Biomicroscopy I - Solutions Exercise Sheet 8

November 4, 2025

1 Fourier transform with a 2-lens, $4f$ system

A. Refer to Figure 1.

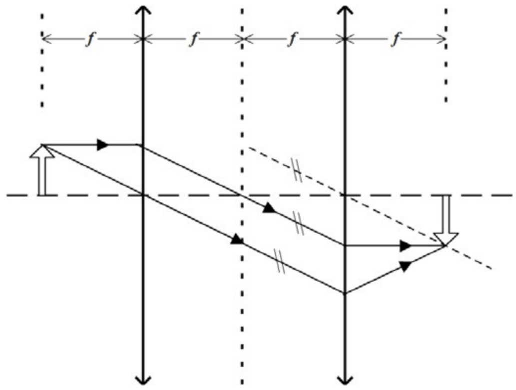

Figure 1: Ray tracing of a $4f$ system

B. The Fourier plane of the $4f$ system is found in the middle of the two lenses:

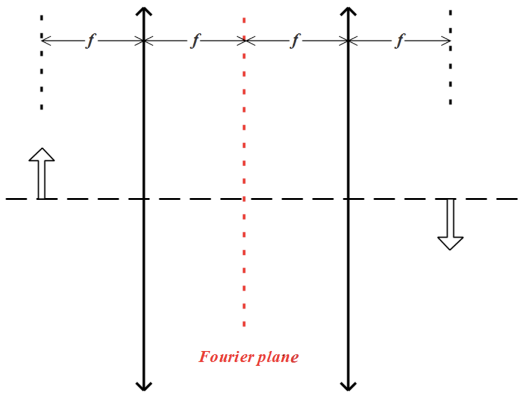

Figure 2: Fourier plane in a $4f$ system

C. Recall that the Fourier transform $\mathcal{F}$ of a function f is defined as

$$\mathcal{F}\{f(x)\} = F(p_x) = \int_{\infty}^{\infty} f(x) e^{-j2\pi p_x} dx$$

In particular, the Fourier transform of $f(x) = \cos(2\pi\nu x)$ is

$$\mathcal{F}\{\cos(2\pi\nu x)\} = F(p_x) = \frac{1}{2}\left(\delta(p_x - \nu) + \delta(p_x + \nu)\right)$$

which is the expected pattern at the back focal plane of the first lens (see Figure 3). A sinusoidal signal of the same frequency ν as the input is seen at the back focal plane of the second lens.

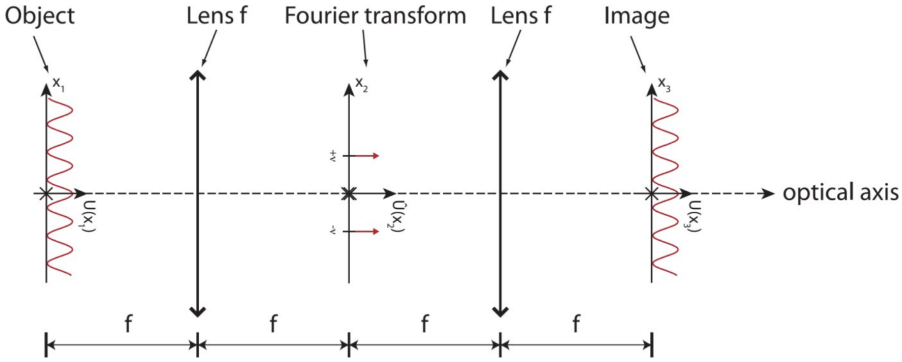

Figure 3: Sinusoidal object in a $4f$ system

D. Recall that the Fourier transform of a rectangular function of width a is $\Pi\left(\frac{x}{a}\right)$ is a sinc function:

$$F(p_x) = \mathcal{F}\{\Pi\left(\frac{x}{a}\right)\} = a \cdot \mathrm{sinc}(ax)$$

where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. Therefore, in the back focal plane of the first lens one expects a sinc function $(F(p_x))$ and in the back focal plane of the second lens one expects another rectangle.

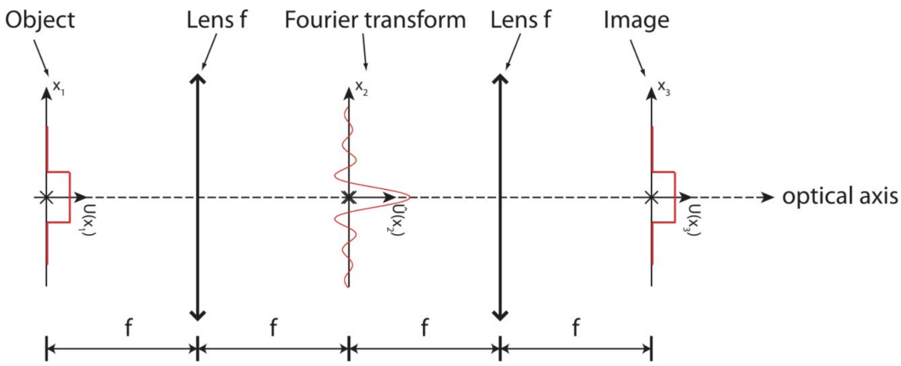

Figure 4: Rectangular object in a $4f$ system

When the width of the rectangular function increases by twice, the width of the central peak in the sinc function decreases by half. Similarly, when the width of the rectangular decreases by half, the width of the central peak in the sinc increases by twice.

2 'Fourier house' with a 2-lens, $4f$ system

Suppose the sinusoidal shadings in the house have a period of T with an offset 1:

$$\mathrm{U}(x_1) = \sin\left(\frac{2\pi}{T}x\right) + 1$$

A. We observe the Fourier transform of the different sinusoidal patterns in the back focal plane of the first lens:

$$\hat{\mathrm{U}}(p_x) = \frac{1}{i\lambda f}\left\{\frac{i}{2}\left[\delta\left(p_x + \frac{1}{T}\right) - \delta\left(p_x - \frac{1}{T}\right)\right] + \delta(p_x)\right\}.$$

Therefore a sinusoidal pattern with period T and an offset is transformed into three Dirac deltas (δ) in the Fourier plane. The Dirac deltas are aligned on a line having the same direction as the variation of the pattern, which is perpendicular to the lines of the pattern, as shown in Figure 5.

The central Dirac delta represents the mean object brightness. The horizontal features represent the sky, the vertical features the ground, and the diagonals features the roof and the body of the house. The shape of the object regions leads to the blurring around the spots.

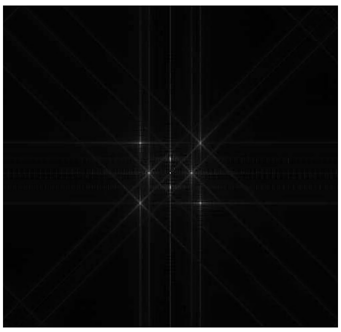

Figure 5: Spatial spectrum of Fourier house

B. In the back focal plane of the second lens, as seen in Exercise 1, you will see the house reversed (magnification $m = -1$) with identical size as shown in Figure 6:

Figure 6: Image of the Fourier house in a $4f$ system

C. To see only the roof, we place a filter in the Fourier plane which is a simple amplitude mask with three holes shown in Figure 7. The circular holes transmit the three spots representing the roof, while blocking the rest of the spatial spectrum. Figure 8 shows the resulting image. The pattern of the roof is clearly visible. The shape of the roof is blurred because the high-frequency content of the edges is removed.

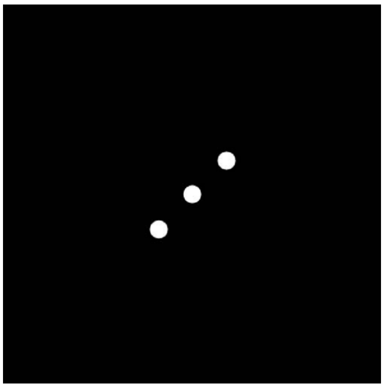

Figure 7: Amplitude filter for the roof

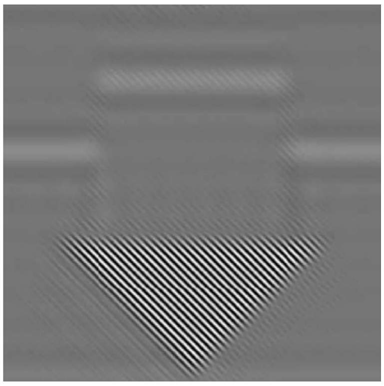

Figure 8: Image of house with roof filter

D. Similarly, to see only the sky, we can use the mask shown in Figure 9. The result is shown in Figure 10.

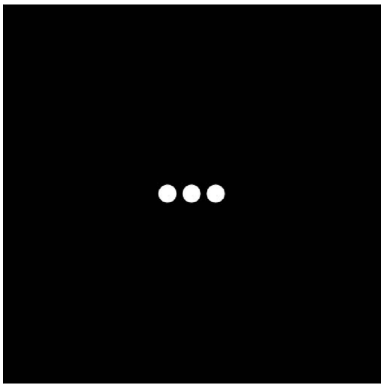

Figure 9: Filter used to obtain only the sky

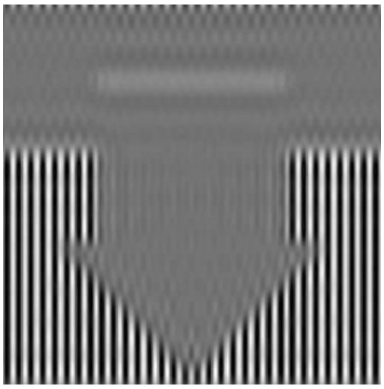

Figure 10: Result using the filter of Figure 9

3 Point spread function (PSF) in a $4f$ configuration

A. As soon as the lenses are infinitely large and no aperture is the present in the optical pathway, all of the diffraction orders path through the system and you can see the Fourier transform of the Dirac function in the Fourier plane which is unity. Reverse Fourier transform of the unity is observed in the Image plane, which is identical to the input Dirac function.

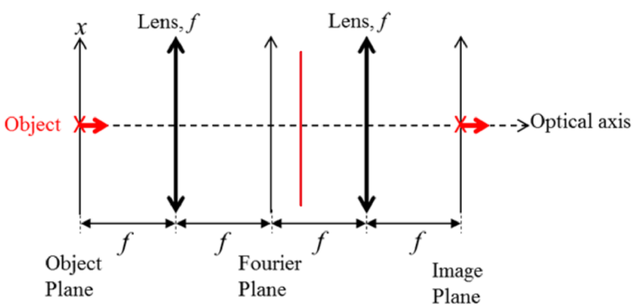

Figure 11: Original object, Fourier plane image and converted image in a 4-f system with infinitely large aperture.

B. The aperture crops the unity into a rectangular function in the Fourier plane, thus the observation on the image plane is now the Fourier transformation of a rectangular function. By recalling previous exercise (set 7): $\mathcal{F}\{\text{rect}\left(\frac{x}{a}\right)\}(p_x) = a \cdot \text{sinc}\left(ap_x\right)$ we can sketch resulting image.

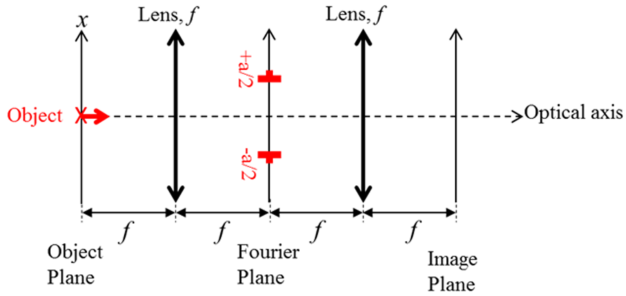

Figure 12: Original object, Fourier plane image and converted image in a 4-f system with an aperture.

The inserted aperture serves as a low-pass filter, which removes higher frequencies from the original image in the Fourier plane. The resulting image of a point source thus gets blurred and the resolution gets reduced.