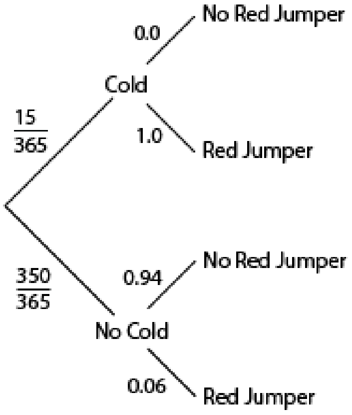
Exercise Set 2 - Solution

1 Refresher on probabilities

a)



b) P(Jumper) = P(Cold) × P(Jumper|Cold) + P(No Cold) × P(Jumper|No Cold) = $15/365 \times 1.0 + 0.06 \times (1 - 15/365) = 9.9\%$

c) 0 (since Mr Schmitt always wears a red jumper when he has a cold).

d)

$$\text{P(Cold} \mid \text{Red Jumper)} = \frac{15/365 \times 1.0}{15/365 + 0.06 \times (1 - 15/365)} = 42\%$$

e) They are not independent. You can prove this either via $P(\text{Jumper}) \times P(\text{Cold}) \neq P(\text{Jumper} \cap \text{Cold})$ or via $P(\text{Jumper}|\text{Cold}) \neq P(\text{Jumper})$.

2 Bonus round

a) The die can show any number between 1 and 6. The minimal gain thus is 100CHF and the maximum is $6^2 * 100\,\text{CHF} = \mathbf{3600\,CHF}$.

b) In the following, we decide to define our random variable X such that "i eyes on the die" maps to the numerical value i^2. We could also have defined X such that it gives the number of eyes itself, and then compute the expectation value of X^2.

The expectation value is computed as

$$\mathbb{E}[X] = \sum_{i=1}^{6} X(\omega_i)P(\omega_i) = \sum_{i=1}^{6} x_i p_i = \sum_{i=1}^{6} 100 i^2 \frac{1}{6} = \frac{100}{6} \sum_{i=1}^{6} i^2$$

This can be directly computed, or the Gauss rule can be used: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

This leads to $\mathbb{E}[X] = \frac{100}{6} * 91 \approx \mathbf{1517\,CHF}$.

It ends up below half of the maximum (which would be 1800CHF), because the slow increase of x^2 biases the expectation value towards lower numbers.

c) The average of the square of two numbers is not the same as the square of the average. For example, already for the numbers 1 and 2, the average would be 1.5 and the average suqared is $[(1+2)/2]^2 = 2.25$. On the other hand, the average of the squares is $(1^2 + 2^2)/2 = 5/2 = 2.5$.

So we cannot simply calculate the mean number of eyes (which would be 3.5) and then square the result.

In general, when you have a *nonlinear* function $f(X)$ then $\mathbb{E}[f(X)]$ is usually equal $f(\mathbb{E}[X])$ (but it can happen to be equal sometimes, and will be equal if all the possible outcomes of X are the same ("degenerate distribution")

3 The probability distribution of dice

a) If the die is fair, you can obtain each number with a probability $\frac{1}{6}$. The mean, the variance and the standard deviation are:

$$\begin{aligned}
\mu = \mathbb{E}(X) &= \sum_{n=1}^{6} P(X=n)\cdot n = \sum_{n=1}^{6} \frac{n}{6} = 3.5 \\
\sigma^2 = \mathbb{V}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sum_{n=1}^{6} \frac{n^2}{6} - \mathbb{E}(X)^2 = 2.92 \\
\sigma &= \sqrt{\mathbb{V}(X)} = 1.71
\end{aligned}$$

b) Unlike X_1 and X_2 the probability distribution for each die, S isn't uniform, as many dice combinations give 8, but only one gives 2.

$S = n$	2	3	4	5	6	7	8	9	10	11	12
Probability $P(S = n)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
CDF $P(S \leq n)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$

The mean, the variance and the standard deviation are:

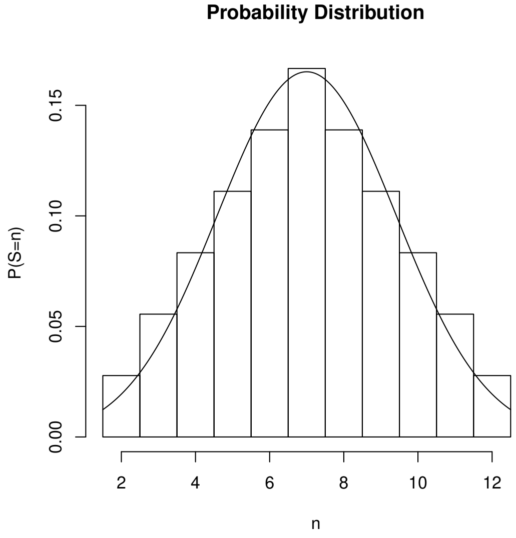
$$\begin{aligned}
\mu_s = \mathbb{E}(S) &= \sum_{n=2}^{12} P(S=n)\cdot n = 7 = \mathbb{E}(X_1) + \mathbb{E}(X_2) \\
\sigma_s^2 = \mathbb{V}(S) &= \mathbb{E}(S^2) - \mathbb{E}(S)^2 = \sum_{n=2}^{12} P(S=n)\cdot n^2 - \mathbb{E}(S)^2 = 5.83 \\
\sigma_s &= \sqrt{\mathbb{V}(S)} = 2.42
\end{aligned}$$

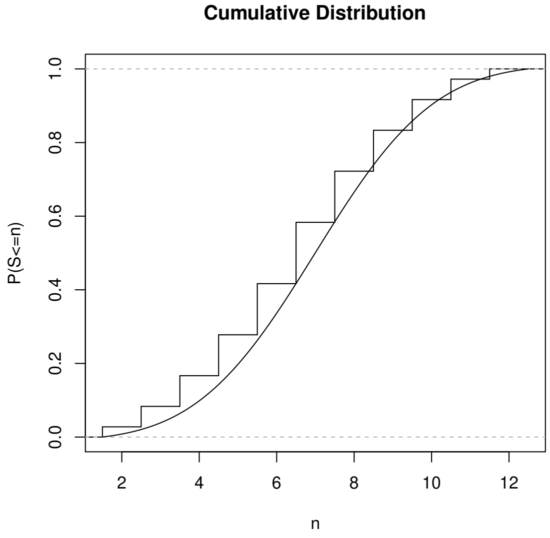
c) See the graphs below. The smooth lines on the graphs represent the Gaussian distribution which we will learn about later.

d) To decide if the game is fair ($\mathbb{E}(G) = 0$), you have to compute the expected gain. Let G be the random variable, which indicate your gain or loss (+2 or −1 CHF).

$$\mathbb{E}(G) = 2\cdot P(S \geq 10) - 1\cdot P(S < 10) = 2\cdot(1 - P(S \leq 9)) - P(S \leq 9) = -0.5\,\text{CHF}$$

So, in average, you will lose money with this game.





4 Bernoulli distribution and coin game

a)

$$P(10\text{ "heads"}) = \left(\frac{1}{2}\right)^{10} = 0.098\%$$

$$P(10\text{ "heads" or }10\text{ "tails"}) = P(10\text{ "heads"}) + P(10\text{ "tails"}) = \left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} = 0.195\%$$

$$P(5\text{ "heads" and }5\text{ "tails"}) = \binom{10}{5} \cdot \left(\frac{1}{2}\right)^{5} \cdot \left(\frac{1}{2}\right)^{5} = \frac{10!}{5! \cdot 5!} \cdot \left(\frac{1}{2}\right)^{10} = 24.6\%$$

b) We use the Bernoulli formula, with $n = 10$ the number of throws, k the number of "heads", $p = 0.5$ the probability a single throw gives "heads" and $1 - p$ (sometimes denoted q) the probability it gives "tails".

1)

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad\qquad P(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$$

2) The expectation value is the sum of the possible values for X, weighted by $P(X = n)$.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^{n} k \cdot P(X = k) = \sum_{k=0}^{n} k \cdot \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= np \cdot \sum_{k=0}^{n} k \cdot \frac{(n-1)!}{(n-k)!k!} p^{k-1} (1-p)^{n-k} \\
&= np \cdot \sum_{k=1}^{n} \frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \cdot \sum_{k'=0}^{n-1} \binom{n-1}{k'} p^{k'} (1-p)^{(n-1)-k'} \\
&= np \cdot \sum_{k'=0}^{n'} \binom{n'}{k'} p^{k'} (1-p)^{n'-k'} \\
&= np \cdot P(\tilde{X} \leq n') = np = 5
\end{aligned}$$

The **standard deviation** can be found using the following:

$$\begin{aligned}
\sigma &= \sqrt{\mathbb{V}(X)} \\
\mathbb{V}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
\mathbb{E}(X^2) &= \sum_{k=0}^{n} k^2 \cdot P(X = k) = \sum_{k=0}^{n} k^2 \cdot \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= np \cdot \sum_{k=1}^{n} k \cdot \frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \cdot \sum_{k'=0}^{n'} (1+k') \cdot \binom{n'}{k'} p^{k'} (1-p)^{n'-k'} \\
&= np + np \cdot n'p \cdot \sum_{k'=1}^{n'} \frac{(n'-1)!}{((n'-1)-(k'-1))!(k'-1)!} p^{k'-1} (1-p)^{(n'-1)-(k'-1)} \\
&= np + n(n-1)p^2 \cdot \sum_{k''=0}^{n''} \binom{n''}{k''} p^{k''} (1-p)^{n''-k''} \\
&= np + n(n-1)p^2 \\
\\
\mathbb{V}(X) &= np + n(n-1)p^2 - (np)^2 = np - np^2 \\
\\
\sigma &= \sqrt{np(1-p)} = 1.58
\end{aligned}$$

3) Indeed, σ is zero for $p = 0$ and $p = 1$. The standard deviation measures a degree of variability or randomness. If it is zero, there is no variability and the results are predetermined and identical. Indeed, this is the case. A coin with $p = 0$ will always show "tails" and a coin with $p = 1$ always "heads". Only if $0 < p < 1$, this is a random process.

4) If $p = \frac{1}{6}$, "heads" will intuitively appear every 6 throws, on average. The proof is a bit more advanced. It is obtained with the expectancy calculus shown below. The random variable Y is the number of throws until the first "heads". We use $q = (1 - p)$. To obtain $Y = k$, one must fail $k - 1$

times and succeed once.

$$\begin{aligned}
P(Y=k) &= q^{k-1}p = \left(\frac{5}{6}\right)^{k-1}\frac{1}{6} \\
\mathbb{E}(Y) &= \sum_{k=1}^{\infty} P(Y=k)\cdot k \\
&= p\sum_{k=1}^{\infty} kq^{k-1} = p\sum_{k=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}q}q^k = p\cdot\frac{\mathrm{d}}{\mathrm{d}q}\sum_{k=1}^{\infty}q^k \\
&= p\cdot\frac{\mathrm{d}}{\mathrm{d}q}\frac{1}{1-q} = p\frac{1}{(1-q)^2} \\
&= \frac{1}{6}\frac{36}{(6-5)^2} = 6
\end{aligned}$$