Solution for Assignment #1
Dielectric interfaces and slab waveguide

Problem 1 (40 points, bonus):

Consider an electromagnetic wave impinging on a dielectric interface between two materials as described in Fig. 1 below:

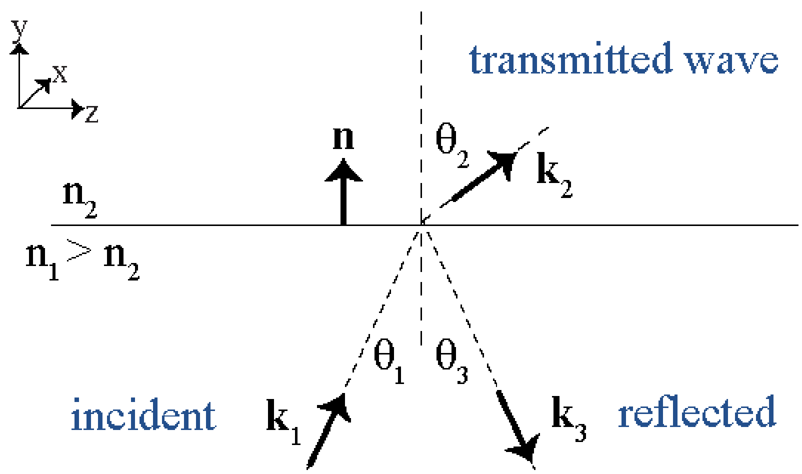

Figure 1: Reflection and transmission at an interface between two media.

By applying the following set of boundary conditions to the two possible polarization (TE and TM, see Fig. 2):

$$(\mathbf{E}_i + \mathbf{E}_r - \mathbf{E}_t) \times \mathbf{n} = 0 \quad (1)$$
$$(\mathbf{H}_i + \mathbf{H}_r - \mathbf{H}_t) \times \mathbf{n} = 0 \quad (2)$$
$$(n_1^2\mathbf{E}_i + n_1^2\mathbf{E}_r - n_2^2\mathbf{E}_t) \cdot \mathbf{n} = 0 \quad (3)$$
$$(\mathbf{B}_i + \mathbf{B}_r - \mathbf{B}_t) \cdot \mathbf{n} = 0. \quad (4)$$

Show that the reflection and transmission coefficients for the two polarizations in terms of the material refractive indices, incidence, reflection and refractive angles, are expressed as:

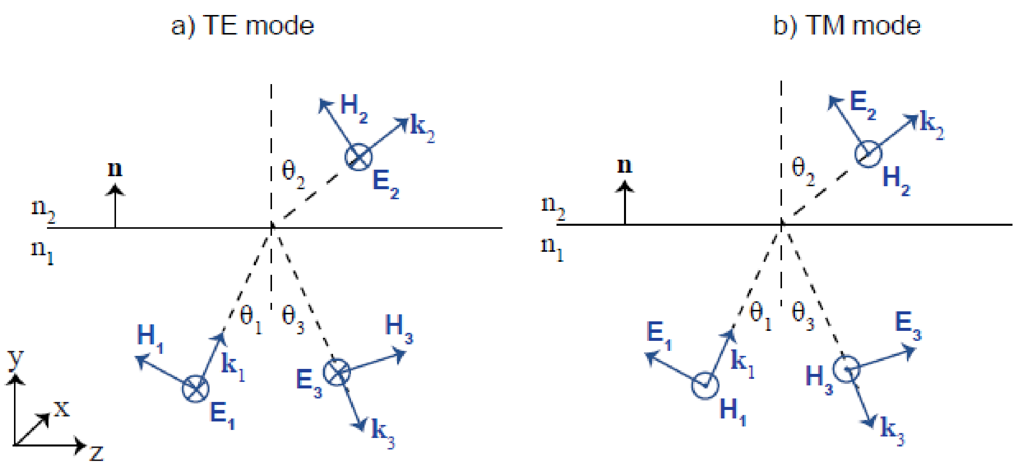

Figure 2: TE and TM polarized waves.

$$r_{TE} = \frac{|\mathbf{E}_3|}{|\mathbf{E}_1|} = \frac{k_{1y} - k_{2y}}{k_{1y} + k_{2y}} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \quad \text{(10 points)} \tag{5}$$

$$t_{TE} = \frac{|\mathbf{E}_2|}{|\mathbf{E}_1|} = 1 + r_{TE} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \quad \text{(10 points)} \tag{6}$$

$$r_{TM} = \frac{|\mathbf{E}_3|}{|\mathbf{E}_1|} = \frac{n_2^2 k_{1y} - n_1^2 k_{2y}}{n_2^2 k_{1y} + n_1^2 k_{2y}} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \quad \text{(10 points)} \tag{7}$$

$$t_{TM} = \frac{|\mathbf{E}_2|}{|\mathbf{E}_1|} = \frac{n_1}{n_2}(1 + r_{TM}) = \frac{2n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \quad \text{(10 poitns)} \tag{8}$$

Assume all materials with an unitary magnetic permeability, i.e., $\mu_{1,2} = 1$, whereas the refractive index is simply the square root of the electrical permittivity i.e, $n_{1,2} = \sqrt{\epsilon_{1,2}}$.

Solution:

1.**Parallel Polarization (TM):** The wave vector of the incident wave can be decomposed into its two Cartesian components as:

$$\mathbf{k_1} = k_1 \cos\theta_1 \hat{z} + k_1 \sin\theta_1 \hat{y} \tag{9}$$

For a parallel polarization (i.e., electric field lying on the plane of incidence), which is a transverse magnetic (TM) polarization, it is more convenient to operate with the magnetic field components:

$$\mathbf{H_i} = \mathbf{H_1} = H_i e^{j\mathbf{k_1}\cdot\mathbf{r}} \hat{x} = \frac{E_i}{\eta_1} e^{j\mathbf{k_1}\cdot\mathbf{r}} \hat{x} \tag{10}$$

$$\mathbf{H_t} = \mathbf{H_2} = H_t e^{j\mathbf{k_2}\cdot\mathbf{r}} \hat{x} = \frac{E_t}{\eta_2} e^{j\mathbf{k_2}.\mathbf{r}} \hat{x} \tag{11}$$

$$\mathbf{H_r} = \mathbf{H_3} = H_r e^{j\mathbf{k_3}\cdot\mathbf{r}} \hat{x} = \frac{E_r}{\eta_2} e^{j\mathbf{k_3}\cdot\mathbf{r}} \hat{x} \tag{12}$$

where $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $\mathbf{k_{1,2}} = \mathbf{k_{i,t}} = k_{1,2} (\hat{z}\cos\theta_{1,2} + \hat{y}\sin\theta_{1,2})$, $\mathbf{k_3} = \mathbf{k_r} = k_3 (\hat{z}\cos\theta_3 - \hat{y}\sin\theta_3)$, and $\eta_{1,2} = \sqrt{\frac{\mu_{1,2}}{\epsilon_{1,2}}} = \frac{1}{n_{1,2}}\sqrt{\frac{\mu_0}{\epsilon_0}}$ is the wave impedance for a non-ferromagnetic material ($\mu_r = 1$). Let us also consider the k-vectors be like $k_{1,2,3\,z} = k_{1,2,3}\cos\theta_{1,2,3}$ and $k_{1,2,3\,y} = k_{1,2,3}\sin\theta_{1,2,3}$. From the Maxwell equation we can relate the electric field components to those of the magnetic field as it follows:

$$\mathbf{E_i} = \mathbf{E_1} = E_i\left[-k_{1y}\hat{z} + k_{1z}\hat{y}\right]e^{i\mathbf{k_1}\cdot\mathbf{r}} \tag{13}$$
$$\mathbf{E_t} = \mathbf{E_2} = E_t\left[-k_{2y}\hat{z} + k_{2z}\hat{y}\right]e^{i\mathbf{k_2}\cdot\mathbf{r}} \tag{14}$$
$$\mathbf{E_r} = \mathbf{E_3} = E_r\left[-k_{3y}\hat{z} - k_{3z}\hat{y}\right]e^{i\mathbf{k_3}\cdot\mathbf{r}} \tag{15}$$

which, by considering $E_i = E_0$, $E_t = t_{TM}E_0$ and $E_r = r_{TM}E_0$, and expanding the scalar products, will be re-written like

$$\mathbf{E_i} = \mathbf{E_1} = E_0\left[k_{1y}\hat{z} - k_{1z}\hat{y}\right]e^{i\left(k_{1z}z + k_{1y}y\right)} \tag{16}$$
$$\mathbf{E_t} = \mathbf{E_2} = t_{TM}E_0\left[k_{2y}\hat{z} - k_{2z}\hat{y}\right]e^{i\left(k_{2z}z + k_{2y}y\right)} \tag{17}$$
$$\mathbf{E_r} = \mathbf{E_3} = r_{TM}E_0\left[-k_{3y}\hat{z} - k_{3z}\hat{y}\right]e^{i\left(k_{3z}z - k_{3y}y\right)} \tag{18}$$

At the interface $y = 0$ the resultant electric field tangential component must be conserved. This means that the tangential component of the resultant of all electric fields at the boundary of the $y \geqslant 0$ semi-plane, must be equal to the tangential component of the resultant of all electric fields at the boundary of the $y \leqslant 0$ semi-plane:

$$E_{1z}|_{y=0} + E_{3z}|_{y=0} = E_{2z}|_{y=0} \tag{19}$$

and by substituting the expressions for the electric fields:

$$E_{1z}e^{ik_{1z}z} + E_{3z}e^{ik_{3z}z} = E_{2z}e^{k_{2z}z} \;\rightarrow\; E_{1z}e^{ik_1 z\cos\theta_1} + E_{3z}e^{ik_3 z\cos\theta_3} = E_{2z}e^{ik_2 z\cos\theta_2} \tag{20}$$

This condition should hold anywhere at the interface, i.e., it must be an invariant of z. Therefore, the arguments of the exponential terms must be all equal:

$$ik_1 z\cos\theta_1 = ik_3 z\cos\theta_3 = ik_2 z\cos\theta_2 \tag{21}$$

In other words, the z component of the k-vector must be conservative:

$$k_{1z} = k_{2z} = k_{3z} \tag{22}$$

If now we consider equation (20), after imposing all the arguments in the exponents equal to each other, we have:

$$E_0\left(k_{1y} - r_{TM}k_{3y}\right) = E_0 k_{2y}t_{TM} \;\rightarrow\; k_{2y}t_{TM} + k_{3y}r_{TM} = k_{1y} \tag{23}$$

Now, boundary conditions for tangential magnetic fields can be written as:

$$(\mathbf{H_i} + \mathbf{H_r} - \mathbf{H_t})\times\mathbf{n} = 0 \rightarrow\; H_{1x}|_{y=0} + H_{3x}|_{y=0} = H_{2x}|_{y=0} \rightarrow\; n_1 E_{1x}e^{jk_1 z\cos\theta_1} + n_1 E_{3x}e^{jk_3 z\cos\theta_3} = n_2 E_{2y}e^{jk_2 z\cos\theta_2} \tag{24}$$

So, we must have:

$$n_1 E_0\left(1 + r_{TM}\right) = n_2 E_0 t_{TM} \rightarrow\; \frac{n_1}{n_2}(1 + r_{TM}) = t_{TM} \tag{25}$$

Now, we simply have a linear system of equations to solve:

$$k_{2y}t_{TM} + k_{3y}r_{TM} = k_{1y} \tag{26}$$
$$\frac{n_1}{n_2}(1 + r_{TM}) = t_{TM} \tag{27}$$

whose solution will be

$$r_{TM} = \frac{|\mathbf{E}_3|}{|\mathbf{E}_1|} = \frac{n_2^2 k_{1y} - n_1^2 k_{2y}}{n_2^2 k_{1y} + n_1^2 k_{2y}} = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2} \tag{28}$$
$$t_{TM} = \frac{|\mathbf{E}_2|}{|\mathbf{E}_1|} = \frac{n_1}{n_2}(1 + r_{TM}) = \frac{2n_1\cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2} \tag{29}$$

We note that these coefficients have been derived as ratios between electric fields, but they do not hold for magnetic fields, since they would assume a slightly different form in the latter case. Refer to the script for a difference between E-field and H-fields cases.

Perpendicular Polarization (TE):
For a perpendicular polarization, i.e. for a wave whose electric field polarization is orthogonal to the plane of incidence, one can write the electric fields as:

$$\mathbf{E_i} = \mathbf{E_1} = E_0 e^{j\mathbf{k_1}\cdot\mathbf{r}}\hat{x} \tag{30}$$
$$\mathbf{E_t} = \mathbf{E_2} = t_{TE} E_0 e^{j\mathbf{k_2}\cdot\mathbf{r}}\hat{x} \tag{31}$$
$$\mathbf{E_r} = \mathbf{E_3} = r_{TE} E_0 e^{j\mathbf{k_3}\cdot\mathbf{r}}\hat{x} \tag{32}$$

and the magnetic fields are written via Maxwell equations as:

$$\mathbf{H_i} = \mathbf{H_1} = H_0\left[-k_{1y}\hat{z} + k_{1z}\hat{y}\right]e^{j\mathbf{k_1}\cdot\mathbf{r}} \tag{33}$$
$$\mathbf{H_t} = \mathbf{H_2} = t_{TE}H_0\left[-k_{2y}\hat{z} + k_{2z}\hat{y}\right]e^{j\mathbf{k_2}\cdot\mathbf{r}} \tag{34}$$

$$\mathbf{H_r} = \mathbf{H_3} = r_{TE}H_0\left[-k_{3y}\hat{z} - k_{3z}\hat{y}\right]e^{j\mathbf{k_3}\cdot\mathbf{r}} \tag{35}$$

With similar consideration as the TM case, we achieve:

$$\mathbf{H_i} = \mathbf{H_1} = H_0\left[k_{1y}\hat{z} - k_{1z}\hat{y}\right]e^{j\left(k_{1z}z + k_{1y}y\right)} \tag{36}$$
$$\mathbf{H_t} = \mathbf{H_2} = t_{TE}H_0\left[k_{2y}\hat{z} - k_{2z}\hat{y}\right]e^{j\left(k_{2z}z + k_{2y}y\right)} \tag{37}$$
$$\mathbf{H_r} = \mathbf{H_3} = r_{TE}H_0\left[-k_{3y}\hat{z} - k_{3z}\hat{y}\right]e^{j\left(k_{3z}z - k_{3y}y\right)} \tag{38}$$

Considering the Equations (30) to (35), we can apply the same approach for calculating reflection and transmission coefficients of TE polarized light, thus getting:

$$r_{TE} = \frac{|\mathbf{E}_3|}{|\mathbf{E}_1|} = \frac{k_{1y} - k_{2y}}{k_{1y} + k_{2y}} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} \tag{39}$$

$$t_{TE} = \frac{|\mathbf{E}_2|}{|\mathbf{E}_1|} = 1 + r_{TE} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} \tag{40}$$

Problem 2 (30 points):

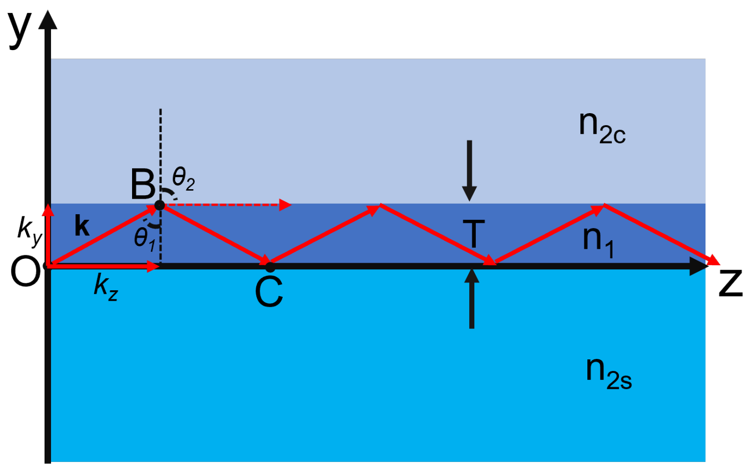

Figure 3: Cross section of a dielectric slab waveguide. In the case of total internal reflection, the electric field of the refracted wave is evanescent and does not propagate. This is indicated by the dashed ray forming the angle θ_2.

1) Consider the slab waveguide depicted in Fig. 3. Calculate the total phase accumulated by a ray with TE polarization, while propagating along a complete round-trip transverse trajectory, as a function of the angle θ_1. In particular, assume θ_1 being larger than the critical angle, so that θ_2 does not exhibit real values (total internal reflection condition). A round-trip transverse travel is, e.g., the path connecting points O, B and C: the wave acquires a finite longitudinal displacement (along z), yet on the transverse axis (y) it comes back to a point with the same original ordinate. The ray is characterized by the wave vector $|\ \mathbf{k}\ | = n_1 k_0$ (initially moving within the core layer), decomposed into its two components k_y and k_z. Let n_{2c}, n_1, n_{2s} be the refractive indices of the cladding, core (core) and substrate layers, respectively. Assume the origin of reference axis system y-z to be located at the interface between the core and substrate. [Note that in the case of total internal reflection, the wave experiences a further phase change any time it hits an interface (such as points B and C, while the origin O is simply treated as a starting point). To calculate this extra phase accumulation utilize the Fresnel reflection coefficients retrieved in the previous problem (conveniently adjusted for the total internal reflection case, i.e. $\cos\theta_2 = \sqrt{1-\sin^2\theta_2}$ being purely imaginary, since $(1-\sin^2\theta_2) < 0$)]. (25 points)
2) Impose that the total phase calculated at the previous point to be equal to an integer multiple of 2π. What kind of relationship does it lead to? (5 points)

Solution:

Let us start by reconsidering Eq. 39, where we explicit the total dependence on the incident angle θ_1, by using Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$:

$$r_{TE} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{n_1\cos\theta_1 - n_2\sqrt{1-\sin^2\theta_2}}{n_1\cos\theta_1 + n_2\sqrt{1-\sin^2\theta_2}} = \frac{n_1\cos\theta_1 - n_2\sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_1}}{n_1\cos\theta_1 + n_2\sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_1}} \tag{41}$$

We recall that, in general the condition of total internal reflection implies the non-existence on an exiting, transmitted ray. When this occurs, we have that $\theta_2 = \pi/2$ and Snell's law returns the critical incidence angle θ_{1cr}:

$$n_1\sin\theta_{1cr} = n_2\sin(\pi/2) \Rightarrow \theta_{1cr} = \arcsin(n_2/n_1) \tag{42}$$

We now note that for any incident angle $\theta_1 > \theta_{1c}$, the quantity under the square root in Eq. 41 becomes negative. As such, the reflection coefficient becomes complex and can be rewritten as it follows:

$$r_{TE}(\theta_1 > \theta_{1c}) = \frac{n_1 \cos\theta_1 - i n_2 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}}{n_1 \cos\theta_1 + i n_2 \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}} = |r_{TE}| \exp\left[-2i \arctan\left(\frac{\sqrt{(n_1 \sin\theta_1)^2 - n_2^2}}{n_1 \cos\theta_1}\right)\right] \tag{43}$$

where we express the coefficient in terms of modulus and phase. What is important to note here is the fact that, conversely to the case of a transmitted ray, in the total internal reflection, the reflected ray does experience a phase change which depends on both the refractive indices, as well as the incident angle. If the second dielectric is replaced by a metal ($n_2 \to \infty$), Eq. 43 reveals that this phase is simply a π-rotation (mirrored propagation). This preliminary digression served to calculate the extra phase contribution that a ray accumulates in a single round-trip travel within the core of the slab waveguide. In particular, the totally internally reflected ray hits two interfaces: core/cladding and core/substrate. We can write a TE reflection coefficient for each of them

$$r_{12c} = |r_{12c}| \exp\left[-2i \arctan\left(\frac{\sqrt{(n_1 \sin\theta_1)^2 - n_{2c}^2}}{n_1 \cos\theta_1}\right)\right] = |r_{12c}| \exp[-i\phi_{12c}] \tag{44}$$

$$r_{12s} = |r_{12s}| \exp\left[-2i \arctan\left(\frac{\sqrt{(n_1 \sin\theta_1)^2 - n_{2s}^2}}{n_1 \cos\theta_1}\right)\right] = |r_{12s}| \exp[-i\phi_{12s}] \tag{45}$$

Now we are ready to write the entire phase accumulated $\Delta\phi$ by the ray along the path OBC marked in Fig. 3:

$$\Delta\phi = n_1 k_y T(\text{y-axis projection, moving up}) + \phi_{12c} + n_1 k_y T(\text{y-axis projection, moving down}) + \phi_{12s} = \tag{46}$$

$$= k_0 T n_1 \cos\theta_1 + \phi_{12c} + k_0 T n_1 \cos\theta_1 + \phi_{12s} \tag{47}$$

where $\phi_{12c/s}$ represent the phase of the coefficient in Eqs. 44 and 45. Plugging all the quantities and expressing $\cos\theta_1$ in terms of $\sin\theta_1$, we have:

$$\Delta\phi = 2k_0 T n_1 \sqrt{1 - \sin^2\theta_1} - 2\arctan\left(\frac{\sqrt{(n_1 \sin\theta_1)^2 - n_{2c}^2}}{n_1 \sqrt{1 - \sin^2\theta_1}}\right) - 2\arctan\left(\frac{\sqrt{(n_1 \sin\theta_1)^2 - n_{2s}^2}}{n_1 \sqrt{1 - \sin^2\theta_1}}\right). \tag{48}$$

Let us define the quantity $n_1 \sin\theta_1 = n_{eff}$, as the *effective refractive index* and rewrite Eq. 48 accordingly:

$$\Delta\phi = 2k_0 T \sqrt{n_1^2 - n_{eff}^2} - 2\arctan\left(\frac{\sqrt{n_{eff}^2 - n_{2c}^2}}{\sqrt{n_1^2 - n_{eff}^2}}\right) - 2\arctan\left(\frac{\sqrt{n_{eff}^2 - n_{2s}^2}}{\sqrt{n_1^2 - n_{eff}^2}}\right). \tag{49}$$

If the condition of the transverse resonance must be established in order to allow for the propagation of a mode inside the waveguide, then the total phase accumulated by a ray along a complete round-trip should be an integer multiple m of 2π, that is:

$$\Delta\phi = 2\pi m \tag{50}$$

By using Eq. 48 along with the trigonometry identity $\arctan(a) + \arctan(1/a) = \pi/2$, we finally get:

$$k_0 T \sqrt{n_1^2 - n_{eff}^2} - \arctan\left(\frac{\sqrt{n_1^2 - n_{eff}^2}}{\sqrt{n_{eff}^2 - n_{2c}^2}}\right) - \arctan\left(\frac{\sqrt{n_1^2 - n_{eff}^2}}{\sqrt{n_{eff}^2 - n_{2s}^2}}\right) = (m+1)\pi. \tag{51}$$

The last relation is named the *characteristic or eigenequation of the waveguide* for TE modes. Once all the design parameters of the waveguide are chosen (n_1, n_{2c}, n_{2s}, T) it can be solved (numerically) for n_{eff} by imposing an increasing value for m, so to determine if and how many modes are supported.

Problem 3 (70 points):

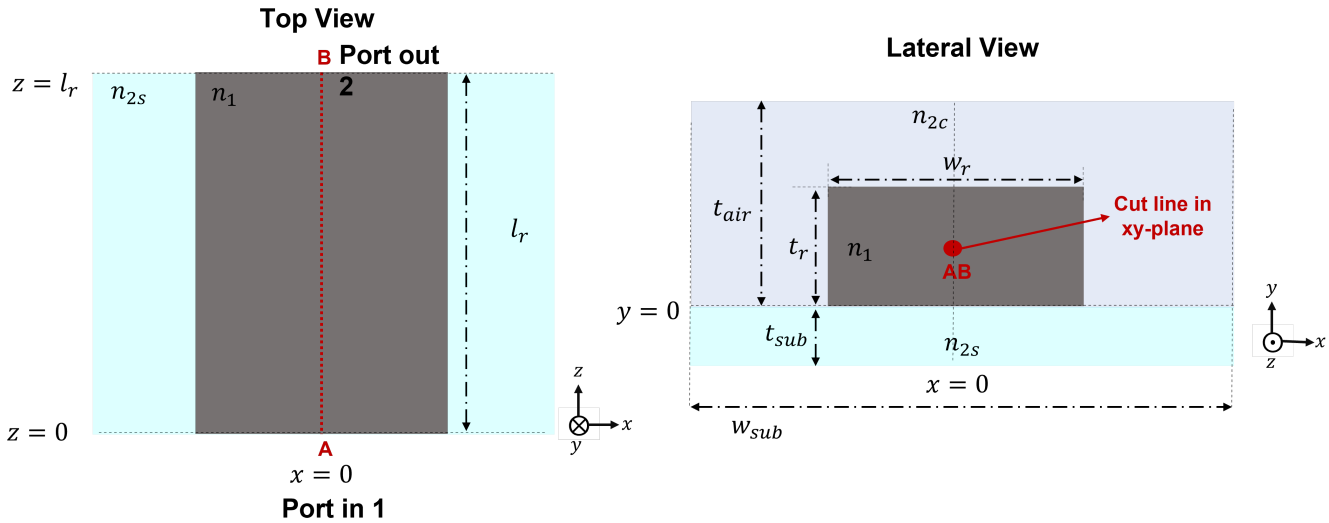

Figure 4: Top view and cross section of a ridge waveguide.

Consider a ridge waveguide, such as that depicted in Figure 4. The parameters in the figure are here listed:
$w_r = 1.2\ \mu m$
$t_r = 0.4\ \mu m$
$t_{sub} = 6\ \mu m$
$t_{air} = 6\ \mu m$
$l_r = 5\ \mu m$
$n_1 = 2.2$
$n_{2c} = 1$
$n_{2s} = 1.5$
$w_{sub} = 10\ \mu m$
Build this structure following the instructions in the manual provided with this assignment.
You may need to adjust the mesh (number of cells per wavelength) to speed up the simulations. Try to keep the total mesh cell numbers (bottom right corner of cst window) smaller than $250,000$.

a) **From the frequency domain solution**, compute the first 3 port modes at the wavelength of 1.5 μm. Calculate the corresponding effective refractive indices and reports the numbers in the solutions. For each mode address the following points:

- Visualize the electric field **components** (X, Y and Z) associated to the mode. Determine which is the dominant polarization of the mode and generate the corresponding 2D plot (*Contour*). (**20 points**)
- Generate the 2D map of the **absolute** value of the electric field associated to the mode (*Contour*). (**10 points**)
- Generate the 2D map of the electric fields **lines** associated to the mode (*Arrows*). (**10 points**)

b) **From the frequency domain solution**, repeat the simulations at various wavelengths (1.1, 1.3, 1.5, 1.7 μm), and trace the trends of the effective and group refractive indices as a function of wavelength . **(10 points)**
Note that the concept of effective and group refractive indices will be explained in the lectures in the coming weeks, but in these simulations you can already obtain the results and reflect on why they show such a trend.

Suggestion: Run separate simulations for each requested wavelength. Use the following wavelength ranges *in your simulations:* [1.1, 1.11], [1.3, 1.31], [1.5, 1.51], *and* [1.7, 1.71], *respectively. To calculate the effective refractive index, you can refer to Section 2.11 of the CST manual (page 18).*

c) **From the time domain solution**, plot the 2D profile of the dominant electric field component in the x–z plane (through the middle of the waveguide along the thickness direction). Then, generate the 1D plot along the AB cutline, as shown in Figure 4 at the middle of the waveguide. (**20 points**)
Suggestions: to generate a 1D cutting line, please refer to Section 3.8 of the CST manual (page 27).

Solution:

(a) With the current waveguide design, the dominant mode polarization is along the x-axis, as shown in Fig.5.The corresponding effective refractive indices of the first three modes are reported in Fig.6.

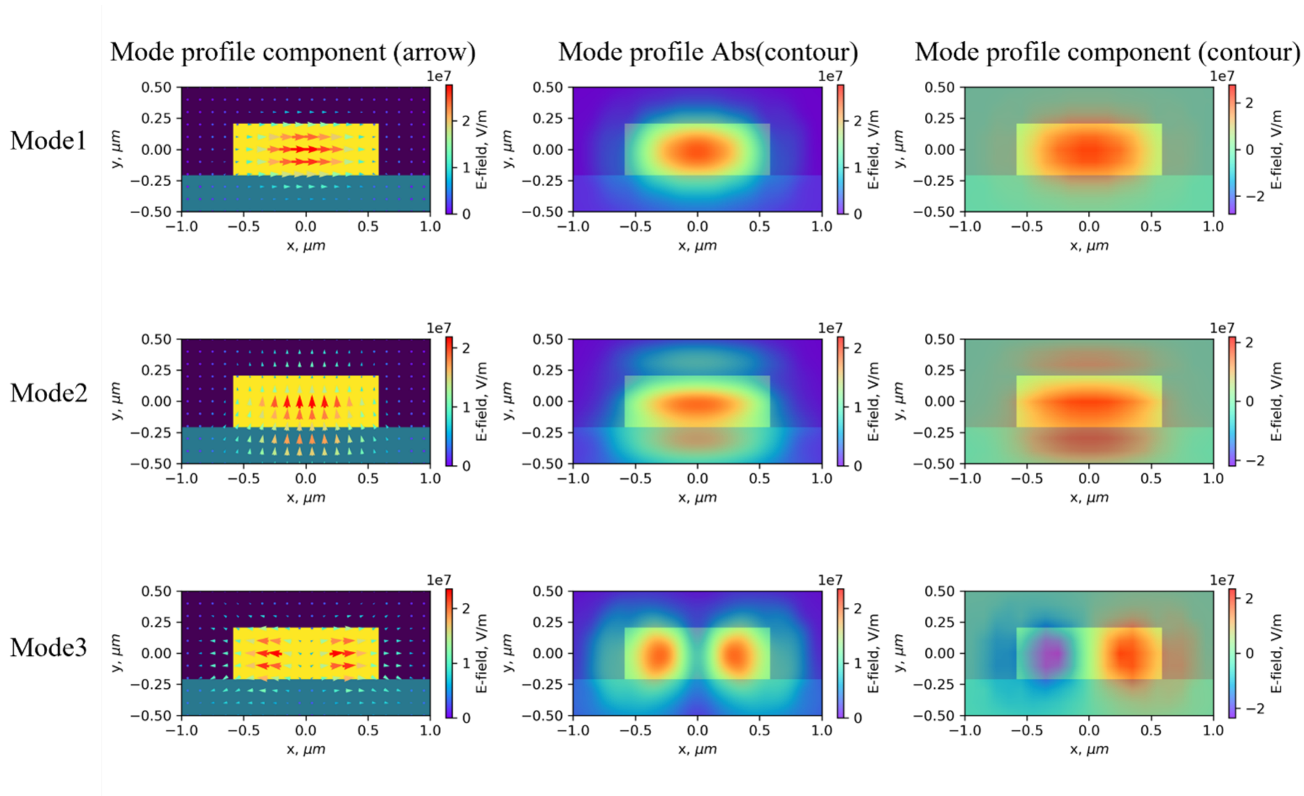

Figure 5: Polarization (left) and mode profiles of the absolute value of the electric field (middle) and the x-component (right) of the first three modes.

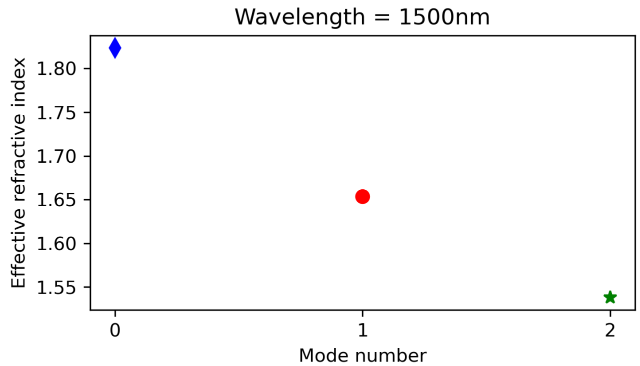

Figure 6: Effective refractive indices of the first three modes.

(b) The effective and group refractive indices at various wavelengths are shown in Fig. 7. As the wavelength increases, the modes spread further into the surrounding low-index materials, leading to a decrease in the effective refractive index. Since dispersive materials are not included in the simulated system, the group index remains constant across different wavelengths.

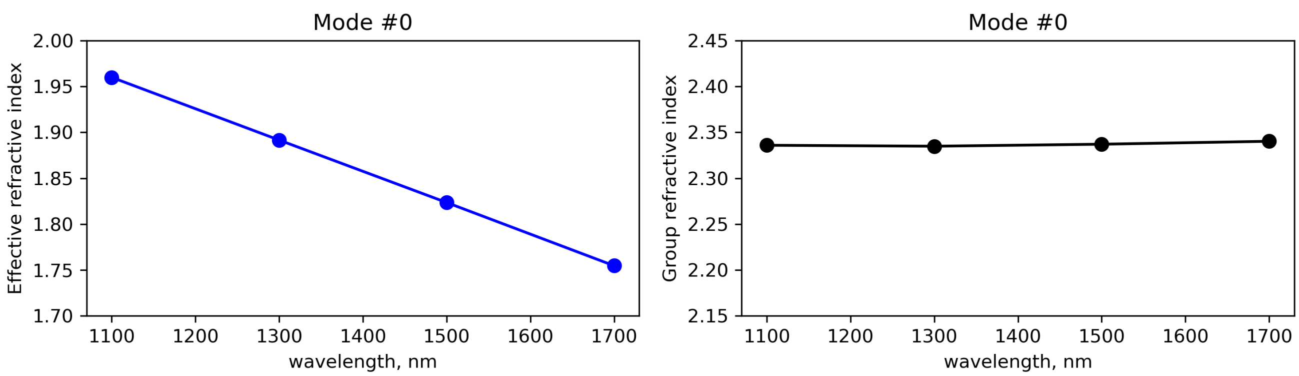

Figure 7: Effective and group refractive indices at various wavelengths.

(c) The dominant electric field component is the x-componnet. Its 2D field distribution along the x-z cutting plane is shown in Fig. 8, and the 1D field along the cuttring line is shown in Fig. 9

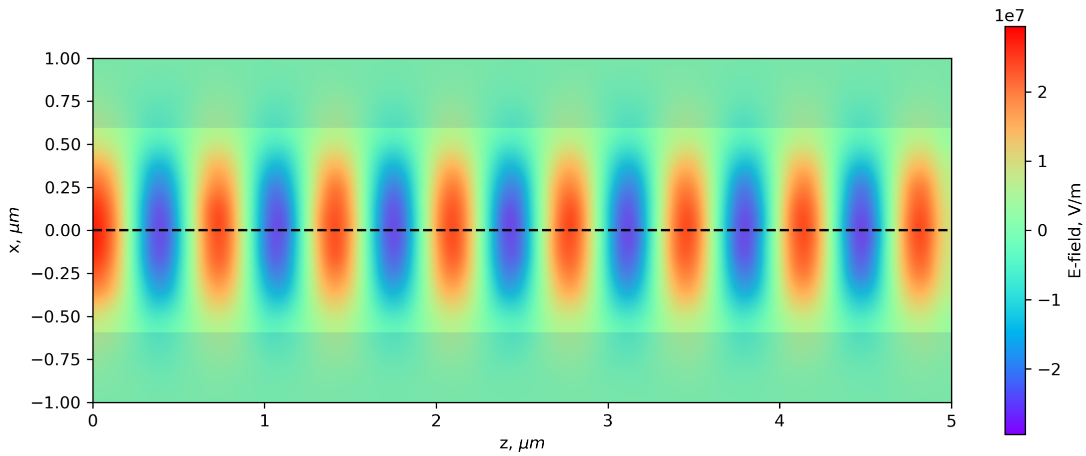

Figure 8: 2D cut-plane of the x-component of the electric field.

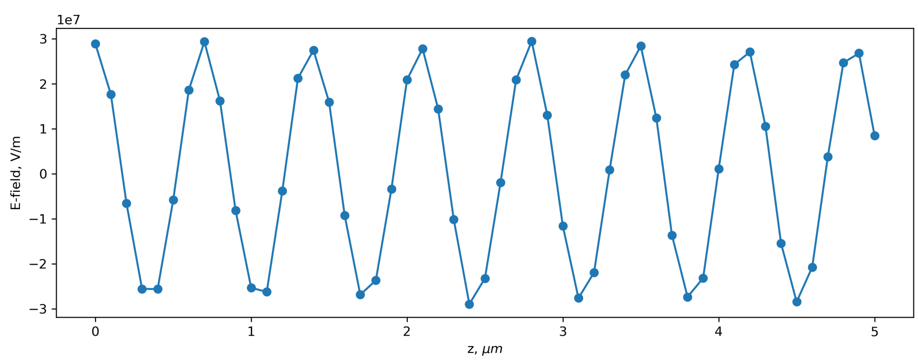

Figure 9: x-component of the electric field along the AB cutline.