Homework 1

Exercise 1

In thermodynamics a function of the form of $y = f(x) = x \ln x$ is used extensively. Calculate the first and second derivatives of the function.

Exercise 2

You are given the function $y = f(x) = 1/6x^3 - x^2 - 6x + 2$.

(a) Calculate the local minimum and maximum.

(b) Calculate the inflection point.

Exercise 3

In this exercise, we introduce some standard notation used in thermodynamics for concepts that you already know: In Analysis II, we wrote vector fields like

$$f(x, y) = \begin{pmatrix} x^2 + 3y \\ x + 5y \end{pmatrix}. \tag{1}$$

In thermodynamics, we write this vector field as

$$\delta f = (x^2 + 3y)\mathrm{d}x + (x + 5y)\mathrm{d}y \tag{2}$$

and call this object a **differential form**. The symbol δ in front of δf indicates that this is a differential form.
If we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ like $f(x, y) = 3x^2 + 4xy - 3y^2$, we can generate a vector field by taking the gradient

$$\nabla f(x, y) = \begin{pmatrix} 6x + 4y \\ 4x - 6y \end{pmatrix}. \tag{3}$$

In thermodynamics, we say that we compute the **differential of** f, also called the **total derivative** of f, written as

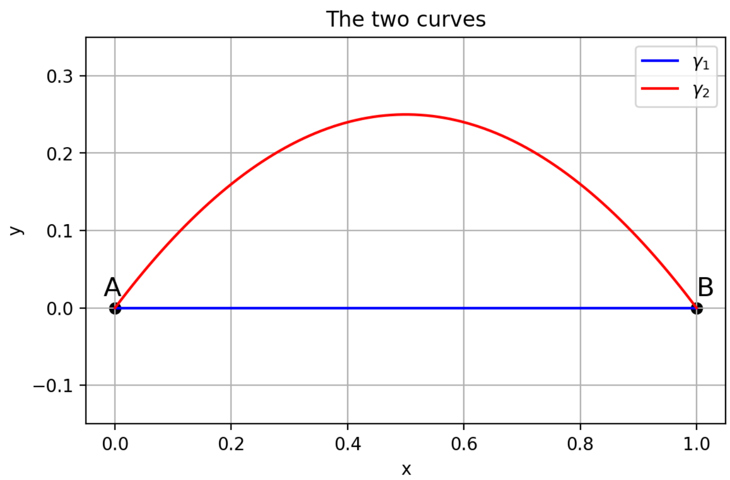
$$\mathrm{d}f = (6x + 4y)\mathrm{d}x + (4x - 6y)\mathrm{d}y. \tag{4}$$

We can compute the integral of a differential form along a curve $\gamma : I \to \mathbb{R}^n$ in the same way as for vector fields along curves since we are only changing the notation. Answer the following questions:

(a) Let $f(x, y) = x^3 - y$ and consider the integral of its total derivative along two different curves

$$I_1 = \int_{\gamma_1} \mathrm{d}f, \qquad I_2 = \int_{\gamma_2} \mathrm{d}f, \tag{5}$$

where $\gamma_1 : [0, 1] \to \mathbb{R}^2, \gamma_1(s) = (s, 0)$ is the straight line segment from $A = (0, 0)$ to $B = (1, 0)$ and $\gamma_2 : [0, 1] \to \mathbb{R}^2, \gamma_2(s) = (s, s - s^2)$ is a parabolic curve with the **same endpoints** (see figure).
Try to answer this question without doing any calculations: Is $I_1 = I_2$?



(b) Can you explain why? (you will need a theorem about line integrals)

(c) Let $\delta g = xy\mathrm{d}x + y^2\mathrm{d}y$ be a differential form that can not be written as a total derivative of a function and consider the two integrals

$$I_3 = \int_{\gamma_1} \delta g, \qquad I_4 = \int_{\gamma_2} \delta g. \tag{6}$$

Again, without doing any calculations: What can you say about the relation between I_3 and I_4?

Exercise 4

The ideal gas law states $pV = nRT$.

(a) Calculate the following partial derivatives:

$$\left(\frac{\partial p}{\partial T}\right)_V, \left(\frac{\partial T}{\partial V}\right)_p, \left(\frac{\partial V}{\partial p}\right)_T$$

(b) What does the following expression equal to?

$$\left(\frac{\partial p}{\partial T}\right)_V \cdot \left(\frac{\partial T}{\partial V}\right)_p \cdot \left(\frac{\partial V}{\partial p}\right)_T =?$$

(c) Calculate the total derivative of the pressure as a function of T and V.

$$\mathrm{d}p =?$$