Biomicroscopy I - Solutions Exercise Sheet 3

September 23, 2025

1 Thin lenses: imaging

Let us start by writing the transfer matrix $T(L_1L_2)$ from lens L_1 to L_2. We can decompose it into several submatrices:

- The thin lens L_1:

$$L_1 = \begin{bmatrix} 1 & 0 \\ -\Phi_1 & 1 \end{bmatrix} \text{ with } \Phi_1 = \frac{1}{f_1}$$

- The free space propagation T_1 in between:

$$T_1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

- And the thin lens L_2:

$$L_2 = \begin{bmatrix} 1 & 0 \\ -\Phi_2 & 1 \end{bmatrix} \text{ with } \Phi_2 = \frac{1}{f_2}$$

The transfer matrix $T(L_1L_2)$ is then:

$$T(L_1L_2) = L_2 \cdot T_1 \cdot L_1 = \begin{bmatrix} 1 & 0 \\ -\Phi_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\Phi_1 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 - d\Phi_1 & d \\ d\Phi_1\Phi_2 - \Phi_1 - \Phi_2 & 1 - d\Phi_2 \end{bmatrix}$$

We now have the transfer matrix $T(L_1L_2)$, which describes the system defined by the two lenses.

A. Let us now add free space propagation before and after the matrix derived above:

$$C = \begin{bmatrix} 1 & \tau_2 \\ 0 & 1 \end{bmatrix} \cdot T(L_1L_2) \cdot \begin{bmatrix} 1 & \tau_1 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 - d\Phi_1 + \tau_2(d\Phi_1\Phi_2 - \Phi_1 - \Phi_2) & \tau_1\left(1 - d\Phi_1 + \tau_2(d\Phi_1\Phi_2 - \Phi_1 - \Phi_2)\right) + d + \tau_2(1 - d\Phi_2) \\ d\Phi_1\Phi_2 - \Phi_1 - \Phi_2 & (\tau_1 + \tau_2)\left(d\Phi_1\Phi_2 - \Phi_1 - \Phi_2\right) + 1 + d\Phi_2 \end{bmatrix}$$

where $t_1 = \tau_1$ is the object position with respect to the first lens L_1 and $t_2 = \tau_2$ the image position with respect to the second lens L_2. Due to the conjugation between the image and object planes, we can impose the imaging condition to retrieve the position of the image:

$$C = \begin{bmatrix} -0.2\tau_2 & 5 + 2\tau_2 - 0.2\tau_1\tau_2 \\ -0.2 & 2 - 0.2\tau_1 \end{bmatrix}$$

$$C_{12} = 5 + 2\tau_2 - 0.2\tau_1\tau_2 = 0$$

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Using this expression, we can obtain the position of the image, t_2:

$$t_2 = \frac{-5}{2 - 0.2t_1} = 5\text{cm}$$

We can also retrieve the magnification, m, given by the coefficient C_{11}:

$$m = C_{11} = -0.2\tau_2 = -1$$

Similarly to the previous part, if we move the object 2.5cm towards L_1:

$$t_1 = 15\text{cm} - 2.5\text{cm} = 12.5\text{cm}$$
$$t_2 = \frac{-5}{2 - 0.2t_1} = 10\text{cm}$$
$$m = -0.2\tau_2 = -2$$

So, the image is located 10cm to the right of L_2. In this case, when the object is shifted 2.5cm towards the lens, the image is shifted by 5cm to the right of the previous image. Therefore, the axial magnification is $\frac{5}{2.5} = 2$

Similarly, when displacing 2.5cm away from the lens L_1:

$$t_1 = 15\text{cm} + 2.5\text{cm} = 17.5\text{cm}$$

$$t_2 = \frac{-5}{2 - 0.2t_1} = 3.33\text{cm}$$
$$m = -0.2\tau_2 = -0.67$$

The axial magnification in this case is $\frac{1.67}{2.5} = 0.67$.

2 Thin lenses: Magnification

A. Let us write the different matrices necessary to calculate the conjugate matrix of the system:

- Free space propagation Object-Lens:

$$T_0 = \begin{bmatrix} 1 & \tau_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} \text{ with } \tau_0 = d = 60 \text{ cm}$$

- Thin lens L :

$$R = \begin{bmatrix} 1 & 0 \\ -\phi & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.05 & 1 \end{bmatrix} \text{ with } \phi = \frac{1}{f} = 0.05 \text{ cm}^{-1}$$

- Free space propagation Lens-Image:

$$T_1 = \begin{bmatrix} 1 & \tau_1 \\ 0 & 1 \end{bmatrix} \text{ with } \tau_1 = x \text{ and } x \text{ is the position of the image after the lens}$$

The product of those matrices gives the conjugate matrix as

$$C = T_1 \cdot R \cdot T_0 = \begin{bmatrix} 1 & \tau_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\phi & 1 \end{bmatrix} \begin{bmatrix} 1 & \tau_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \tau_1\phi & \tau_1 \\ -\phi & 1 \end{bmatrix} \begin{bmatrix} 1 & \tau_0 \\ 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 - \tau_1\phi & \tau_0 (1 - \tau_1\phi) + \tau_1 \\ -\phi & -\phi\tau_0 + 1 \end{bmatrix}$$

Applying the imaging condition on the matrix element c_{12}, i.e. $c_{12} = 0$, we get

$$\tau_0 (1 - \tau_1\phi) + \tau_1 = 0$$

Which leads to the result

$$\tau_1 = x = \frac{\tau_0}{\tau_0\phi - 1} = \frac{60}{0.05 \cdot 60 - 1} = 30 \text{ cm}$$

B. The magnification m is given by the matrix element c_{11} as

$$m = c_{11} = 1 - \tau_1\phi = -0.5$$

Conversely, knowing the magnification m, the equation in B) and expression for τ_1 from A) we can solve it for d to retrieve the distance of the object to the lens to get required magnification.

$$\tau_1 = \frac{1-m}{\phi} = \frac{\tau_0}{\tau_0\phi - 1} \Rightarrow \tau_0 = d = \frac{m-1}{m} \cdot \frac{1}{\phi} = 30 \text{ cm.}$$

3 Two thin lenses

The ABCD of the optical system can be found as:

$$T(ES) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{bmatrix}$$
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0.5 & 2 \\ -0.125 & 1.5 \end{bmatrix}$$

By using the formulas for cardinal plane distances derived during the lecture:

4 Thick lens

We have $h_0 = 2$ cm, $d_0 = 8$ cm, $d_1 = 2.8$ cm, $r_1 = 2.4$ cm, $r_2 = -2.4$cm, $n_{\text{glass}} = 1.6, n_{\text{air}} = 1$.

Translation matrix: $T_j = \begin{bmatrix} 1 & d_j \\ 0 & 1 \end{bmatrix}$

Distances	Directed distance	with ABCD elements	Found Values
Focal point F_o	$\overline{EF_o}$	$\frac{D}{C}$	-12 cm
Focal length f_o	$\overline{H_oF_o}$	$\frac{1}{C}$	-8 cm
Principal plane H_o	$\overline{EH_o}$	$\frac{D-1}{C}$	-4 cm
Focal point F_i	$\overline{SF_i}$	$-\frac{A}{C}$	4 cm
Focal length f_i	$\overline{H_iF_i}$	$-\frac{1}{C}$	8 cm
Principal plane H_i	$\overline{SH_i}$	$\frac{1-A}{C}$	-4 cm

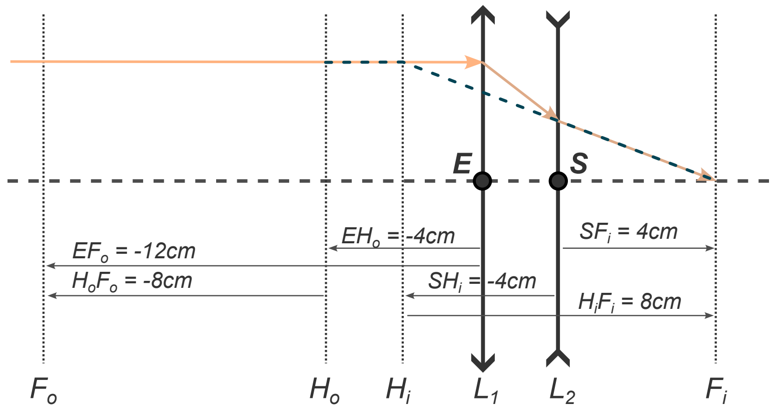

Figure 1: Cardinal and focal planes. The dashed line represents the auxiliary (helper)ss ray, while the yellow one is the actual light propagation.

Refraction matrix: $R_j = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2-n_1)}{n_2 r} & \frac{n_1}{n_2} \end{bmatrix}$

$$T_0 = \begin{bmatrix} 1 & d_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}, T_1 = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2.8 \\ 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1 & 0 \\ -\frac{n-1}{nr_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{0.25}{1.6} & \frac{1}{1.6} \end{bmatrix}, R_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1-n}{r_2} & n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.25 & 1.6 \end{bmatrix}$$

a) Conjugate Matrix:

$$C = T_2R_2T_1R_1T_0 = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.56 & 6.25 \\ -0.39 & -2.56 \end{bmatrix} = \begin{bmatrix} 0.56 - 0.39d_2 & 6.25 - 2.56d_2 \\ -0.39 & -2.56 \end{bmatrix}$$

$$c_{12} = 6.25 - 2.56d_2 = 0$$

$$d_2 = 2.44 \text{ cm}$$

b) Lateral magnification: We have $\begin{bmatrix} h_i \\ \theta_i \end{bmatrix} = C \begin{bmatrix} h_o \\ \theta_o \end{bmatrix}$ with $C = \begin{bmatrix} -0.39 & 0 \\ -0.39 & -2.56 \end{bmatrix}$

i.e., $h_i = c_{11}h_o$ which defines the lateral magnification $m = c_{11} = -0.39$. Thus the image height is 0.78 cm and the image is upside down. The indices o and i stand for object and image respectively.

c) Focal length: With ABCD calculation, we have

$$
\begin{aligned}
T(ES) &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} = R_2 T_1 R_1 \\
&= \begin{bmatrix} 1 & 0 \\ -0.25 & 1.6 \end{bmatrix} \begin{bmatrix} 1 & 2.8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{0.25}{1.6} & \frac{1}{1.6} \end{bmatrix} = \begin{bmatrix} 0.56 & 1.75 \\ -0.39 & 0.56 \end{bmatrix} \\
\overline{EF_o} &= \frac{D}{C} = \frac{0.56}{-0.39} = -1.44 \text{ cm} \\
\overline{SF_i} &= -\frac{A}{C} = \frac{0.56}{0.39} = 1.44 \text{ cm}
\end{aligned}
$$

The focal planes are 1.44 cm in front of the first lens surface, and 1.44 cm behind the last surface.

d) Principal Planes: As we found the ABCD matrix of the lens as:

$$T(ES) = \begin{bmatrix} 0.56 & 1.75 \\ -0.39 & 0.56 \end{bmatrix}$$

With the formulas for principal plane distances:

$\overline{EH_o} = \frac{(D-1)}{C} = 1.12$ cm (to the right of the first lens surface); $\overline{SH_i} = \frac{(1-A)}{C} = -1.12$ cm (to the left of the second lens surface).