

Biomicroscopy I - Solutions Exercise Sheet 9

November 11, 2025

1 Fourier transform with a lens

A. The mathematical representation of the input function $f(x)$ of the apertures in the object plane:

- i. $f_i(x) = \text{rect}\left(\frac{x}{a}\right)$
- ii. $f_{ii}(x) = f_1(x+b) + f_1(x-b) = \text{rect}\left(\frac{x+b}{a}\right) + \text{rect}\left(\frac{x-b}{a}\right)$
- iii. $f_3(x) = f_1(x) + f_1(x+b) + f_1(x-b) = \text{rect}\left(\frac{x}{a}\right) + \text{rect}\left(\frac{x+b}{a}\right) + \text{rect}\left(\frac{x-b}{a}\right)$

B. The image formed at Fourier plane is a Fourier transform of an entry object.

Reminder:

- The Fourier transform is a linear operator:

$$\mathcal{F}\{f(x) + g(x)\} = \mathcal{F}\{f(x)\} + \mathcal{F}\{g(x)\}$$

- The Fourier transform of the rectangular function rect is:

$$f(x) = \text{rect}\left(\frac{x}{a}\right) \xleftrightarrow{\mathcal{F}} F(p_x) = a \cdot \text{sinc}(ap_x),$$

where

$$\text{sinc}(p_x) = \frac{\sin(\pi p_x)}{\pi p_x}$$

- Translation property of the Fourier transform:

$$f(x - x_0) \xleftrightarrow{\mathcal{F}} e^{-i2\pi p_x x_0} \cdot F(p_x)$$

- The definition of $\cos(z)$ for complex numbers:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Therefore, the Fourier transforms of the apertures would be:

i.

$$F_1(p_x) = a \cdot \text{sinc}(ap_x)$$

ii.

$$\begin{aligned} F_2(p_x) &= a \cdot \text{sinc}(ap_x)e^{i2\pi p_x b} + a \cdot \text{sinc}(ap_x)e^{-i2\pi p_x b} \\ &= 2a \cdot \text{sinc}(ap_x) \cdot \cos(2\pi p_x b) \end{aligned}$$

iii.

$$\begin{aligned} F_3(p_x) &= a \cdot \text{sinc}(ap_x) + a \cdot \text{sinc}(ap_x)e^{i2\pi p_x b} + a \cdot \text{sinc}(ap_x)e^{-i2\pi p_x b} \\ &= a \cdot \text{sinc}(ap_x) \cdot (1 + 2 \cos(2\pi p_x b)) \end{aligned}$$

The images formed at the Fourier plane of the lens for $a = 1$ and $b = 2$ are shown below in Figure 1.

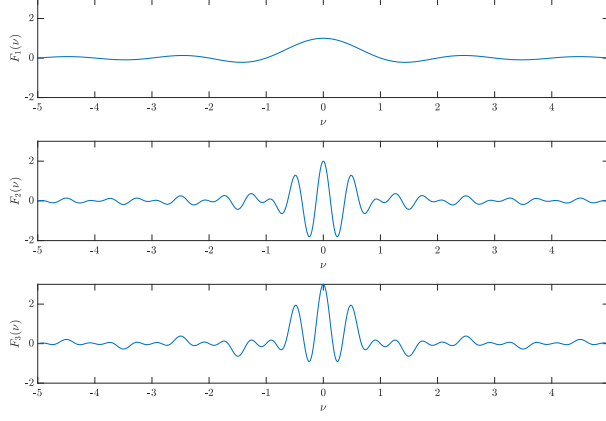


Figure 1: Images of the apertures formed in the Fourier plane.

2 Resolution in $4f$ scheme

In order to define the resolution of the system (or the minimum distance between two monochromatic sources) we will use Rayleigh d_{min}^R and Abbe d_{min}^A criteria:

$$d_{min}^R = 0.61 \cdot \frac{\lambda}{\text{NA}} = 0.61 \cdot \frac{\lambda}{n \sin \alpha}; \quad d_{min}^A = 0.5 \cdot \frac{\lambda}{\text{NA}} = 0.5 \cdot \frac{\lambda}{n \sin \alpha},$$

where α is the half-angle of the total angular opening of radiation coming from the source.

A. In the presented $4f$ -scheme the numerical aperture is defined by the height of lenses:

$$\text{NA} = n \sin \alpha = n \cdot \frac{(h_1/2)}{\sqrt{(h_1/2)^2 + f^2}} \approx 0.3$$

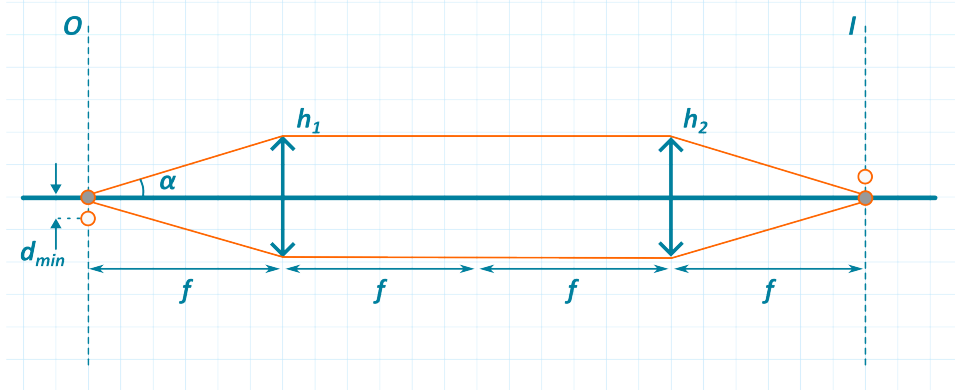


Figure 2: The $4f$ scheme consisting of two thin lenses with marginal rays (in orange) defining the angular opening of the system.

Therefore, the resulting resolution for both criteria could be found:

$$d_{min}^R = 0.61 \cdot \frac{\lambda}{\text{NA}} = 1220 \text{ nm} = 1.22 \text{ } \mu\text{m}; \quad d_{min}^A = 0.5 \cdot \frac{\lambda}{\text{NA}} = 1000 \text{ nm} = 1 \text{ } \mu\text{m}.$$

B. When the refractive index of the lenses change from n to \tilde{n} , the numerical aperture changes as $\tilde{\text{NA}} = \frac{\tilde{n}}{n} \text{NA}$. In this case, $n = 1$, so $\tilde{\text{NA}} = \tilde{n} \text{NA} = 0.45$. The resulting resolution then

increases as well (the distance between the resolved sources is smaller):

$$\tilde{d}_{min}^R = 0.61 \cdot \frac{\lambda}{\tilde{NA}} \approx 813 \text{ nm}; \quad \tilde{d}_{min}^A = 0.5 \cdot \frac{\lambda}{\tilde{NA}} \approx 667 \text{ nm}.$$

For changing wavelength $\tilde{\lambda}$ you can directly insert its value in the resolution limit criteria and also increase the resolution by decreasing wavelength:

$$\tilde{d}_{min}^R = 0.61 \cdot \frac{\tilde{\lambda}}{NA} \approx 976 \text{ nm}; \quad \tilde{d}_{min}^A = 0.5 \cdot \frac{\tilde{\lambda}}{NA} \approx 800 \text{ nm}.$$

- C. Since the full height $h_2' = 12 \text{ cm} > h_1$ then the total angular opening should not be influenced by changing this lens to a bigger one and, therefore, the resolution stays the same.

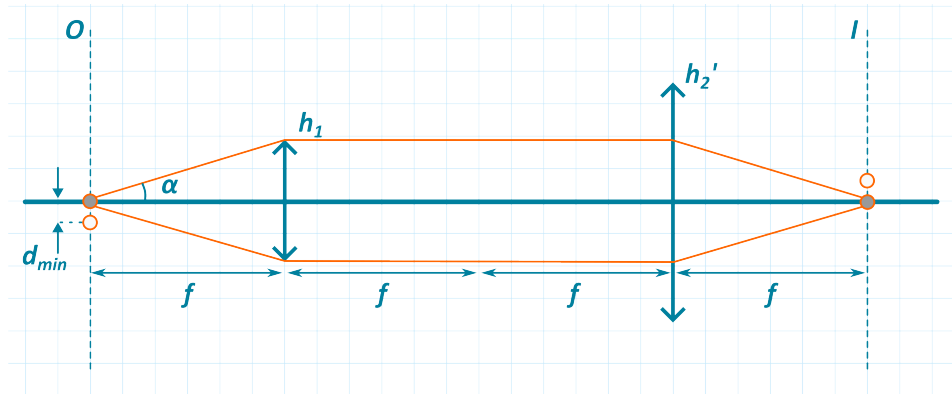


Figure 3: The $4f$ scheme consisting of two thin lenses. The second lens has larger height than the first.

In second case, the full-height $h_2'' = 4 \text{ cm} < h_1$ and, therefore, the angular opening is now defined by the second lens:

$$NA'' = n \sin \alpha'' = n \cdot \frac{(h_2/2)}{\sqrt{(h_2/2)^2 + f^2}} \approx 0.205$$

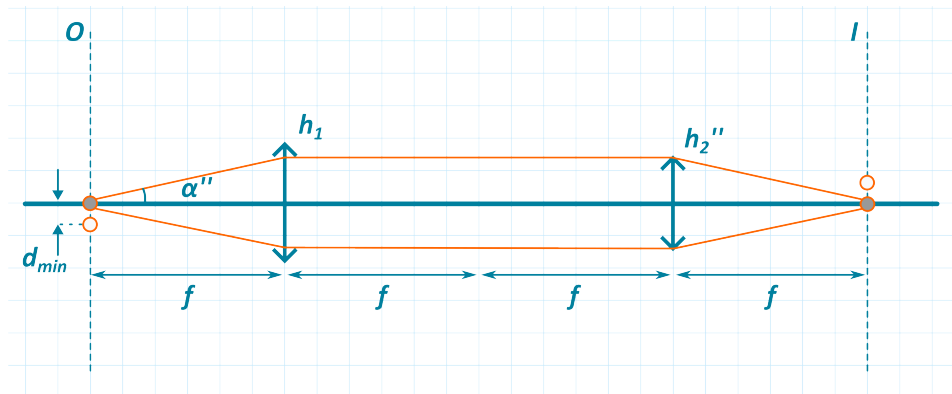


Figure 4: The $4f$ scheme consisting of two thin lenses. The second lens has smaller height than the first.

And corresponding resolution is therefore:

$$d_{min}^R = 0.61 \cdot \frac{\lambda}{NA''} \approx 1.785 \text{ } \mu\text{m}; \quad d_{min}^A = 0.5 \cdot \frac{\lambda}{NA''} \approx 1.463 \text{ } \mu\text{m}.$$

3 Resolution of a human eye

- A. A beam of a laser light focused with a lens of focal length f , is propagating through a circular aperture of diameter D . The beam can be approximated by a plane wave. Therefore, its intensity after the propagation through the lens at the plane at a distance of one focal length f is calculated as:

$$I(x, y) = I_0 \left[\frac{2J_1 \left(\frac{\pi D \rho}{\lambda f} \right)}{\frac{\pi D \rho}{\lambda f}} \right]^2 \quad \text{where } \rho = \sqrt{x^2 + y^2}$$

I_0 is the peak intensity and J_1 is a Bessel function of the first kind. From this formula, the diffraction limited spot size is measured at the radius where this function becomes zero. The radius value is then:

$$\rho = 1.22 \cdot \frac{\lambda \cdot f}{D}$$

which is given by the Rayleigh's Criteria. Using this, we can compute the diameter of the diffraction limited spot:

$$\text{Spot diameter} = 2\rho_s = 2.44 \frac{\lambda f}{D} = 2.44 \frac{500 \text{ nm} \times 20 \text{ mm}}{2 \text{ mm}} = 12.2 \mu\text{m}$$

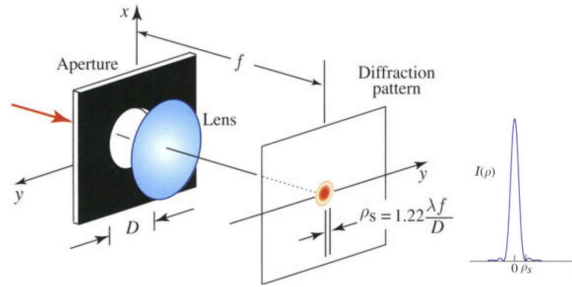


Figure 5: Diffraction pattern at focal length f after a lens with circular aperture.

- B. For a square shaped aperture on a lens with focal length f , the intensity on the plane at the distance of one focal length f can be calculated from the propagation through the rectangular aperture:

$$I(x, y) = I_0 \text{sinc}^2 \frac{D_x x}{\lambda f} \text{sinc}^2 \frac{D_y y}{\lambda f}$$

where D_x and D_y are the aperture width and height, respectively; and $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. For a square aperture, D_x is equal to D_y . The spot size from this equation is measured at the point where this function becomes zero for the first time. This corresponds to:

$$x_0 = \frac{\lambda f}{D}$$

which gives the half size of the spot along x -axis. The diameter of the diffraction limited spot will be:

$$2x_0 = 2 \frac{\lambda f}{D} = 2 \cdot \frac{500 \text{ nm} \cdot 20 \text{ mm}}{2 \text{ mm}} = 10 \mu\text{m} \text{ along the } x\text{-axis}$$

For the y -axis, the spot diameter is the same ($10\mu\text{m}$), since the aperture is square. Since the feature size is smaller for this case, we would have had better resolution in case we had a square iris. Well... We should enjoy what we are given!

P.S. Please **do not** try to focus a laser beam on the back focal plane of your eye:)