence interval? Are 77% or 74% included?

   Let us assume that is was not the result of one single measurement, but the average of ten repeated ones.. How does the interval reduce? How about for 1000 averaged measurements?

f) Let us assume the average over n measurements results in 75%. How many replicates (n) would be sufficient so that 77% would just lie outside the interval, thus supporting the 74% theory?

3 Poisson's law and supernova explosions [normal-advanced / recommended]

Supernovae are big dying stars that emit vast bursts of radiation during their explosion. These are the most energetic cosmic events we know. Sometimes, these explosions are so intense that they become visible by eye even during daylight for periods of approximately one month. Such ultra-bright supernova events are extremely rare, yet so striking that even in long-past history these events have been recorded by multiple civilizations around the globe at that time. The two last well-documented events happened around year 1000 and in 1604. A way to model such extremely rare events is though the Poisson law, defined as follows:

Let X be a discrete random variable that can take any non-negative integer value $k = \{0, 1, 2, ...\}$. The probability distribution given by $P(X = k) = C\frac{\lambda^k}{k!}$ where C and λ are constant numbers with $C, \lambda > 0$.

Hint: You may find it useful to remember that the Maclaurin series (a.k.a. Taylor series starting from $x = 0$ *of the exponential function is* $e^x = \sum_0^\infty \frac{x^n}{n!}$

a) Calculate the value of the constant C such that $P(X)$ is a well-defined probability distribution.

b) Compute the mean of X.

c) In the case of supernova, k is the number of supernovae observed in a century. Our sparse data suggests on average one such event in four centuries. Based on b), calculate λ to match this observation.

d) Assume that a human lives one century. Based on that model, calculate the probability of:

   - exactly two supernovae during your lifetime.
   - at least two events.
   - at least one event.
   - no supernova at all?

4 Aircraft probability of failure [normal-advanced / optional]

Each time you take a commercial flight, each engine has a probability $1/10^6$ to fail. A plane with two engines can still fly if it has one engine left. A four-engine-plane must have at least one engine *on each side* working (i.e. two engines may break, but only if they are on different sides).

a) Which kind of plane is safer (only considering this aspect)?

There are about $24'000$ planes flying every day, 50% of them are bi-engine, the others have four engines.

b) How many planes fail (i.e. cannot fly any more) due to engine issues per year?

c) What sort of distribution does this correspond to? What is the standard deviation?

Justify the applications of the theorem you use.