Microtechnique: 2025-2026 Master Semester 1
Fundamental of integrated photonic components

Solution for Assignment #2

Optical fibers

Problem 1 (30 points):

Boundary conditions of a step-index fiber

Guided modes inside a cylindrical, step-index fiber with cylindrical symmetry ($n = n_1, r \leq a$ and $n = n_2, r > a$), must obey the Helmholtz equation in the cylindrical coordinates:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} + n^2 k_0^2 U = 0 \tag{1}$$

In class, we found that guided modes must be of the shape $U(r,\phi,z) = u(r)e^{il\phi}e^{i\beta z}$, $l = \pm 1, \pm 2, ...$, traveling in the z-direction with propagation constant β. The cylindrical symmetry for the refractive index leads to bounded solutions for the core (with refractive index n_1 and radius of a) and the cladding (with refractive index n_2) given by the Bessel functions:

$$E_z(r,\phi,z) = A_1 J_l(k_\perp r)e^{il\phi}e^{i\beta z}, r \leq a \tag{2}$$

$$E_z(r,\phi,z) = C_1 K_l(\gamma r)e^{il\phi}e^{i\beta z}, r > a \tag{3}$$

and

$$H_z(r,\phi,z) = A_2 J_l(k_\perp r)e^{il\phi}e^{i\beta z}, r \leq a \tag{4}$$

$$H_z(r,\phi,z) = C_2 K_l(\gamma r)e^{il\phi}e^{i\beta z}, r > a \tag{5}$$

where $k_\perp^2 = n_1^2 k_0^2 - \beta^2$ and $\gamma^2 = \beta^2 - n_2^2 k_0^2$. After some math, we find that we can express all other components E_r, E_ϕ, H_r and H_ϕ in terms of derivatives of z-components E_z and H_z as:

$$E_r(r,\phi,z) = \frac{i}{k_0^2\epsilon_r - \beta^2}\left(\frac{\omega\mu_0}{r}\frac{\partial H_z}{\partial \phi} + \beta\frac{\partial E_z}{\partial r}\right) \tag{6}$$

$$E_\phi(r,\phi,z) = \frac{i}{k_0^2\epsilon_r - \beta^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu_0\frac{\partial H_z}{\partial r}\right) \tag{7}$$

$$H_r(r,\phi,z) = \frac{i}{k_0^2\epsilon_r - \beta^2}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\epsilon_0\epsilon_r}{r}\frac{\partial E_z}{\partial \phi}\right) \tag{8}$$

$$H_\phi(r,\phi,z) = \frac{i}{k_0^2\epsilon_r - \beta^2}\left(\frac{\beta}{r}\frac{\partial H_z}{\partial \phi} + \omega\epsilon_0\epsilon_r\frac{\partial E_z}{\partial r}\right) \tag{9}$$

Starting from here, demonstrate that the following transcendental equation describes the discrete set of modes inside a fiber:

$$(\frac{1}{X^2}+\frac{1}{Y^2})^2\frac{\beta^2 l^2}{k_0^2} = (\frac{J_l'(X)}{XJ_l(X)}+\frac{K_l'(Y)}{YK_l(Y)})(n_1^2\frac{J_l'(X)}{XJ_l(X)}+n_2^2\frac{K_l'(Y)}{YK_l(Y)}) \tag{10}$$

where X, Y, and V parameters are defined as: $X = k_\perp a$, $Y = \gamma a$, $V = \sqrt{X^2+Y^2} = k_0 a\sqrt{n_1^2-n_2^2}$. Proceed as follows:

1. Write down the set of four equations that represent the continuity of E_z, H_z, E_ϕ and H_ϕ at $r = a$ and derive equation 10. (**20 points**)

2. Simplify for the case of $l = 0$. Which conditions must be satisfied? What happens in the case that $n_1 \simeq n_2$? (**10 points**)

Solution:

1. We can apply the boundary conditions with regards to the continuity of E_z, H_z, E_ϕ and H_ϕ since they are all tangential to the interface. We have therefore at $r = a$

$$A_1 J_l(k_\perp a) = C_1 K_l(\gamma a), E_z - \text{cont.} \tag{11}$$

$$A_2 J_l(k_\perp a) = C_2 K_l(\gamma a), H_z - \text{cont.} \tag{12}$$

$$\frac{1}{k_0^2 n_1^2-\beta^2}(\frac{\beta}{a} il A_1 J_l(k_\perp a) - \omega\mu_0 A_2 k_\perp J_l'(k_\perp a)) = \frac{1}{k_0^2 n_2^2-\beta^2}(\frac{\beta}{a} il C_1 K_l(\gamma a) - \omega\mu_0 C_2 \gamma K_l'(\gamma a)), E_\phi - \text{cont.} \tag{13}$$

$$\frac{1}{k_0^2 n_1^2-\beta^2}(\frac{\beta}{a} il A_2 J_l(k_\perp a) + \omega\epsilon_0 n_1^2 A_1 k_\perp J_l'(k_\perp a)) = \frac{1}{k_0^2 n_2^2-\beta^2}(\frac{\beta}{a} il C_2 K_l(k_\perp a) + \omega\epsilon_0 n_2^2 C_1 \gamma K_l'(\gamma a)), H_\phi - \text{cont.} \tag{14}$$

By further using that $k_0^2 n_1^2 - \beta^2 = k_\perp^2$ and $k_0^2 n_2^2 - \beta^2 = -\gamma^2$, introducing generic variables $X = k_\perp a$ and $Y = \gamma a$ that trivially satisfy $X^2 + Y^2 = V^2 = k_0^2 a^2 (n_1^2 - n_2^2)$, and substituting $C_1 = \frac{A_1 J_l(k_\perp a)}{K_l(\gamma a)}$ and $C_2 = \frac{A_2 J_l(k_\perp a)}{K_l(\gamma a)}$ we simplify:

$$\frac{1}{X^2}(\beta il A_1 J_l(X) - \omega\mu_0 A_2 X J_l'(X)) = \frac{-1}{Y^2}(\beta il A_1 J_l(X) - \omega\mu_0 Y \frac{A_2 J_l(X)}{K_l(Y)} K_l'(Y)) \tag{15}$$

$$\frac{1}{X^2}(\beta il A_2 J_l(X) + \omega\epsilon_0 n_1^2 A_1 X J_l'(X)) = \frac{-1}{Y^2}(\beta il A_2 J_l(X) + \omega\epsilon_0 n_2^2 \frac{A_1 J_l(X)}{K_l(Y)} Y K_l'(Y)) \tag{16}$$

After some cumbersome reshuffling we get

$$(\frac{1}{X^2}+\frac{1}{Y^2})\beta il A_1 = \omega\mu_0 A_2(\frac{J_l'(X)}{XJ_l(X)}+\frac{K_l'(Y)}{YK_l(Y)}) \tag{17}$$

$$(\frac{1}{X^2}+\frac{1}{Y^2})\beta il A_2 = \omega\epsilon_0 A_1(-n_1^2\frac{J_l'(X)}{XJ_l(X)}-n_2^2\frac{K_l'(Y)}{YK_l(Y)}) \tag{18}$$

and FINALLY the transcendental equation, substituting $\omega^2\mu_0\epsilon_0 = k_0^2$

$$\begin{aligned}
&(\frac{1}{X^2} + \frac{1}{Y^2})^2\frac{\beta^2 l^2}{k_0^2} \\
&= (\frac{J_l'(X)}{XJ_l(X)} + \frac{K_l'(Y)}{YK_l(Y)})(n_1^2\frac{J_l'(X)}{XJ_l(X)} + n_2^2\frac{K_l'(Y)}{YK_l(Y)}) \qquad (19)
\end{aligned}$$

This is the most generic equation that describes the possible solutions for X and Y for any a, n_1 and n_2.

2. We can now solve this for the special case of $l = 0$ (in this course we will restrict our discussion to this, but note that an entire zoology of modes can exist in a fiber).

We can easily see that one of the following two equations need to be fulfilled:

$$\frac{J_0'(X)}{XJ_0(X)} + \frac{K_0'(Y)}{YK_0(Y)} = 0, or \qquad (20)$$

$$n_1^2\frac{J_0'(X)}{XJ_0(X)} + n_2^2\frac{K_0'(Y)}{YK_0(Y)} = 0 \qquad (21)$$

If, in addition, we only consider the subset of fibers with low index contrast where $n_1 \simeq n_2 \simeq n_{eff} = n$ we find therefore

$$\frac{J_0'(X)}{XJ_0(X)} \simeq -\frac{K_0'(Y)}{YK_0(Y)} \qquad (22)$$

We can further simplify by using the identity for the derivatives of the Bessel functions

$$J_l'(X) = \frac{l}{X}J_l(X) - J_{l+1}(X) \qquad (23)$$

$$K_l'(Y) = \frac{l}{Y}K_l(Y) - K_{l+1}(Y) \qquad (24)$$

and find

$$\frac{J_1(X)}{XJ_0(X)} \simeq -\frac{K_1(Y)}{YK_0(Y)} \qquad (25)$$

Problem 2 (70 points):

Simulation of a step-index fiber Simulate a single mode fiber (Throlabs, SMF630HP) as depicted in Figure 1 for the following values: $d_{core} = 3.8\ \mu m$, $d_{clad} = 20\ \mu m$, and $L_{fiber} = 0.1\ \mu m$. Use the dispersion parameters for the dielectric constant ϵ for the core and cladding provided in disp_core.txt and disp_clad.txt. In order to load these files into the definition of new materials in CST, refer to the Manual at page 12. [Suggestion: to reduce computation time, use a mesh size of 2 cells per wavelength for all simulations.]

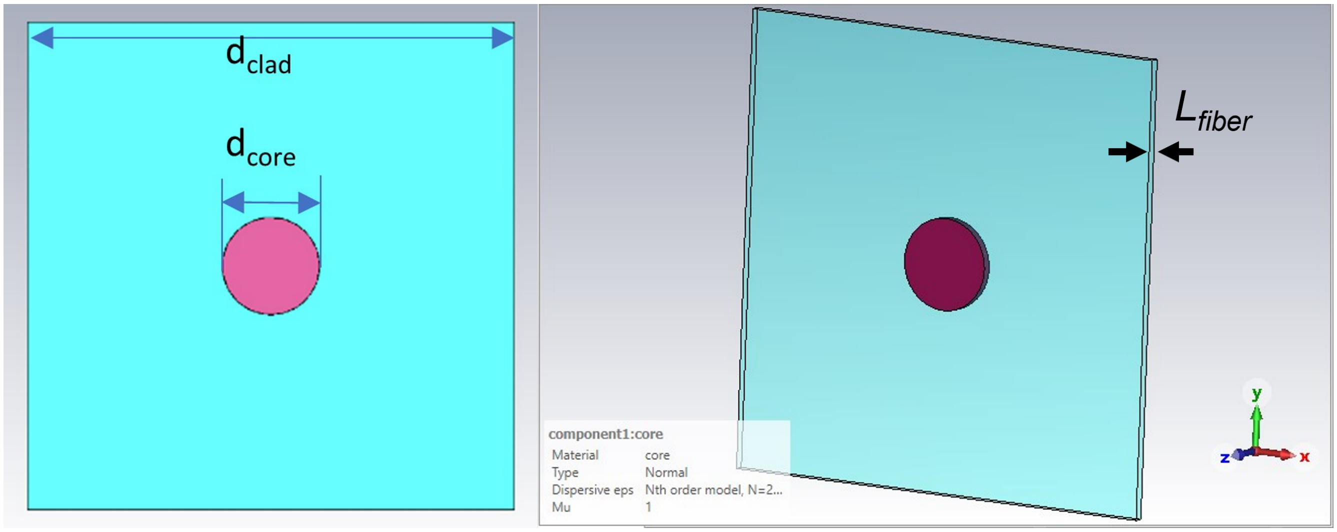

Figure 1: Schematics of the fiber layout.

(a) Plot the 2D mode profiles of the first three modes (in both *Arrows* and *Contour* visual mode) at $\lambda = 0.686\ \mu$m. Which one is (are) the fundamental mode(s)? (**20 points**).

(b) Plot the line cuts of the mode profiles (absolute value, 1D plot) of these first three modes at (X, Y=0, Z=0) in one single panel. Compare the three cases (**20 points**).

(c) Similarly to the simulation exercise for the ridge waveguide, repeat the simulation for five wavelengths (0.6, 0.625, 0.65, 0.675, 0.7) μm and plot the effective refractive index (n_{eff}) for the fundamental mode(s) of each wavelength. (**15 points**)

(d) Repeat the simulation to now retrieve the group refractive index (n_g) of the fundamental mode as a function of the wavelength and plot it. [In Setup Solver, use 5 samples Equidistant in the wavelength range 0.6 and 0.7 μm] (**15 points**).

Solution:

(a) 2D mode profiles:

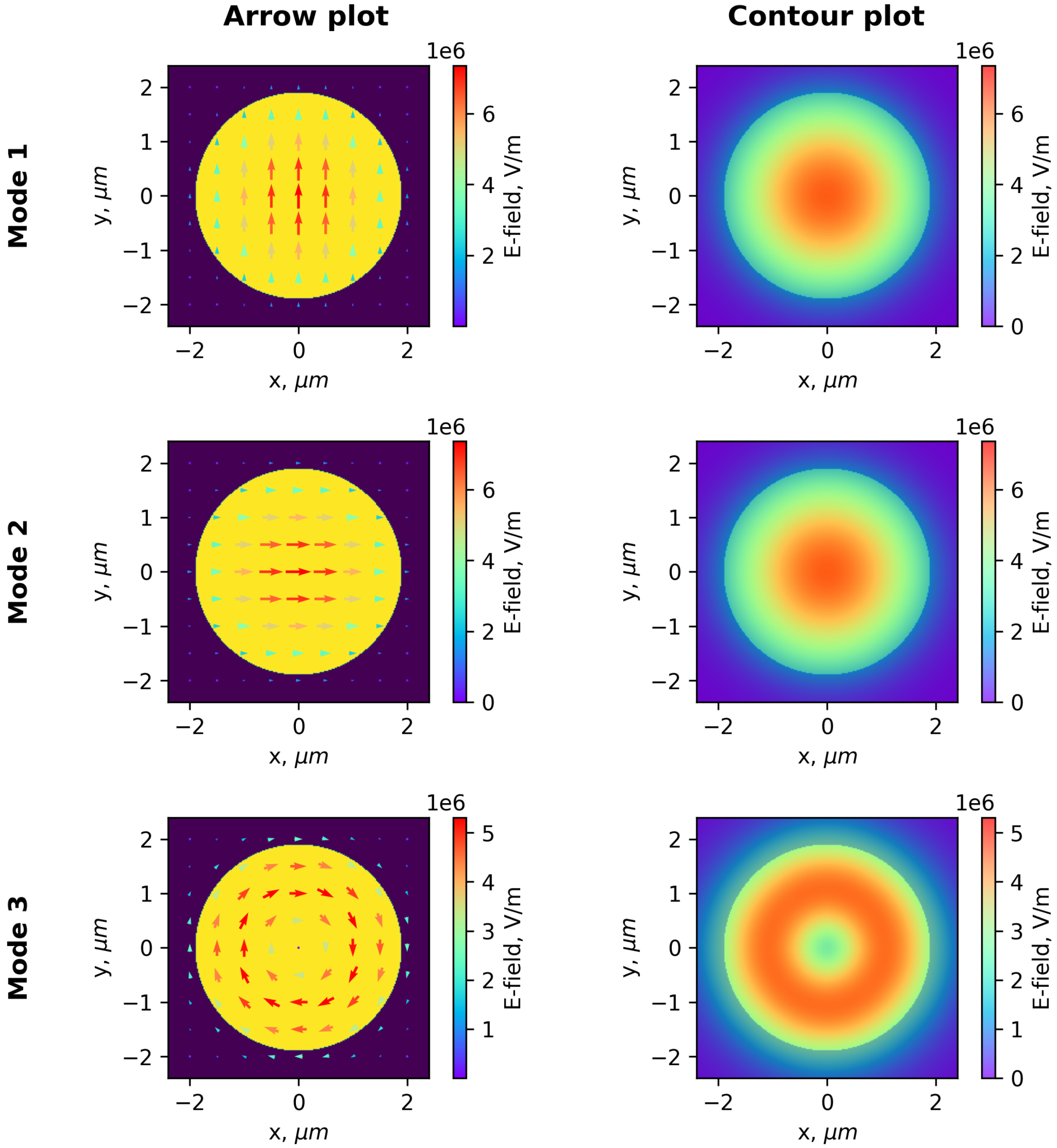

Figure 2: 2D mode profile (in arrow and contour visual mode) for the first three modes of a fiber.

(b) 1D profile mode (cross-section at X, Y=0, Z=0):

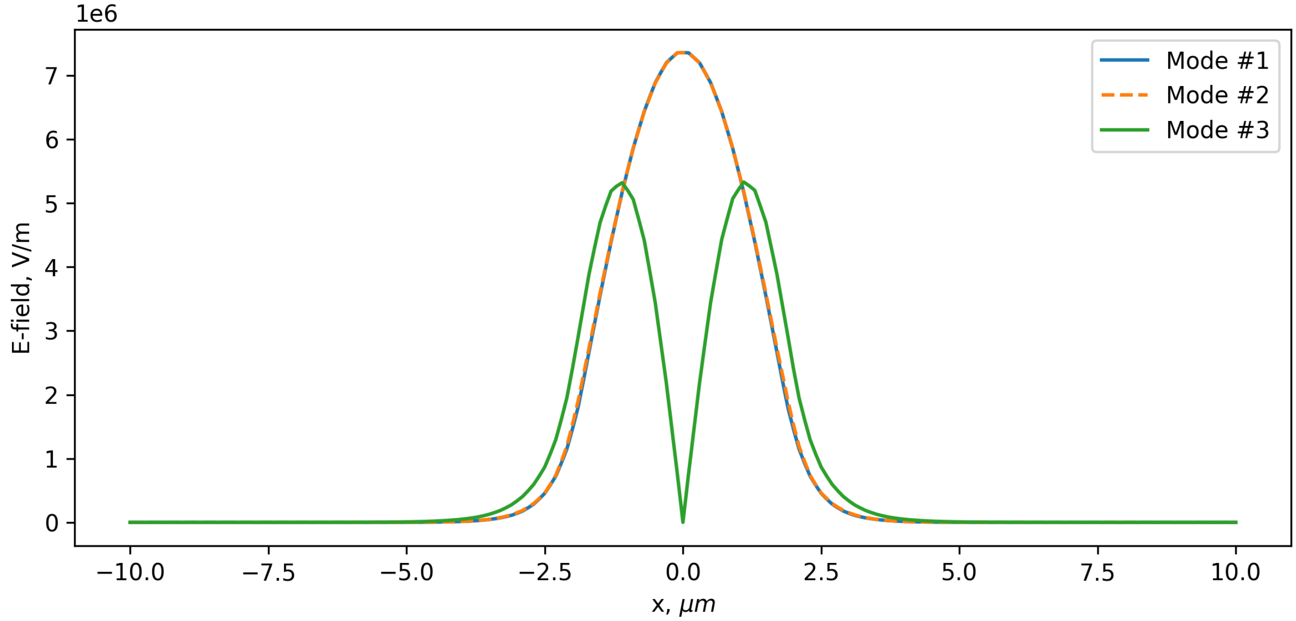

Figure 3: The cross-section (1D plot) of the first three modes on the same plot at (X, Y=0, Z=0). The first two modes (red and green curves) are degenerate and their electric field amplitudes (absolute value) are the same. The electric field amplitude of the third mode is the blue curve. As shown, the third mode has a lower amplitude but has a longer extension compared to the first two degenerate modes.

(c) n_{eff} and n_g directly calculated from CST as a function of the wavelength

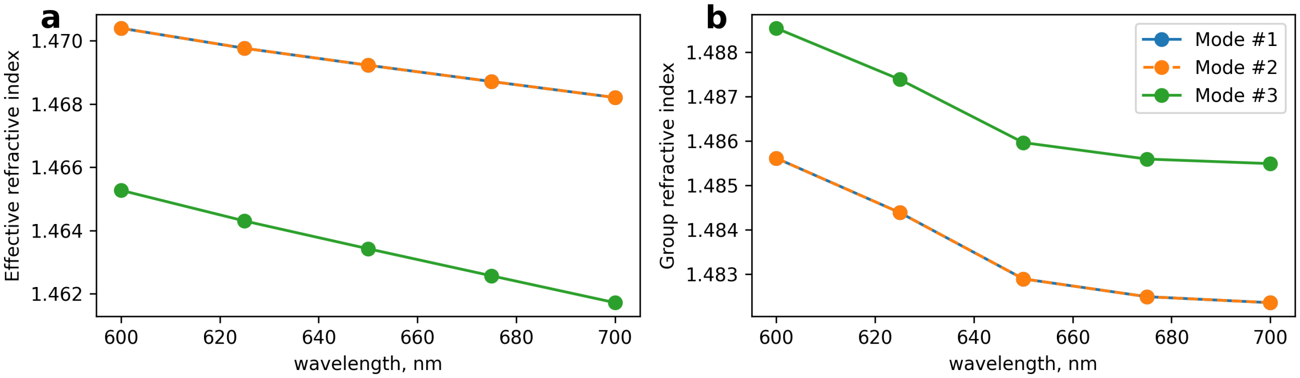

Figure 4: n_{eff} (red open circle) and n_g (blue open circle) vs. wavelength. The group index is retrieved numerically via a numerical algorithm.