Biomicroscopy II: Fourier optics Toolbox – lecture notes

Laboratoire d'optique biomédicale, EPFL – Arno Bouwens

October 24, 2011

1 The wave nature of light

To investigate the behavior of optical systems, our approach so far has been to consider light to consist of **rays**. Although this picture is very useful and intuitive, we need to improve it a little further to understand why a microscope has a limited resolution.

1.1 The wave equation

Any electromagnetic effect, such as light, obeys Maxwell's equations. In their simplest form, these equations reduce to

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} \tag{1}$$

where v is the speed of light and $E(z,t)$ represents the electric field at some point z and at time t. The simplest propagating wave that satisfies this equation[1] is the harmonic wave

$$E(x,t) = A\cos(kz - \omega t) \tag{2}$$

The argument $\omega t - kz$ is called the *phase*. Let's first examine why (2) can be called a *propagating* wave. When the argument of the cosine function is held constant, (2) is of course nothing more than a static cosine function. But what does it mean to hold $kz - \omega t$ constant?

$$kz - \omega t = B \quad \Rightarrow \quad z = \frac{\omega}{k}t + B \tag{3}$$

Clearly, to hold the argument constant, z must increase with time (with speed $v = \frac{\omega}{k})^2$. In other words, the cosine function moves through space with a speed $v = \frac{\omega}{k}$. ω is called the pulsation, it is related to the frequency f of light: $\omega = 2\pi f$. k is the wave-number, which is related to the wavelength λ through $k = 2\pi/\lambda$.

1.2 Which wave representation?

There is a problem when using a cosine to represent a wave such as light: we have to remember and apply tons of trigoniometric formulas… We need a smarter approach. The real part of

$$E(z,t) = Ae^{i(\omega t - kz)}, \tag{4}$$

is exactly equation (2)! Using this complex notation has the advantage that we won't need trigoniometric formulas. We just need to remember that the physical electric field corresponds to the real part. Figure 1 shows a graphical representation of equation (4) at a fixed position is space $z = 0$: the *phasor*. The intensity of a light wave can be calculated from $I = |E|^2$.

[1] You can easily check that (2) satisfies (1) by substituting it in the wave equation.

[2] This is the same v as in equation (1). Again this can be seen by substituting (2) in (1).

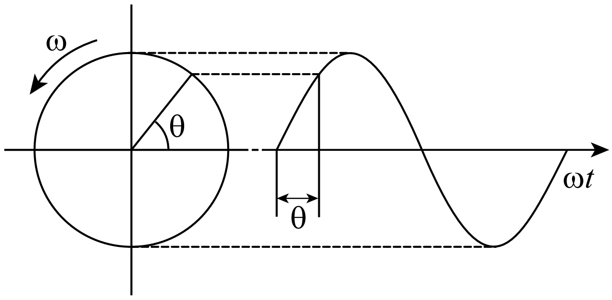

Figure 1: A light wave can be represented by a *phasor*. The length of the arrow represents the amplitude of the wave, the angle θ represents the phase. In our case, $\theta = \omega t$, the arrow circles around with time.

1.3 Three dimensional waves

Up until now, we've considered one-dimensional waves. What about three-dimensional waves? First, we need to ask ourselves how we are going to picture three-dimensional waves. A very helpful concept to do this is the *wavefront*. A wavefront is defined as a surface of constant phase. This is nothing more than the 3-D extension of what we did in equation (3). Let us look at two fundamental types of 3-D waves: the plane wave and the spherical wave.

1.3.1 The plane wave

For plane waves, the wavefronts are obviously (infinitely large) planes, as illustrated in figure 2. The planes are perpendicular to the wave vector $\vec{k}$. Let us take $\vec{k} = k\vec{1}_z$ along the z-direction. The plane wave can now be written as

$$E(x, y, z, t) = Ae^{i(kz-\omega t)}. \tag{5}$$

Note that in general, the plane wave can be written as $E(x, y, z, t) = Ae^{i(\vec{k}.\vec{r}-\omega t)}$, with $\vec{r} = (x, y, z)$.

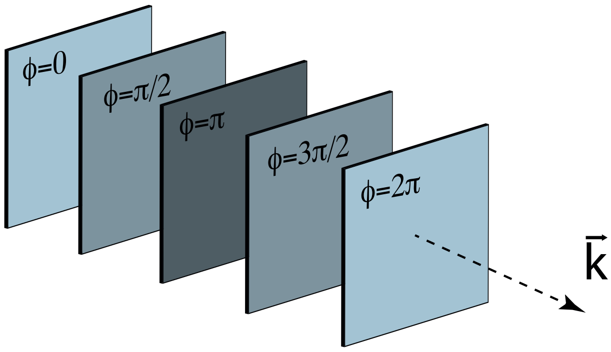

Figure 2: Wavefronts of a plane wave.

1.3.2 The spherical wave

The second fundamental type of wave we will consider is the spherical wave. Here, the wavefronts are spheres with a common center. They can be expressed mathematically using

$$E(r, t) = \frac{A}{r}e^{i(kr-\omega t)}, \tag{6}$$

where r is the distance to the center. Can you find a physical reason for the presence of the factor $1/r$?

2 The relation between waves and rays

We now have two models for light: one where light is represented by wave fronts and one where light is said to consist of rays. How do wavefronts interact with optical elements such as a lens?

2.1 Wavefronts and a lens

Imagine a plane wave incident on a positive lens, as in figure 3. Since the lens is positive, we know that the center is thicker than the edges of the lens. Also, propagation in glass delays light with respect to propagation in air. Therefore, the part of the plane wave on the optical axis will experience more delay (more glass) than the part of the plane wave that travels through the edge of the lens.

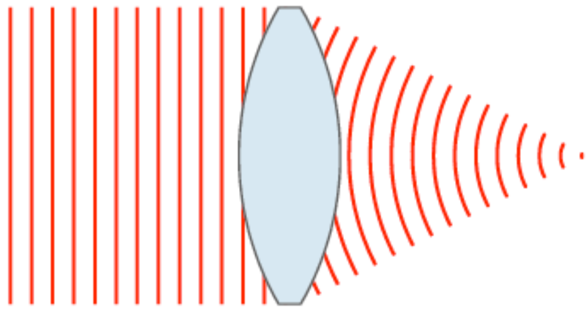

Figure 3: A plane wave incident on a positive lens.

Clearly, the positive lens transformed the plane wave in a spherical wave. By comparing this situation with the ray picture of a positive lens, we find that **rays are orthogonal lines to the wavefronts**. The plane wave therefore corresponds to parallel rays and the spherical wavefront to converging or diverging rays.

2.2 Wavefronts and ABCD matrix optics

How can the transformation of wavefronts by lenses be described by ABCD matrix optics? First, remember that an ABCD matrix M gives the relation between the height and angle of a ray at the input and at the output of an optical system.

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} y_2 = Ay_1 + B\theta_1 \\ \theta_2 = Cy_1 + D\theta_1 \end{array} \tag{7}$$

For a spherical wave with it's center on the optical axis, we find that it's radius can be found from $R_i = y_i/\theta_i$. [3]

$$R_2 = \frac{y_2}{\theta_2} = \frac{Ay_1 + B\theta_1}{Cy_1 + D\theta_1} = \frac{A\frac{y_1}{\theta_1} + B}{C\frac{y_1}{\theta_1} + D}, \tag{8}$$

or

$$R_2 = \frac{AR_1 + B}{CR_1 + D}. \tag{9}$$

Let us examine the consequences of (9) for a translation and a thin lens.

[3] For positive y, positive θ indicates a diverging ray, hence R will be positive for diverging waves and negative for converging rays.

a) Translation (propagation in air)

$$M = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad R_2 = R_1 + t \tag{10}$$

This makes a lot of sense (check for yourself using simple sketches of diverging or converging rays)!

b) Thin positive lens

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \qquad \Rightarrow \qquad R_2 = \frac{R_1}{1 - R_1/f} \qquad \Rightarrow \qquad \frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \tag{11}$$

This last expression confirms our intuition about the positive lens in figure 3. In the limit of $R_1 \to \infty$ (a plane wave), $R_2 = -f$ (a converging spherical wave).

3 Fourier analysis

There is one last - and very important - tool that needs to be introduced if we want to do some serious analysis of the wave nature of light: the Fourier transform.

3.1 The Fourier transform

The Fourier transform (FT) can tell us which frequencies are present in a given function, and with what strength to contribute to the function. In other words the FT (denoted $\mathcal{F}\{\bullet\}$) decomposes a function in harmonic functions (of the form $e^{i\omega t}$).

$$\mathcal{F}\{g(t)\} = \hat{G}(f) = \int_{-\infty}^{+\infty} g(t) e^{-i2\pi f t} dt \tag{12}$$

3.2 Some properties of the Fourier transform

a) linearity: $\mathcal{F}\{g(t) + h(t)\} = \hat{G}(f) + \hat{H}(f)$

b) translation: $\mathcal{F}\{g(t - t_0)\} = e^{-2\pi i t_0 f}\hat{G}(f)$

c) modulation: $\mathcal{F}\left\{g(t)e^{i2\pi\gamma t}\right\} = \hat{G}(f - \gamma)$

d) scaling: $\mathcal{F}\{g(at)\} = \frac{1}{|a|}\hat{G}\left(\frac{f}{a}\right)$

e) duality: if $\mathcal{F}\{\mathcal{F}\{g(t)\}\} = g(-t)$

f) convolution: $\mathcal{F}\{g(t) * h(t)\} = \hat{G}(f)\hat{H}(f)$

g) harmonic function: $\mathcal{F}\{\exp(i2\pi\gamma t)\} = \int_{-\infty}^{+\infty} e^{-i2\pi(f-\gamma)t}dt \equiv \delta(f - \gamma)$

If this is the first time you see some of these properties, try to prove them for yourself using equation (12). What are the FTs of $\cos(2\pi\gamma t)$ and $\sin(2\pi\gamma t)$.

3.3 Spatial frequencies

Probably you have used FTs before, in the case of time-varying signals. Here, we will apply FTs to spatially varying functions and therefore we will be investigating *spatial frequencies*. Consider some spatial function $U(x)$, we will denote the spatial frequency corresponding to x with p_x.

$$\mathcal{F}\{U(x)\} = \hat{U}(p_x) = \int_{-\infty}^{+\infty} U(x) e^{-i2\pi p_x x} dx \tag{13}$$

What do spatial frequencies look like in 2-D?

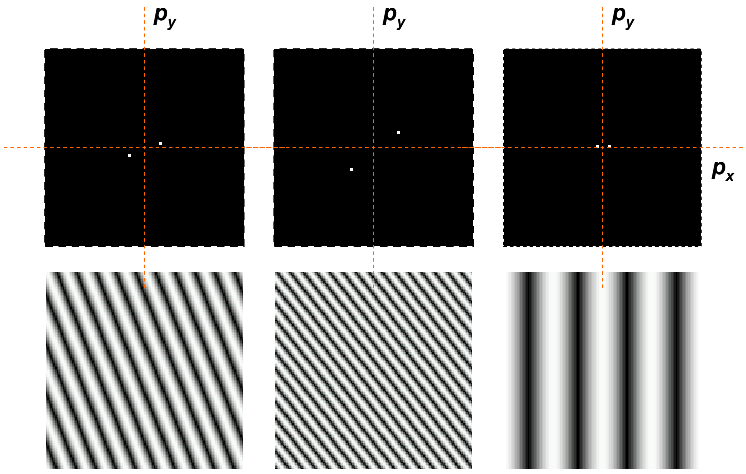

Figure 4: Spatial frequencies.

3.4 Some examples

See exercises.