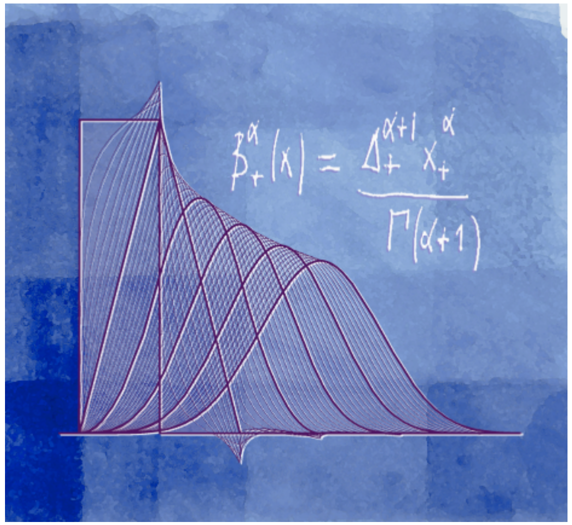
Image Processing

Chapter 4

Morphological Processing

Prof. Michael Unser, LIB
Prof. Dimitri Van De Ville, MIP



November 2025

OUTLINE

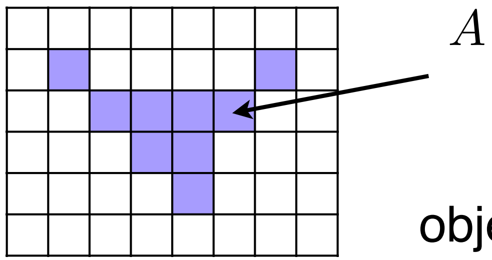
- Morphology: introduction
- Basic definitions
- Erosion and dilation
- Opening and closing
- Distance map and watershed
- Graylevel morphology
- Morphological filtering

Morphology: introduction

Deals with shape and structure [Serra, Matheron]

- Language: set theory
  - Binary images (bitmap)

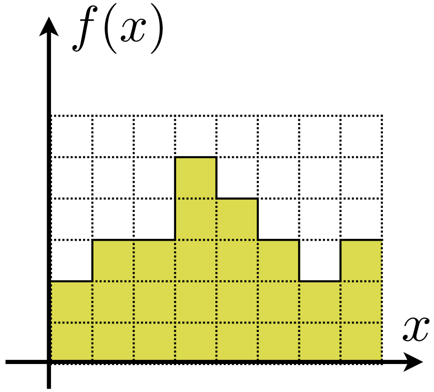
    Sets of points in 2D space ($\mathbb{Z}^2$)



object *vs.* background

  - Quantized graylevel images

    Sets of points in $\mathbb{Z}^3$



- Type of transformations
  - Set-theoretic: union, intersection, etc.
  - With structuring element: dilation, erosion

Morphology: application areas

Classification of objects or image features based on shape.

- Examples
  - Extraction of objects with a specific shape
  - With a size smaller or greater than a limit
  - Contour detection
- Typical image processing tasks where it can be useful
  - Preprocessing: noise reduction, simplification
  - Feature detection
  - Segmentation: contour extraction
  - Post-processing: shape cleaning and simplification
- Main application area
  - Material sciences, mineralogy, granulometry
  - Medicine and biology: cell counting, cytology, gel electrophoresis, micro-arrays
  - Machine vision

Basic definitions

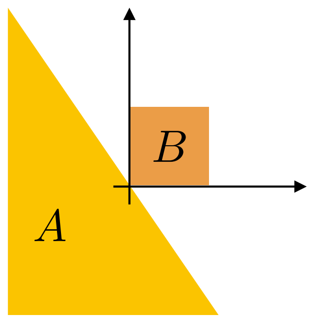
- Universal set

  $\mathbb{E}$ is the set of every possible element (e.g., $\mathbb{E} = \mathbb{Z}^2$ or $\mathbb{E} = \mathbb{R}^2$)

- Sets and subsets

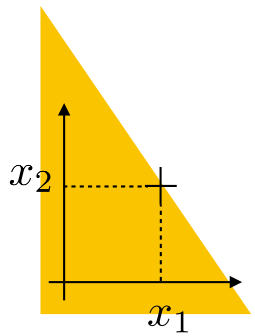
  Sets: $A, B \subset \mathbb{E}$

  Elements: $a = (a_1, a_2) \in A, \quad b = (b_1, b_2) \in B$

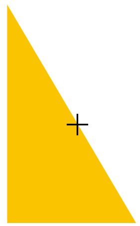


- Translation by $x = (x_1, x_2)$

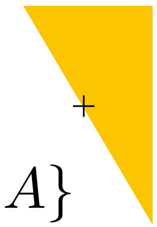
  $$(A)_x = \{c | c = a + x, \text{ for } a \in A\}$$



Basic definitions (Cont'd)



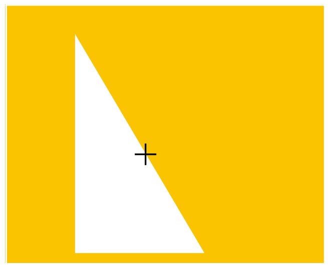
- Reflection or symmetry



$$A^{\mathrm{s}} = \{x \in \mathbb{E} \mid x = -a, \text{ for } a \in A\}$$

- Complement

$$A^{\mathrm{c}} = \{x \in \mathbb{E} \mid x \in \mathbb{E} \backslash A\}$$

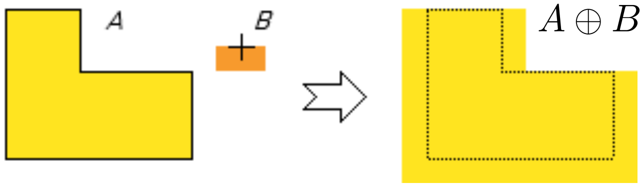


Dilation and erosion in R^2

- Structuring element

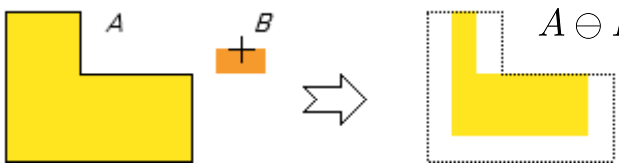
- Dilation

$$A \oplus B = \{x \in \mathbb{E} \mid (B^s)_x \cap A \neq \emptyset\} = \bigcup_{x \in A} (B)_x$$

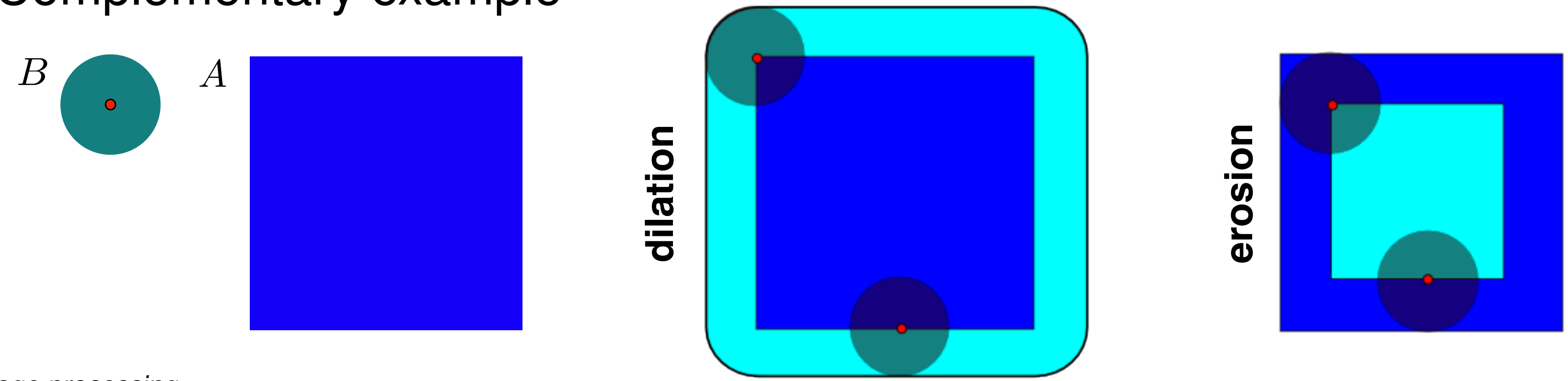


- Erosion

$$A \ominus B = \{x \in \mathbb{E} \mid (B)_x \subseteq A\} = \bigcap_{x \in B^s} (A)_x$$



- Complementary example



Dilation and erosion in R^2

- Structuring element

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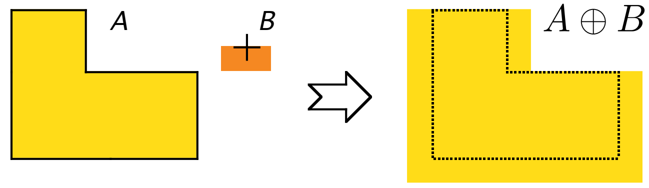
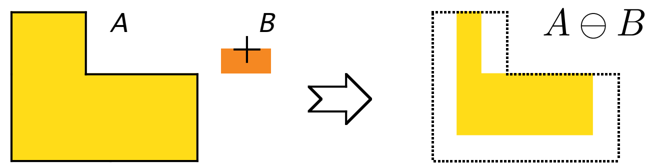
$$A \oplus B = \{x \in \mathbb{E} \mid (B^{\mathrm{s}})_x \cap A \neq \emptyset\} = \bigcup_{x \in A} (B)_x$$



- Erosion

$$A \ominus B = \{x \in \mathbb{E} \mid (B)_x \subseteq A\} = \bigcap_{x \in B^{\mathrm{s}}} (A)_x$$



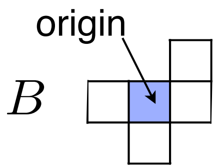
- Duality relations

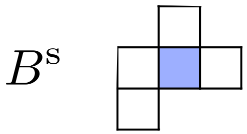
$$A \ominus B = (A^{\mathrm{c}} \oplus B^{\mathrm{s}})^{\mathrm{c}}$$ (Erosion = “Dilation” of complement)

$$A \oplus B = (A^{\mathrm{c}} \ominus B^{\mathrm{s}})^{\mathrm{c}}$$ (Dilation = “Erosion” of complement)

Dilation and erosion in Z^2

- Structuring element

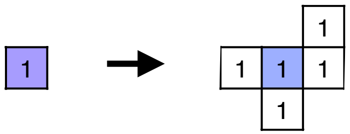




- Dilation

$$A \oplus B = \{x \in \mathbb{E} \mid (B^s)_x \cap A \neq \emptyset\} \quad \Leftrightarrow \quad A \oplus B = \bigcup_{x \in A} (B)_x$$

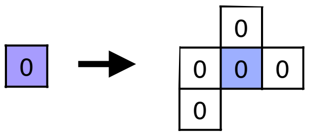
Parallel implementation:
(once 1, remains 1)



- Erosion

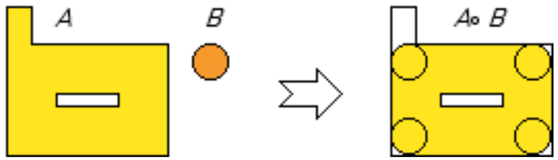
$$A \ominus B = \{x \in \mathbb{E} \mid (B)_x \subseteq A\} \quad \Leftrightarrow \quad A \ominus B = \left(\bigcup_{x \in A^c} (B^s)_x \right)^c$$

Parallel implementation:
(once 0, remains 0)



Opening

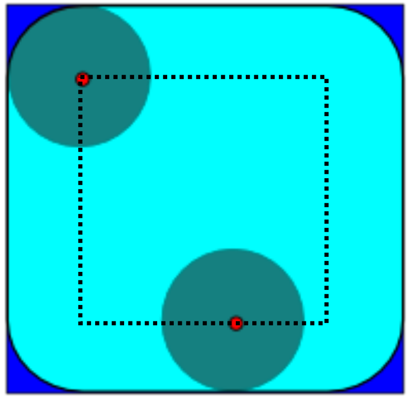
- Opening operator



$$A \circ B = (A \ominus B) \oplus B$$

  - Interpretation 1: smallest set that has a given erosion $A \ominus B$
  - Interpretation 2: Union all B's included in A

$$A \circ B = \bigcup_{x \in \mathbb{E}} \{ (B)_x | \, (B)_x \subseteq A \}$$

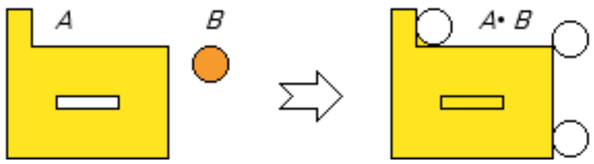


- Properties
  - Subset $A \circ B \subseteq A$
  - Invariance to origin $\forall x, \; A \circ (B)_x = A \circ B$
  - Idempotence $(A \circ B) \circ B = (A \circ B)$
  - Order preservation $C \subseteq D \quad \Rightarrow \quad (C \circ B) \subseteq (D \circ B)$

Closing

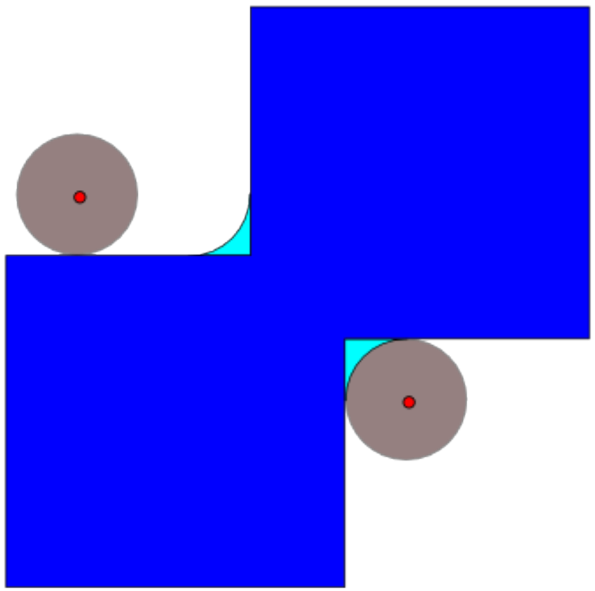
- Closing operator

$$A \bullet B = (A \oplus B) \ominus B$$



  - Interpretation 1: : largest set that has a given dilation $A \oplus B$
  - Interpretation 2: : Complement of union all B^{s} included in A^{c}

$$A \bullet B = \left(\bigcup_{x \in \mathbb{E}} \{ (B^{\mathrm{s}})_x | (B^{\mathrm{s}})_x \subseteq A^{\mathrm{c}} \} \right)^{\mathrm{c}}$$



- Properties
  - Superset $A \subseteq A \bullet B$
  - Invariance to origin $\forall x,\ A \bullet (B)_x = A \bullet B$
  - Idempotence $(A \bullet B) \bullet B = (A \bullet B)$
  - Order preservation $C \subseteq D \quad \Rightarrow \quad (C \bullet B) \subseteq (D \bullet B)$

Duality relations

- Erosion and dilation

$$(A \ominus B)^{\mathrm{c}} = A^{\mathrm{c}} \oplus B^{\mathrm{s}}$$

$$(A \oplus B)^{\mathrm{c}} = A^{\mathrm{c}} \ominus B^{\mathrm{s}}$$

- Opening and closing

$$A \circ B = (A^{\mathrm{c}} \bullet B^{\mathrm{s}})^{\mathrm{c}}$$

$$A \bullet B = (A^{\mathrm{c}} \circ B^{\mathrm{s}})^{\mathrm{c}}$$

Distance Map and Watershed

$$A_0[\mathbf{k}] = \begin{cases} 1, & \mathbf{k} \in \textbf{object} \\ 0, & \mathbf{k} \in \textbf{background} \end{cases}$$

$n = 0$

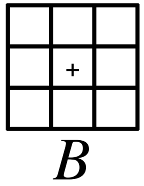
initialize D_{map}

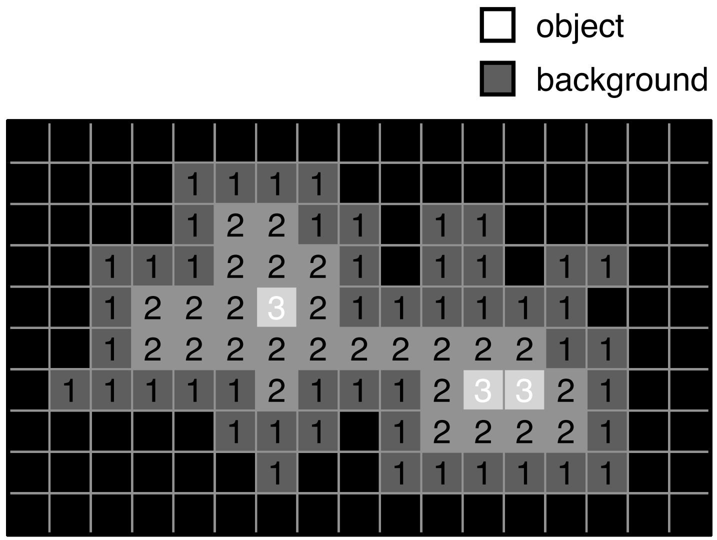
while $\max(A_n) = 1$ **do**

  $A_{n+1} = \textbf{erode}(A_n, B)$

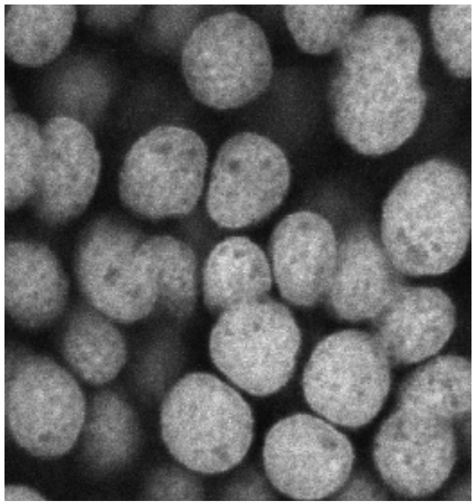
  $D_{\mathrm{map}} = D_{\mathrm{map}} + (n+1)(A_n - A_{n+1})$

  $n = n + 1$

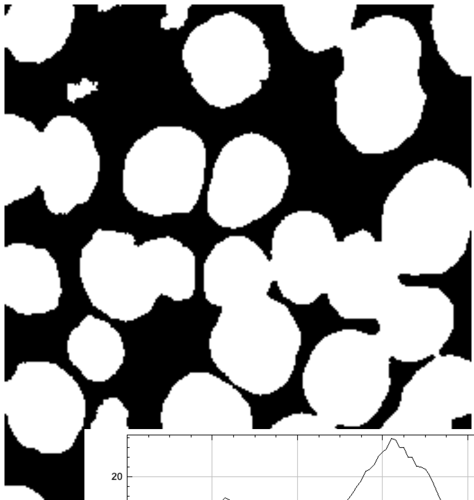




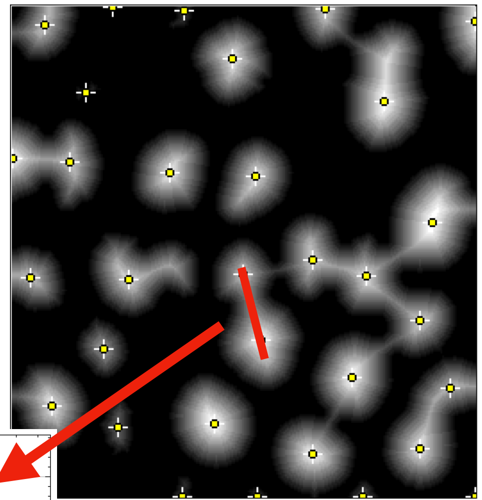
Original



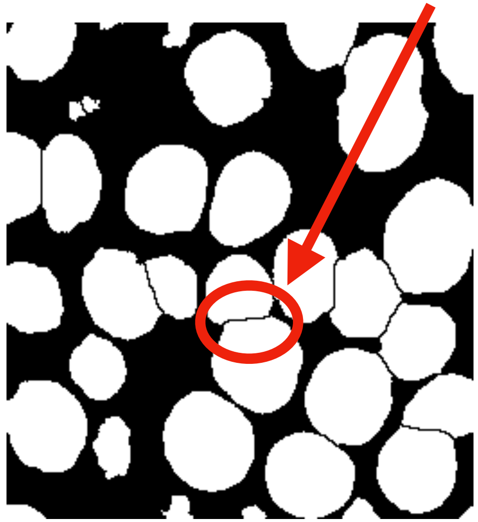
Thresholded image

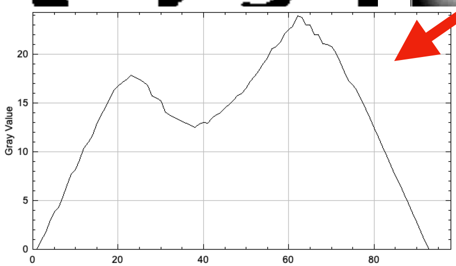


Distance map



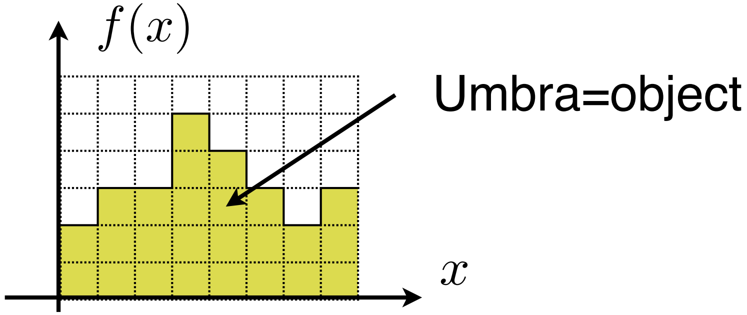
Watershed dam



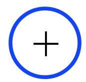


local maxima ➜ seeds of watershed

Graylevel morphology

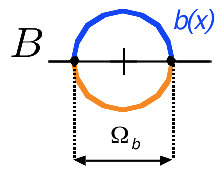


Common structural elements are symmetric

- Horizontal 2D: W-neighborhood (flat top) 
- Volumetric: approximation of a ball (rolling ball) 

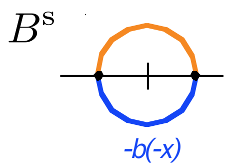
Dilation

$$(f \oplus b)[\boldsymbol{k}] = \max_{\boldsymbol{k}_0 \in \Omega_b} \{ f[\boldsymbol{k} - \boldsymbol{k}_0] + b[\boldsymbol{k}_0] \,|\, (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f \}$$



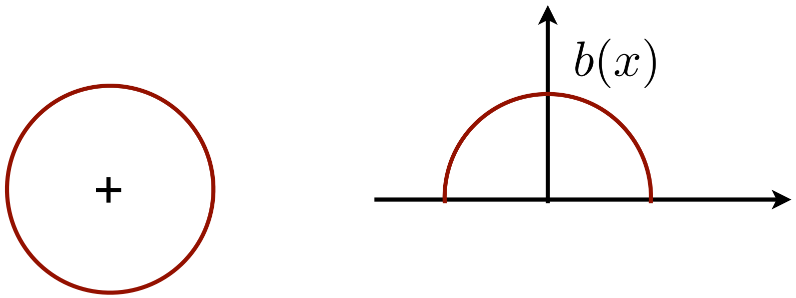
Erosion

$$(f \ominus b)[\boldsymbol{k}] = \min_{-\boldsymbol{k}_0 \in \Omega_b} \{ f[\boldsymbol{k} - \boldsymbol{k}_0] - b[-\boldsymbol{k}_0] \,|\, (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f \}$$



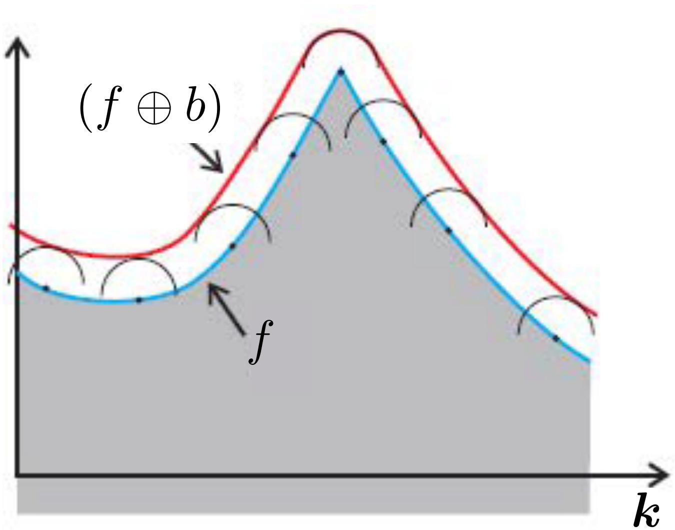
Graylevel dilation and erosion

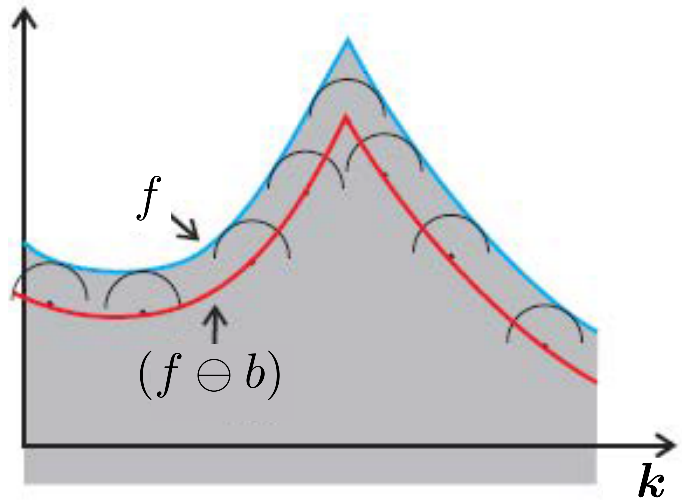
Structuring element:



« Graylevel formula » (gray = function value)

$$\max_{\boldsymbol{k}_0 \in \Omega_b} \{ f[\boldsymbol{k} - \boldsymbol{k}_0] + b[\boldsymbol{k}_0] \,|\, (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f \} \qquad \min_{-\boldsymbol{k}_0 \in \Omega_b} \{ f[\boldsymbol{k} - \boldsymbol{k}_0] - b[-\boldsymbol{k}_0] \,|\, (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f \}$$





« Binary formula » (gray = extra dimension)

Reminder: $A \oplus B = \bigcup_{x \in A} (B)_x$ $\qquad A \ominus B = \{ x \in \mathbb{E} \,|\, (B)_x \subseteq A \}$

Graylevel dilation and erosion

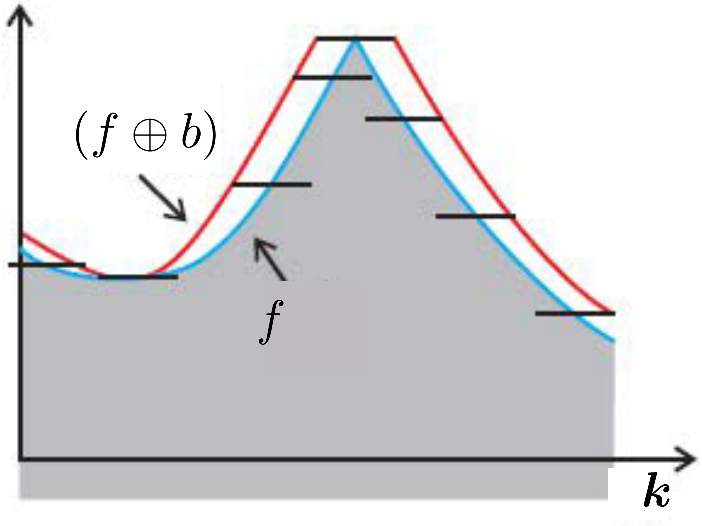
Structuring element: $b(x) = 0$

$\Omega_b = W$

« Graylevel formula » (gray = function value)

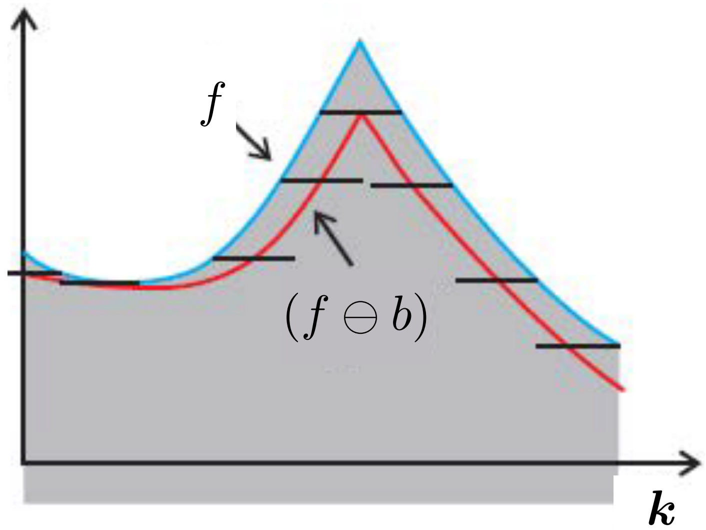
$$\max_{\boldsymbol{k}_0 \in W} \{ f[\boldsymbol{k} - \boldsymbol{k}_0] \,|\, (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f \}$$

max-filter



$$\min_{\boldsymbol{k}_0 \in W} \{ f[\boldsymbol{k} + \boldsymbol{k}_0] \,|\, (\boldsymbol{k} + \boldsymbol{k}_0) \in \Omega_f \}$$

min-filter



« Binary formula » (gray = extra dimension)

Reminder: $$A \oplus B = \bigcup_{x \in A} (B)_x$$ $$A \ominus B = \{ x \in \mathbb{E} \,|\, (B)_x \subseteq A \}$$

Morphological filtering

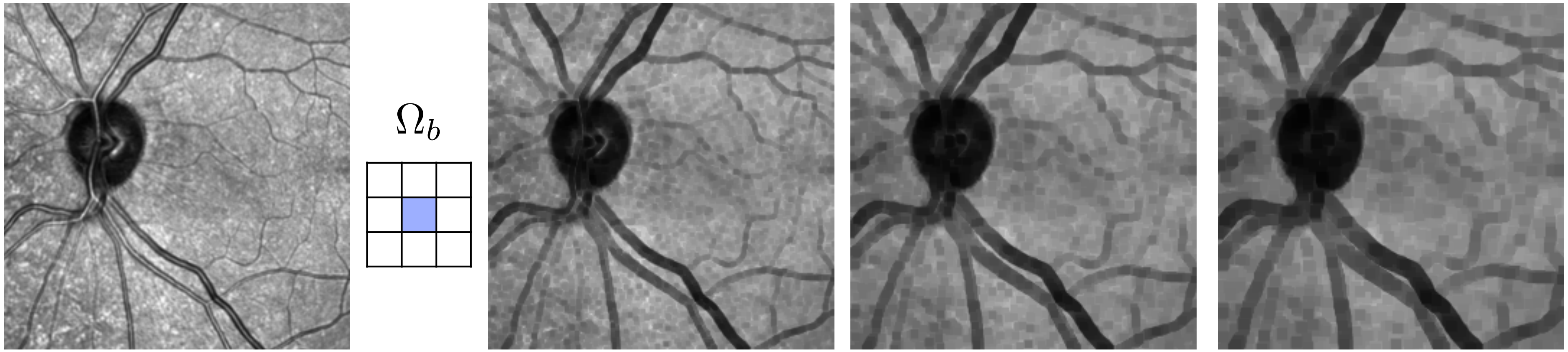
- Special case: $b_0[\boldsymbol{y}] = 0$ and $\Omega_b = W$

Dilation = Max-filter: $$(f \oplus b_0)[\boldsymbol{k}] = \max_{\boldsymbol{k}_0 \in W} \{ f[\boldsymbol{k} - \boldsymbol{k}_0] \mid (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f \}$$

Erosion = Min-filter: $$(f \ominus b_0)[\boldsymbol{k}] = \min_{\boldsymbol{k}_0 \in W} \{ f[\boldsymbol{k} + \boldsymbol{k}_0] \mid (\boldsymbol{k} + \boldsymbol{k}_0) \in \Omega_f \}$$

- Benefit of iteration

Dilation/erosion can be iterated to construct larger equivalent structuring elements



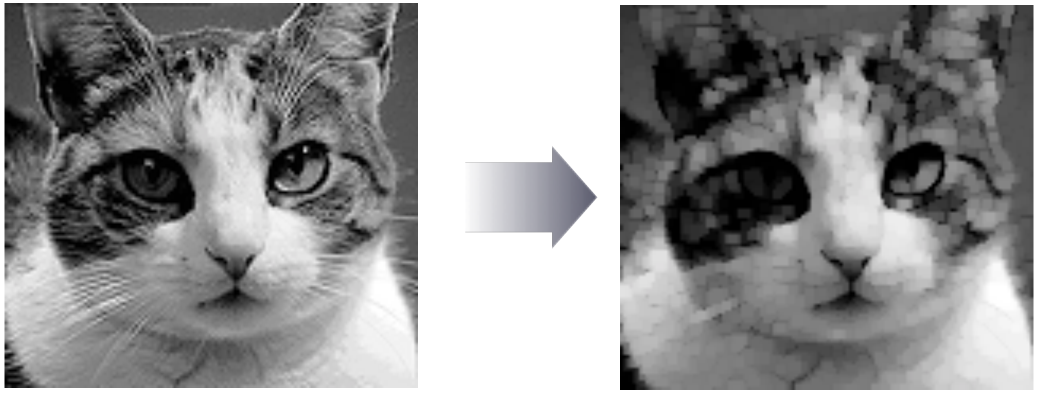
input 240×240 | 3×3 erosion | 2 iterations 5×5 erosion | 3 iterations 7×7 erosion

Morphological filtering (Cont'd)

- Morphological smoothing
  - Opening (*i.e.*, min then max)

$$f \circ b = (f \ominus b) \oplus b$$

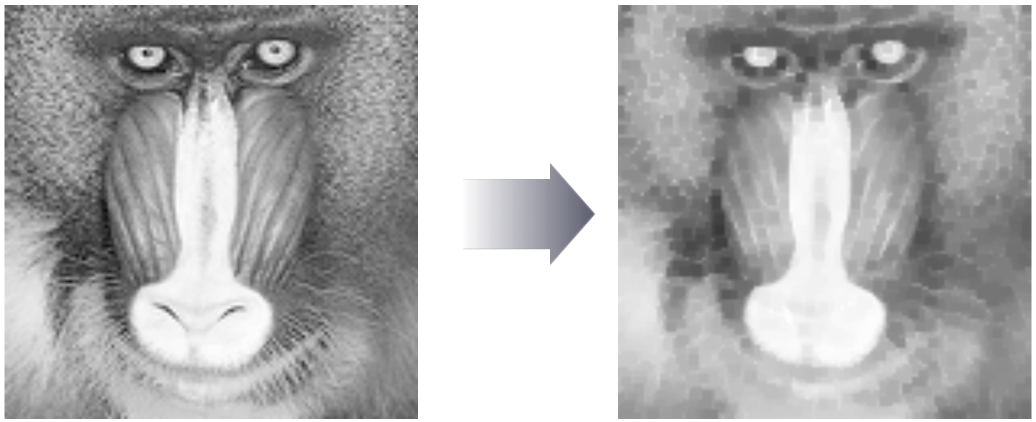
Smoothing by suppression of small bright features



  - Closing (*i.e.*, max then min)

$$f \bullet b = (f \oplus b) \ominus b$$

Smoothing by suppression of small black features



Example of morphological filtering

Original (reduced): 128×128

A: 3×3 min

B: 3×3 max

C: 3×3 max of A

D: 3×3 min of B

Morphological filtering (Cont'd)

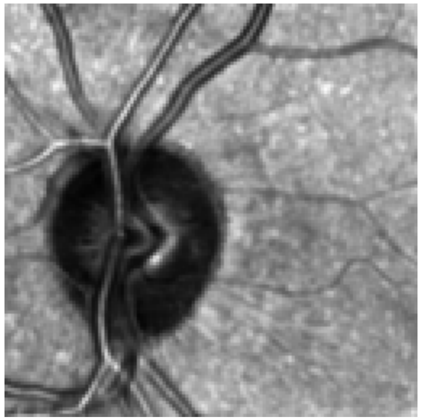
- Morphological gradient

$$g = (f \oplus b) - (f \ominus b)$$

Property: not sensitive to edge direction when using symmetric b

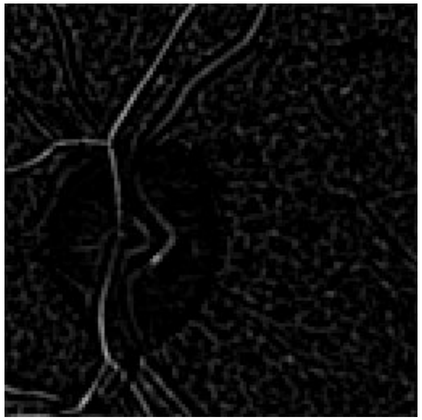
- Top hat

Analog of linear Laplacian



For bright-feature detection

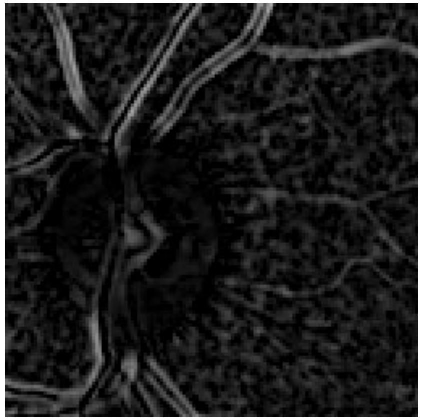
$$g = f - (f \circ b)$$



disk, r=1

For dark-feature detection

$$g = (f \bullet b) - f$$



disk, r=3

References

A.K. Jain, *Fundamentals of Digital Image Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

W.K. Pratt, *Digital Image Processing*. New York: Wiley, 1991.

J. Serra, *Image Analysis and Mathematical Morphology*. Academic Press, 1982.