Biomicroscopy I - Solutions Exercise Sheet 7

November 4, 2025

1 Intensity pattern for two interfering sources

In order to find the full vector of the field let us redraw the vector summation problem in the way presented in Fig. 1. By using law of cosines then one can find the total field amplitude in a way:

$$|A|^2 = |A_1|^2 + |A_2|^2 - 2|A_1||A_2|\cos\alpha = |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos\varphi.$$

By further recalling that $I \sim |A|^2$ we get the desired form for intensity pattern:

$$I = I_1 + I_2 + 2(I_1 I_2)^{\frac{1}{2}}\cos\varphi$$

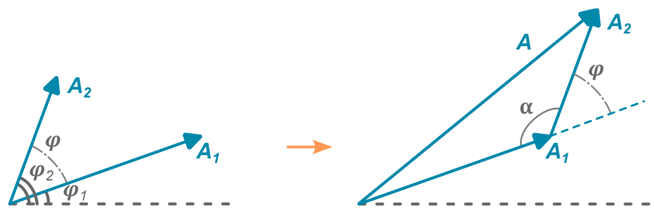

Figure 1: Vector diagram formalism.

2 Double slit interference

A. We need to show that the intensity of two interfering waves at the plane $Z = L$ is

$$I(x) = 2I_0\left(1 + \cos\left(\frac{2\pi d}{\lambda L}x\right)\right)$$

As was obtained in Ex. 1, the interference equation is

$$I = I_1 + I_2 + 2(I_1 I_2)^{\frac{1}{2}}\cos\varphi \tag{1}$$

where I_1 and I_2 denote the intensity of the two interfering waves and φ is the phase difference between the two interfering waves. According to the statement, the two interfering waves have the same intensity value I_0 (*i.e.* $I_1 = I_2 = I_0$). Therefore, we only need to find φ. The phase difference can be computed from the path difference between the two arms given in Figure 2.

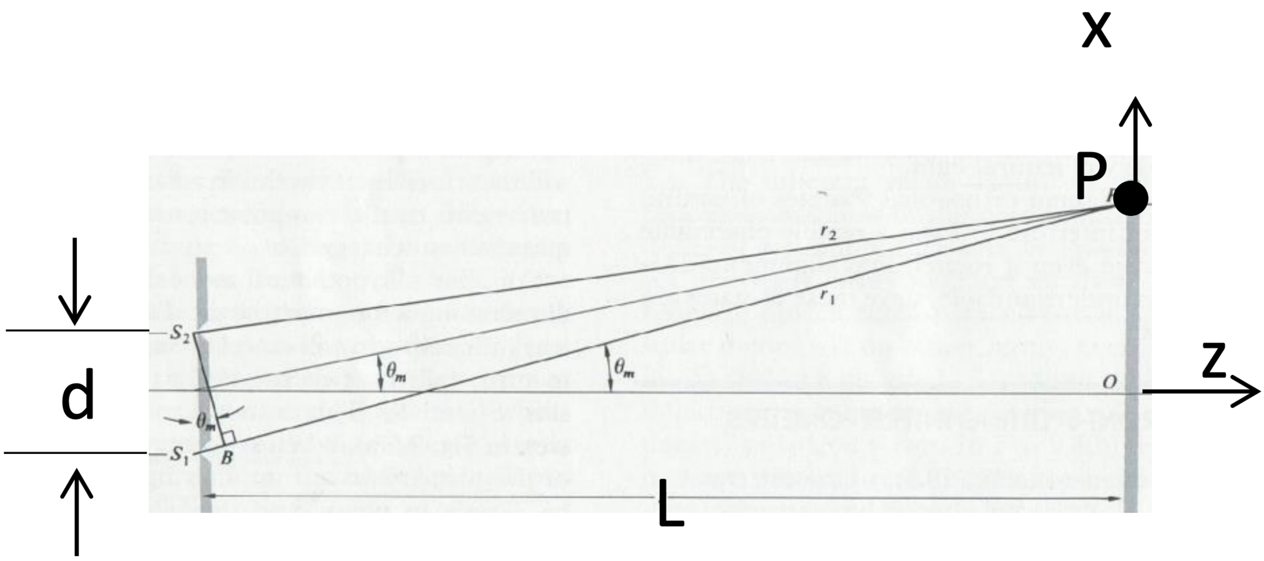

Figure 2: Optical path difference between two arms.

$$\varphi_1 = k \cdot r_1 = \tfrac{2\pi}{\lambda} r_1$$

$$\varphi_2 = k \cdot r_2 = \tfrac{2\pi}{\lambda} r_2$$

$$\varphi = \varphi_2 - \varphi_1 = \tfrac{2\pi}{\lambda} (r_2 - r_1)$$

From triangular symmetry (for the case $L >> x$, $L >> d$),

$$\frac{r_2 - r_1}{d} \approx \frac{x}{L}$$

Therefore,

$$\varphi = \frac{2\pi}{\lambda} \left(\frac{x \cdot d}{L} \right)$$

Now that we have computed the phase difference φ, we can plug it into Equation A. and operate:

$$I = I_0 + I_0 + 2\sqrt{I_0 I_0} \cos\left(\frac{2\pi d}{\lambda L} x \right) = 2I_0 \left[1 + \cos\left(\frac{2\pi d}{\lambda L} x \right) \right] \tag{2}$$

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B. Using Equation 2, which we just derived, we can see that I reaches its maximum value I_{max} when $\cos(\varphi) = 1$. Therefore,

$$I_{max} = 2I_0 [1 + 1] = 4I_0$$

Note that the maximum is reached for $\varphi = 2\pi n$, where $n \in \{0\} \cup \mathbb{Z}^+$, or alternatively when $r_2 - r_1 = n\lambda$ where $n \in \{0\} \cup \mathbb{Z}^+$.

On the other hand, the minimum I_{min} is reached when $\cos(\varphi) = -1$:

$$I_{min} = 2I_0 [1 - 1] = 0$$

Note that the minimum is reached when $\varphi \in (2n+1)\pi$, where $n \in \{0\} \cup \mathbb{Z}^+$, or alternatively when $r_2 - r_1 = (2n+1)\frac{\lambda}{2}$, where $n \in \{0\} \cup \mathbb{Z}^+$.

C. a. The maximum points are reached when

$$\frac{r_2 - r_1}{d} = \frac{x_{max}}{L} \iff x_{max} = \frac{L}{d}(r_2 - r_1)$$

Let x_{max_i} denote the position of the i-th maximum point. Then, $x_{max_i} = \frac{L}{d}\lambda i$. Therefore, the first maximum is located at

$$x_1 = \frac{L}{d}\lambda = \frac{2\text{m}}{1\text{mm}}600\text{nm} = 1.2\text{mm}$$

b. The first minimum point x_{min_1}

$$\frac{r_2 - r_1}{d} = \frac{\lambda}{2}\frac{1}{d} = \frac{x_{min_1}}{L} \rightarrow x_{min_1} = \frac{\lambda L}{2d} = \frac{1.2\text{mm}}{2} = 0.6\text{mm}.$$

c. The third maximum is found at

$$x_{max_3} = 3 \cdot 1.2\text{mm} = 3.6\text{mm}$$

3 Diffraction grating

A. As seen in the lecture, the first diffraction order is given by

$$\sin\theta_1 = \frac{1 \cdot \lambda}{d}.$$

The second order is given by

$$\sin\theta_2 = \frac{2 \cdot \lambda}{d}.$$

In general, the angle of the m-th diffraction order θ_m is given by

$$\sin\theta_m = \frac{m \cdot \lambda}{d}; \quad m \in \mathbb{Z}^+$$

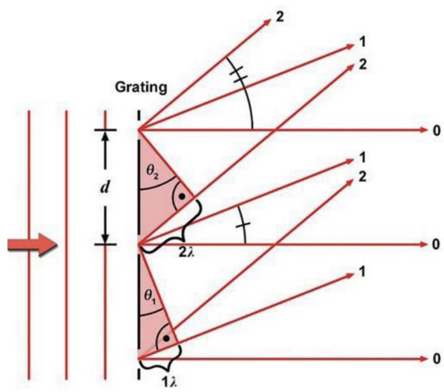

Figure 3: Diffraction grating

B. We have

$$\sin\theta_m = \frac{m \cdot \lambda}{d} = \frac{x}{L}$$

with $m = 1$, $L = 1$ m, $x = 5$ cm, $\lambda = 0.5$ μm

So,

$$d = \frac{\lambda \cdot L}{x} = 10^{-5} \text{ m} = 0.01 \text{ mm}$$

Therefore, there are 100 grating slits per 1 cm.

C. Now, we have $\lambda = 600$ nm, $d = 0.01$ mm, $L = 1$ m, the location of the first diffraction order is given by:

$$x_1 = \frac{\lambda L}{d} = \frac{600 \text{ nm} \cdot 1 \text{ m}}{0.01 \text{ mm}} = \frac{600 \cdot 10^{-9} \text{ m} \cdot 1 \text{ m}}{10^{-5} \text{ m}} = 6 \cdot 10^{-2} \text{ m} = 6 \text{ cm}$$

$$\sin\theta_1 = \frac{\lambda}{d} = \frac{600 \text{ nm}}{0.01 \text{ mm}} = \frac{600 \text{ nm}}{10^4 \text{ nm}} = 0.06$$

$$\theta_1 = \arcsin(0.06) \approx 3.4398°$$

D. Now, $\lambda = 0.5\mu\text{m} = 500$ nm, $d = 0.005$ mm, $L = 1$ m, the location of the first diffraction order is given by:

$$x_1 = \frac{\lambda L}{d} = \frac{500 \text{ nm} \cdot 1 \text{ m}}{0.005 \text{ mm}} = \frac{500 \cdot 10^{-9} \text{ m} \cdot 1 \text{ m}}{5 \cdot 10^{-6} \text{ m}} = 0.1 \text{ m} = 10 \text{ cm}$$

The position of the first diffraction order is farther, thus a denser grating (smaller features) diffracts more.

E. Light with larger wavelength will diffract at larger angle, therefore the larger the wavelength, the further from the centre is the peak.

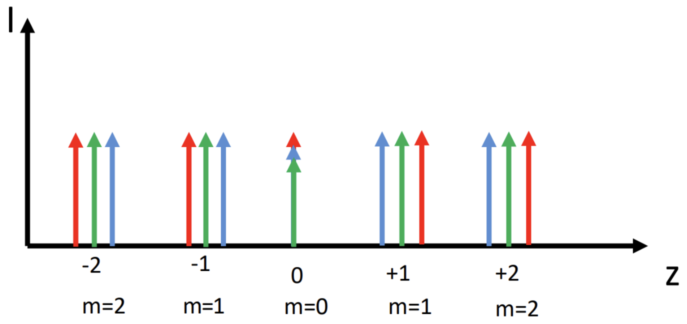

Figure 4: Expected diffraction pattern for different wavelengths

4 1D Fourier transform of periodic functions

The Fourier transform $\mathcal{F}\{f(x)\}$ is defined as:

$$F(p_x) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-i2\pi p_x x}dx \tag{3}$$

Moreover, we are given the following identity:

$$\int_{-\infty}^{\infty} e^{-i2\pi p_x x}dx = \delta(p_x) \tag{4}$$

where $\delta(.)$ denotes the Dirac delta function. First, we note that using variable substitution one can deduce:

$$\int_{-\infty}^{\infty} e^{-i2\pi(p_x-k)x}dx = \delta(p_x - k), \tag{5}$$

where $k \in \mathbb{R}$.

A. $A\cos(2\pi k_0 x)$

$$F(p_x) = A\int_{-\infty}^{\infty} \cos(2\pi k_0 x)\, e^{-i2\pi p_x x}dx = \frac{A}{2}\int_{-\infty}^{\infty} \left[e^{i2\pi k_0 x} + e^{-i2\pi k_0 x}\right] e^{-i2\pi p_x x}dx$$
$$= \frac{A}{2}\int_{-\infty}^{\infty} \left[e^{-i2\pi(p_x-k_0)x} + e^{-i2\pi(k_0+p_x)x}\right] dx = \frac{A}{2}\left[\delta(p_x - k_0) + \delta(p_x + k_0)\right]$$

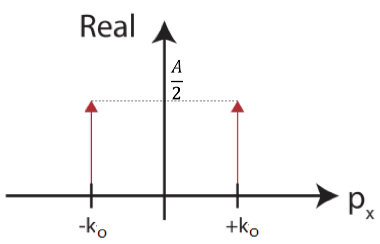

Figure 5: Fourier transform of a cosine function

B. $A\sin(2\pi k_0 x)$

$$F(p_x) = A\int_{-\infty}^{\infty} \sin(2\pi k_0 x)\, e^{-i2\pi p_x x}dx = \frac{A}{2i}\int_{-\infty}^{\infty} \left[e^{i2\pi k_0 x} - e^{-i2\pi k_0 x}\right] e^{-i2\pi p_x x}dx =$$
$$= \frac{A}{2i}\int_{-\infty}^{\infty} \left[e^{-i2\pi(p_x-k_0)x} - e^{-i2\pi(p_x+k_0)x}\right] dx = i\frac{A}{2}\left[\delta(p_x + k_0) - \delta(p_x - k_0)\right]$$

C. $Ae^{i2\pi k_0 x}$

$$F(p_x) = A\int_{-\infty}^{\infty} e^{i2\pi k_0 x}e^{-i2\pi p_x x}dx = A\int_{-\infty}^{\infty} e^{-i2\pi(p_x-k_0)x}dx = A\delta(p_x - k_0)$$

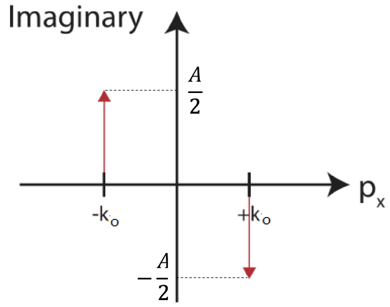

Figure 6: Fourier transform of a sine function

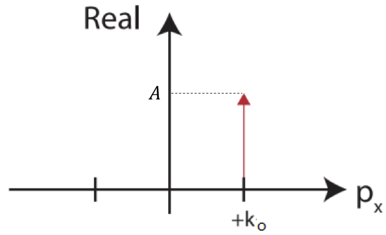

Figure 7: Fourier transform of an exponential function

5 Fourier transform of a unit pulse

The unit pulse is defined as

$$\operatorname{rect}\left(\frac{x}{a}\right) = \Pi\left(\frac{x}{a}\right) = \begin{cases} 1, & \text{if} \quad |x| \leq \frac{a}{2} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Let $f = \Pi$. Its Fourier transform $F = \mathcal{F}(f)$ is:

$$F(p_x) = \int_{-\infty}^{\infty} \Pi\left(\frac{x}{a}\right) e^{-i2\pi p_x x} dx$$

First, we notice that the rectangle function $\Pi\left(\frac{x}{a}\right) = 0$ everywhere but in the interval $\left[-\frac{a}{2}, \frac{a}{2}\right]$,within which $\Pi\left(\frac{x}{a}\right) = 1$. This allows us rewrite the integral as:

$$F(p_x) = \int_{-\frac{a}{2}}^{\frac{a}{2}} 1 \cdot e^{-i2\pi p_x x} dx = \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-i2\pi p_x x} dx$$

Using the fact that $\frac{d(e^{ax})}{dx} = ae^{ax}$, we can solve the integral and operate to simplify the result:

$$\begin{aligned} F(p_x) &= \frac{1}{-i2\pi p_x} \left[e^{-i2\pi p_x}\right]_{-\frac{a}{2}}^{+\frac{a}{2}} = \\ &= \frac{e^{-i\pi p_x a} - e^{i\pi p_x a}}{-i2\pi p_x} = \frac{\sin(a\pi p_x)}{\pi p_x} = \frac{a \sin(a\pi p_x)}{a\pi p_x} = a \cdot \operatorname{sinc}(a p_x). \end{aligned}$$

Therefore, the Fourier transform of the rectangle function $\Pi\left(\frac{x}{a}\right)$ is $F(p_x) = a \cdot \operatorname{sinc}(ap_x)$[1].

[1] You can memorize only the Fourier transform of the rectangular function for $a = 1$ and get the obtained result by using the scaling property.