Biomicroscopy I - Solutions Exercise Sheet 2

September 23, 2025

1 Cartesian oval

All optical path lengths from S to P must be equal. Therefore:

$$l_0 n_1 + l_i n_2 = s_0 n_1 + s_i n_2 = const$$

Drop a perpendicular from A to the optical axis, the point where it touches is B.

$$BP = s_0 + s_i - x$$

Apply Pythagorean Theorem 2 times:

$$l_o = (x^2 + y^2)^{1/2}$$

$$l_i = ((s_0 + s_i - x)^2 + y^2)^{1/2}$$

Substitute to the expression obtained above:

$$n_1(x^2 + y^2)^{1/2} + n_2(y^2 + (s_0 + s_i - x)^2)^{1/2} = const$$

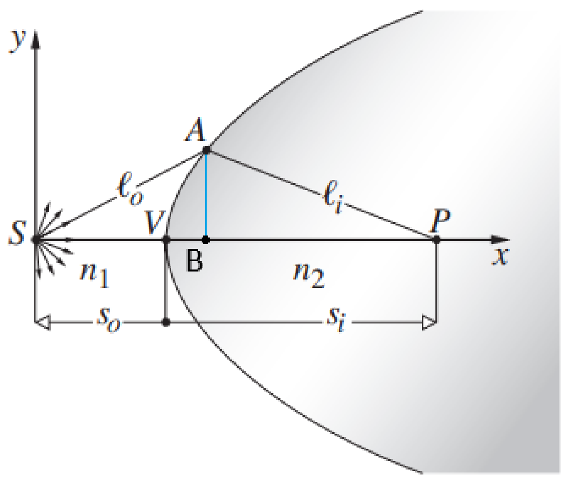
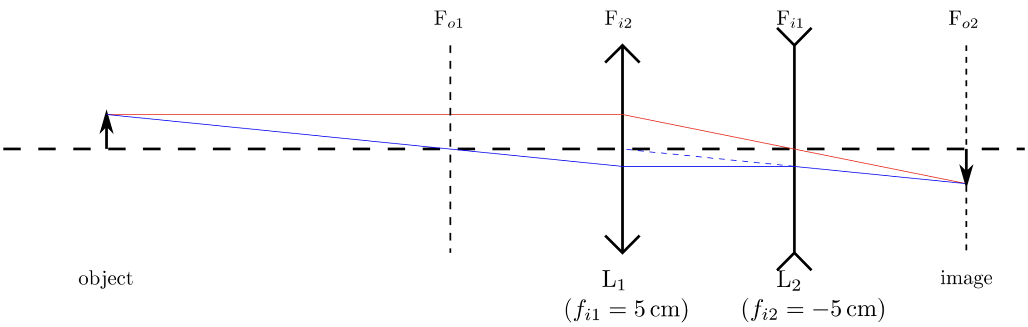


Figure 1: Cartesian oval

2 Thin lenses: Ray tracing

1. By intersecting two rays starting at the same point of the object, we get the corresponding image point. The easiest way in this case is to intersect a horizontal ray (red) and a ray passing through the first focal point (blue) starting from the object. The magnification is $m = -1$. The image is 5 cm to the right of the second lens (L_2).



2. By proceeding similarly, we find that the magnification $m = -2$. The image is located 10cm to the right of the second lens L_2.

3. By proceeding similarly, we find that the magnification $m = -\frac{2}{3}$. The image is located 3.3cm to the right of the second lens L_2.

3 Ray Matrix of a thin lens

When we model a thin lens with the ray-transfer (ABCD) matrix method, light passes through two curved refracting surfaces (Figure 2). Since the thickness of the lens is neglected, we do not include a propagation matrix inside the lens. We need to consider 2 interfaces with curvature of R_1 and R_2 respectively:

$$\begin{bmatrix} y_{out} \\ \theta_{out} \end{bmatrix} = M_{2^{nd}\ curved\ interface} \times M_{1^{st}\ curved\ interface} \begin{bmatrix} y_{in} \\ \theta_{in} \end{bmatrix}$$

Therefore:

$$M_{2^{nd}\ curved\ interface} \times M_{1^{st}\ curved\ interface} = M_{thin\ lens}$$

At the first spherical surface (from n_1 to n_2), the ray matrix is

$$M_{1^{st}\ curved\ interface} = \begin{bmatrix} 1 & 0 \\ \dfrac{n_1 - n_2}{n_2 R_1} & \dfrac{n_1}{n_2} \end{bmatrix}.$$

At the second spherical surface (from n_2 to n_1; for simplification we consider $n_3 = n_1$), the ray matrix is

$$M_{2^{nd}\ curved\ interface} = \begin{bmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_1 R_2} & \dfrac{n_2}{n_1} \end{bmatrix}.$$

The total ray matrix of the thin lens is

$$M_{thin\ lens} = \begin{pmatrix} 1 & 0 \\ \frac{(n_2 - n_1)}{n_1} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) & 1 \end{pmatrix}.$$

The focal length is given by the Lens-Maker's formula:

$$\frac{1}{f} = \frac{(n_2 - n_1)}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

The final expression for the ray matrix of the thin lens is:

$$M_{thin\ lens} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}.$$

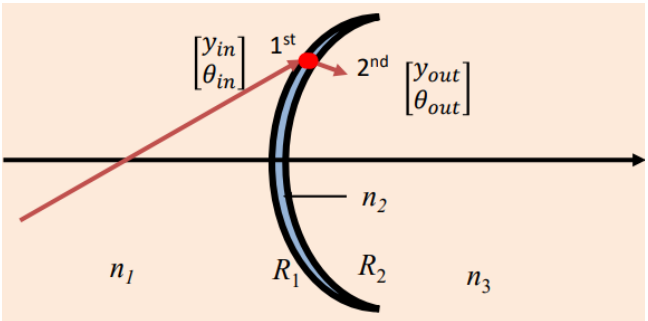


Figure 2: Light propagation trough a thin lens

4 Thin lenses: ABCD matrix

Let us start by writing the transfer matrix $T(L_1L_2)$ from lens L_1 to L_2. We can decompose it into several submatrices:

- The thin lens L_1:

$$L_1 = \begin{bmatrix} 1 & 0 \\ -\Phi_1 & 1 \end{bmatrix} \text{ with } \Phi_1 = \frac{1}{f_1}$$

- The free space propagation T_1 in between:

$$T_1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

- And the thin lens L_2:

$$L_2 = \begin{bmatrix} 1 & 0 \\ -\Phi_2 & 1 \end{bmatrix} \text{ with } \Phi_1 = \frac{1}{f_2}$$

The transfer matrix $T(L_1L_2)$ is then:

$$T(L_1L_2) = L_2 \cdot T_1 \cdot L_1 = \begin{bmatrix} 1 & 0 \\ -\Phi_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\Phi_1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 - d\Phi_1 & d \\ d\Phi_1\Phi_2 - \Phi_1 - \Phi_2 & 1 - d\Phi_2 \end{bmatrix}$$

We now have the transfer matrix $T(L_1L_2)$, which describes the system defined by the two lenses.

5 Consecutive this lenses

We need to prove that a system of two lenses, L_1 and L_2, without any separation ($d = 0$cm), is equivalent to having no lenses if and only if $f_1 = -f_2$. We first notice that using ABCD matrices we can express a lens-free system as the 2×2 identity matrix, $\mathbf{I}_2$.

Let us first prove that if two lenses L_1 and L_2 have focal distances f and $-f$ respectively, then $L_1 \cdot L_2 = \mathbf{I}_2$. By operating the product:

$$L_1 \cdot L_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} + \frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

We now need to prove the other implication: for two arbitrary lenses L_1 and L_2, $\mathbf{I}_2 = L_1 L_2$ only when $f_1 = -f_2$.

Let us start with two arbitrary lenses L_1 and L_2, with focal distances f_1 and f_2. First, we operate the product:

$$L_1 \cdot L_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} - \frac{1}{f_2} & 1 \end{bmatrix}$$

We now impose $L_1 \cdot L_2$ to be equal to $\mathbf{I}_2$:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} - \frac{1}{f_2} & 1 \end{bmatrix} = L_1 \cdot L_2$$

Therefore, we need $-\frac{1}{f_1} - \frac{1}{f_2}$ in order for $L_1 L_2$ to be the 2×2 identity matrix:

$$-\frac{1}{f_1} - \frac{1}{f_2} = 0 \iff -\frac{1}{f_1} = \frac{1}{f_2} \iff f_1 = -f_2$$

So, $L_1 L_2$ is the identity matrix only when $f_1 = -f_2$.

Having proved both implications of the statement, we conclude the proof.