3D rotations and Quaternions

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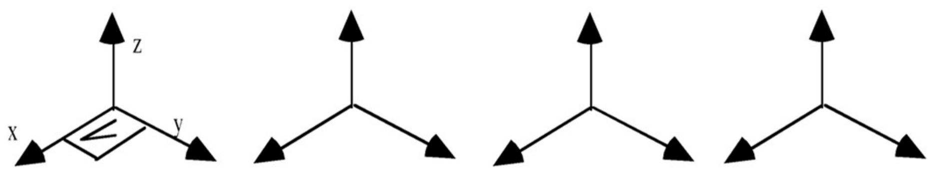

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Objectives of the course

- *Rotations*
  - ~~*Matrices of* ***Direction cosines***~~
  - ***Representation of orientations***
  - ***Rodrigues*** *Formula*
  - *Passage from the rotation matrix to the* ***axis and the angle of rotation***
  - ***Quaternions***

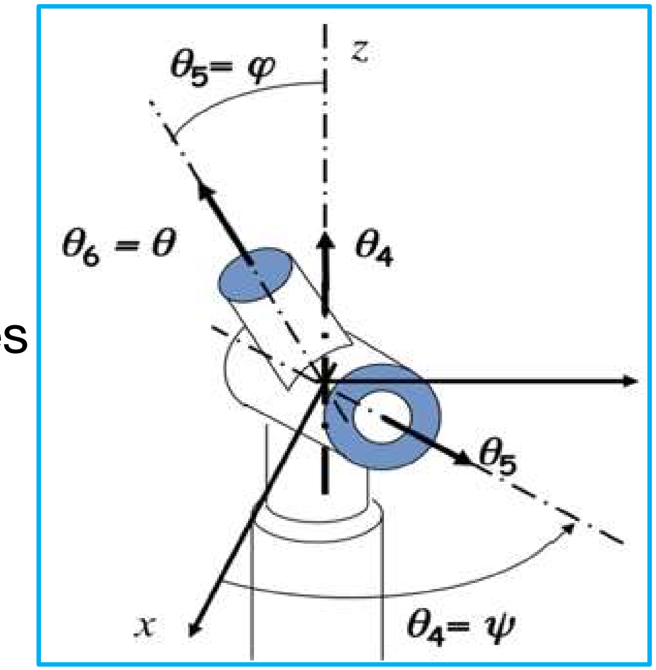
Representation of orientation: Euler angles

Orientation is very often expressed in angles around three axes attached to the moving body.

Historically these are

precession, nutation, proper rotation

(axes linked to the body)



Generalization: There are 12 sets of different sequences

Euler Angles:

Same axis twice

$3 \times 2 \times 1 = 6$ combinations

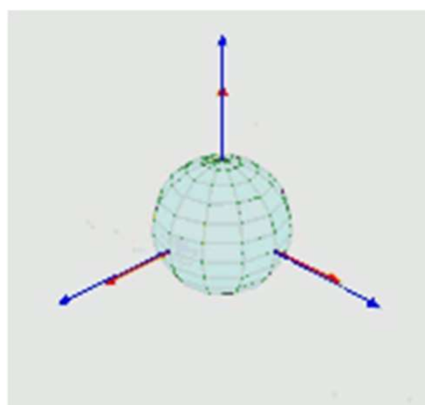
XYX	XYZ	XZX	XZY
YXY	YXZ	YZX	YZY
ZXY	ZXZ	ZYX	ZYZ

Cardan Angles:

All 3 axes

$3 \times 2 \times 1 = 6$ combinations

XYX	XYZ	XZX	XZY
YXY	YXZ	YZX	YZY
ZXY	ZXZ	ZYX	ZYZ



Euler angles, sequence ZXZ

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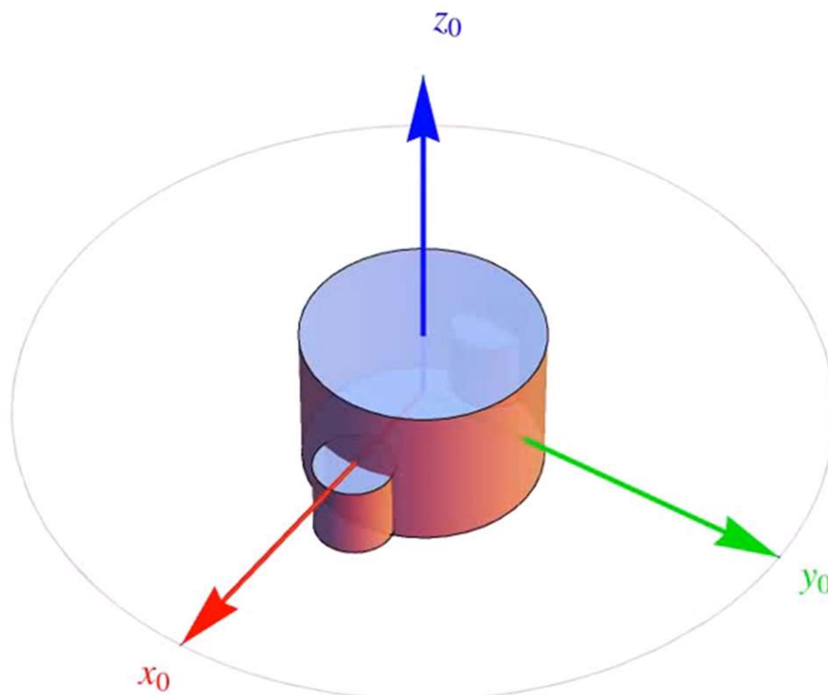


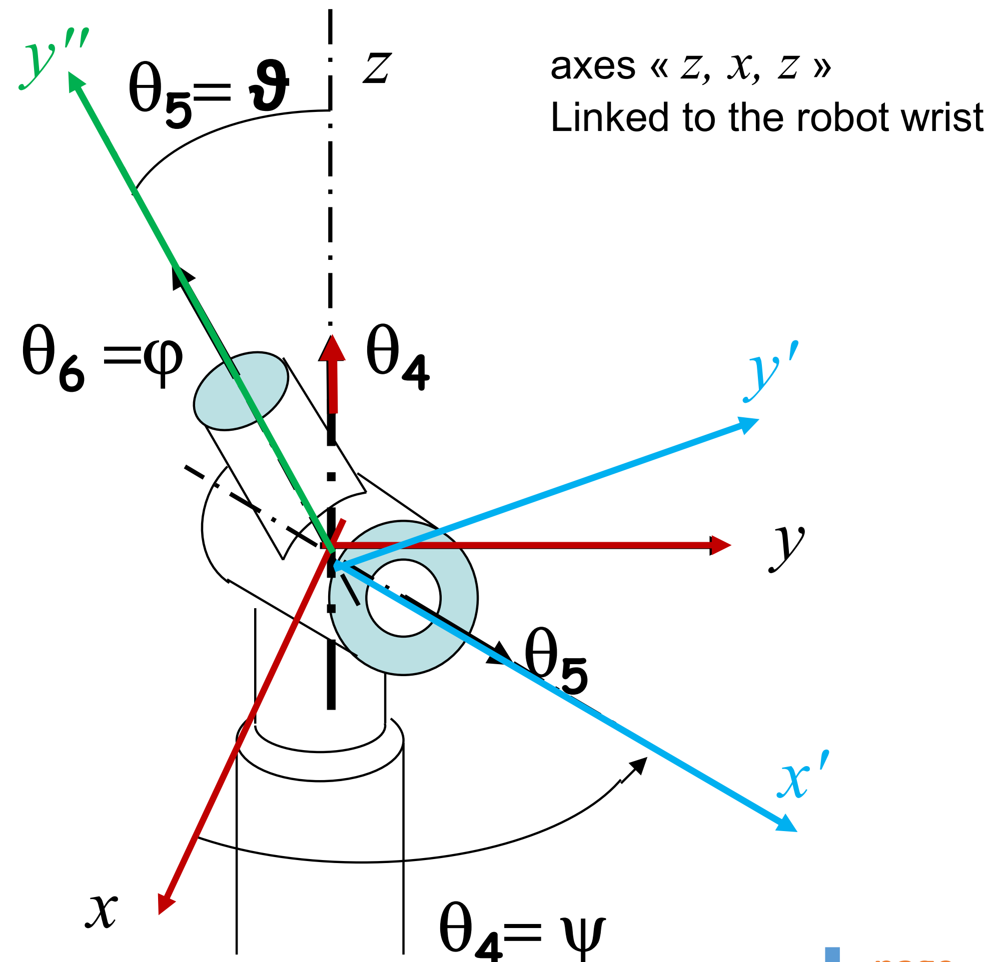
Original Euler angles (z–x–z) convention, used by Leonhard Euler

$(\psi, \vartheta, \varphi)$

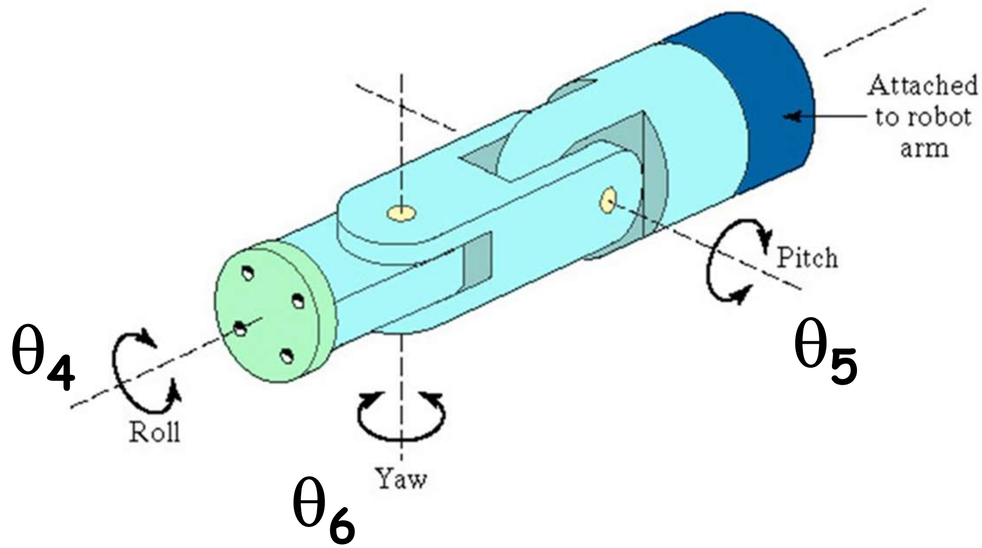
Precession: 1st rotation ψ (or α), **around z**

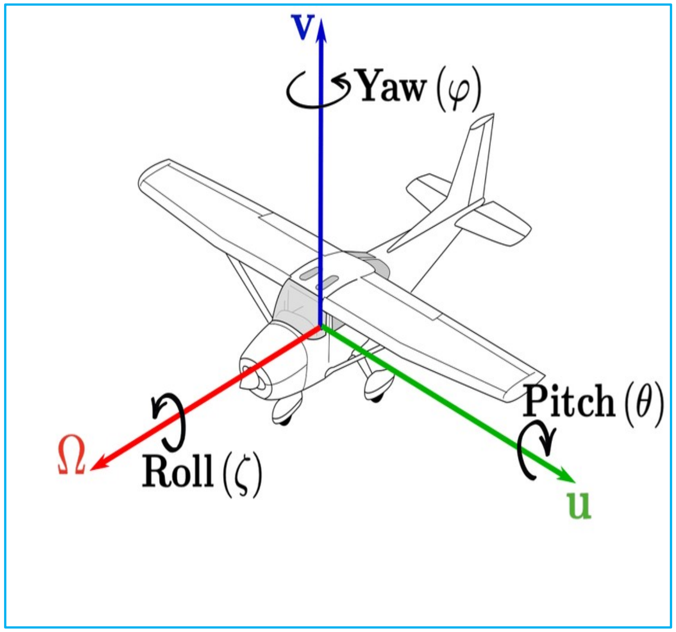
Nutation: 2nd rotation ϑ (or β), **around x'**

Proper rotation: 3rd rotation φ (or β), **around z''**



(yaw–pitch–roll) angles; Wrist angles with 3 concurrent axes





Euler/cardan angles, in robotics and CAD

- **FANUC** & **KUKA** use the fixed XYZ Cardan angle convention,

- **ABB** uses the mobile ZYX Cardan angle convention.
- **Kawasaki**, **Omron** & **Stäubli** use the mobile ZYZ Euler angle convention.

- **CATIA** & **SolidWorks** use the mobile ZYZ Euler angle convention.

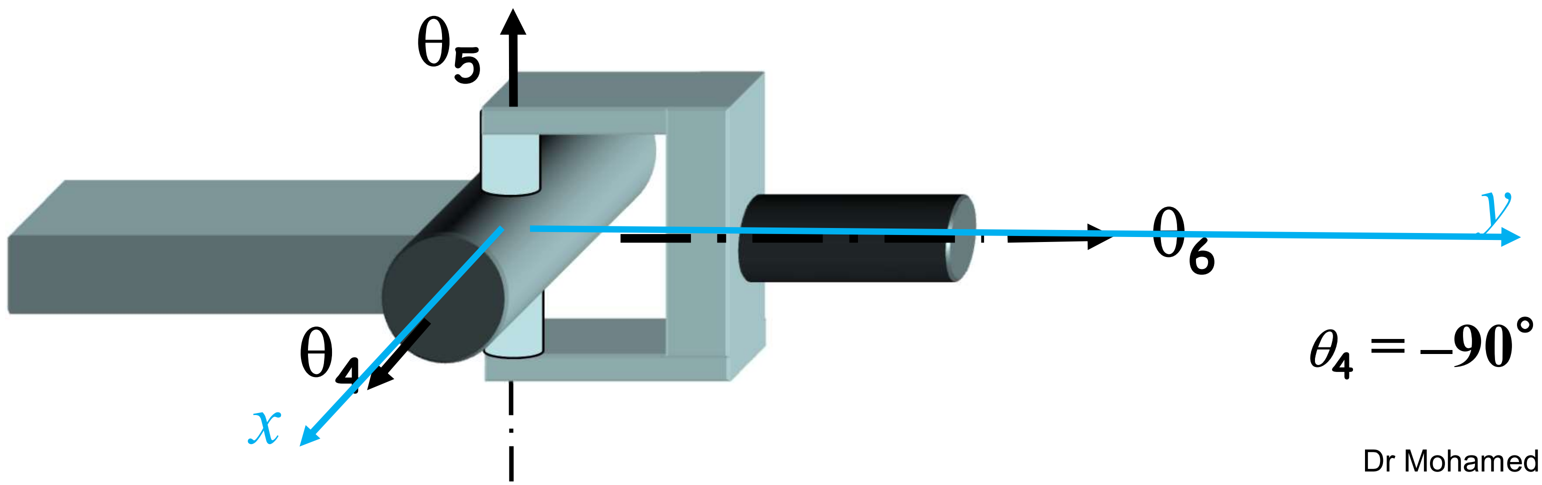
Cardanic wrist:

Roll (Roulis) $\varphi = \theta_6$ %. axis y, direction of approach

Pitch (tangage) $\vartheta = \theta_4$ %. axis x

Yaw (lacet) $\psi = \theta_5$ %. axis z

Angles of Euler, axes linked to the moving body

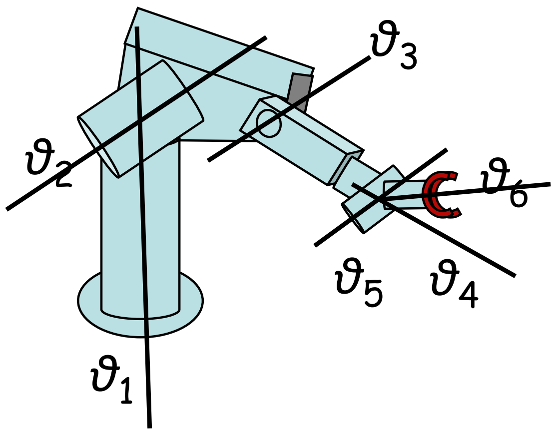


$\theta_4 = -90^\circ$

Finding the cardanic angles $(\varphi, \vartheta, \psi)$, corresponding to a sequence of rotation X-Y-Z

Consider the successive rotations of the Joint angles: $\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6$

$$\underline{P}(\theta_i) = (\mathbf{K_1\ K_2\ K_3\ K_4\ K_5\ K_6}\ )\ \underline{P}_0$$



If we consider only the successive rotations, the total equivalent matrice is as follows:

$$\mathbf{R}_1\ \mathbf{R}_2\ \mathbf{R}_3\ \mathbf{R}_4\ \mathbf{R}_5\ \mathbf{R}_6 = \mathbf{R_{tot}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

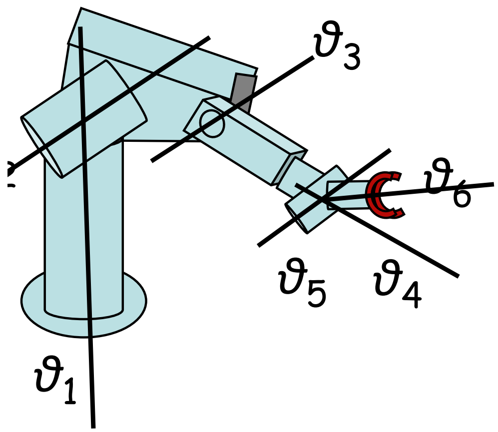
$$\mathbf{R_{tot}} = \mathbf{R_z\ R_y\ R_x}$$

Finding the cardanic angles $(\varphi, \vartheta, \psi)$, corresponding to a sequence of rotation X-Y-Z

$(\varphi, \vartheta, \psi)$ wrt defined in the tool frame, successive rotations X – Y - Z

$$\mathbf{R}_{tot} = \mathbf{R}_z\,\mathbf{R}_y\,\mathbf{R}_x = R_{XYZ}(\psi,\theta,\phi) = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\psi c_\vartheta & c_\psi s_\vartheta s_\varphi - s_\psi c_\varphi & c_\psi s_\vartheta c_\varphi + s_\psi s_\varphi \\ s_\psi c_\vartheta & \dots = r_{22} & \dots = r_{23} \\ -s_\vartheta & c_\vartheta s_\varphi & c_\vartheta c_\varphi \end{bmatrix}$$



Solution:

$\underline{P}(\theta_i) = (\mathbf{K_1\ K_2\ K_3\ K_4\ K_5\ K_6})\ \underline{P}_0$

Translations

Orientations

$$\varphi = \text{Atan2}^*(r_{32}, r_{33})$$

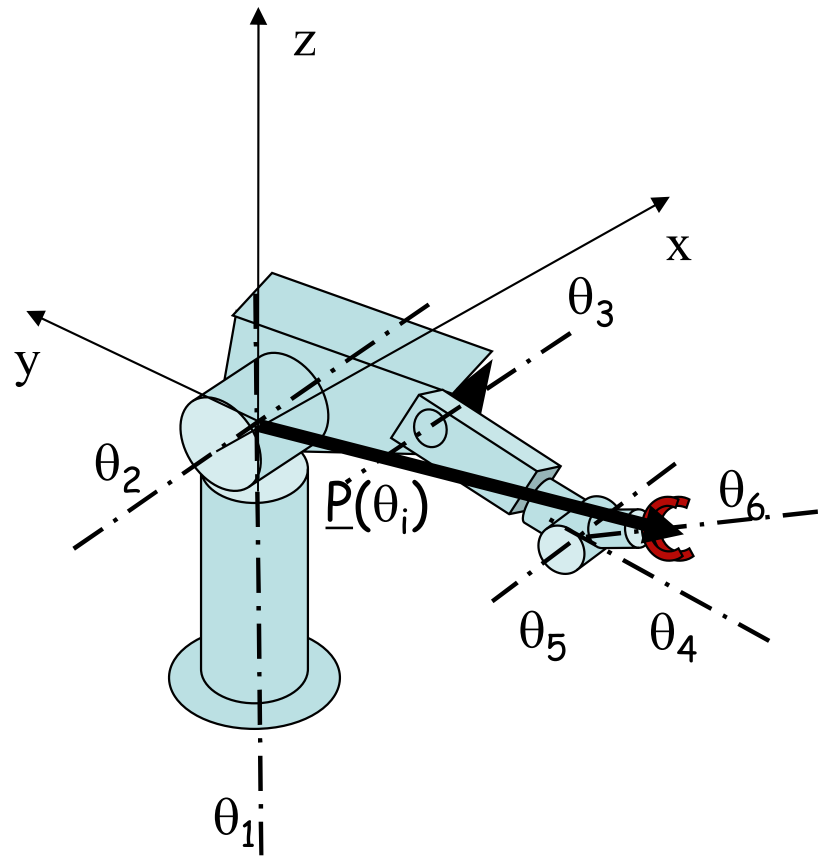
$$\vartheta = A\tan 2(-r_{31}, \pm\sqrt{r^2{}_{32} + r^2{}_{33}})$$

$$\psi = \text{Atan2}(r_{21}, r_{11})$$

*Atan2(x,y) = Arctg (x/y)

Inverse Geometric Model

θ_i ? % defined



We search for $\theta_i(\underline{P})$,

Therefore, the inverse function of the **DGM** $\underline{P}(\theta_i)$

IGM of the 6 DOF robot

Orientation: Angles are Euler angles
with respect to the orientation of the third member.

$$\mathbf{R}_{\mathbf{tot}} = \mathbf{R}_{\mathbf{ee\_base}}\, \mathbf{R}(\theta_4, \theta_5, \theta_6)$$

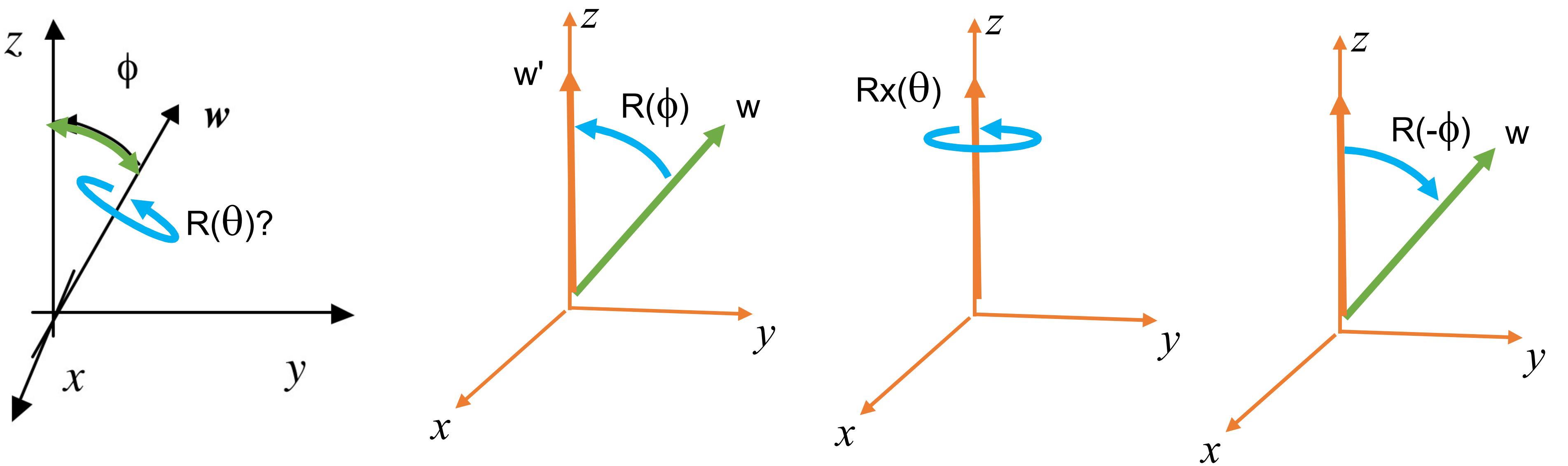
thus

$$\mathbf{R}(\theta_4, \theta_5, \theta_6) = \mathbf{R}_{\mathbf{ee\_base}}{}^{-1}\, \mathbf{R}_{\mathbf{tot}}$$

Rotating an angle ϑ around an axis w

Rotating an angle ϑ around an axis w

Consider the axis w which is not one of the axes of the referential

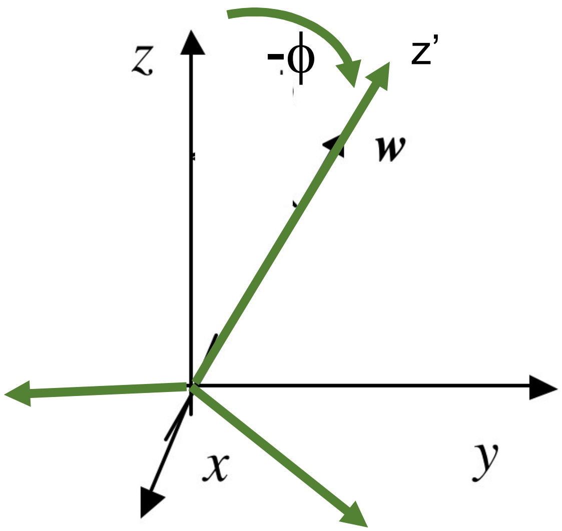


The matrix of rotation around this axis is obtained by combining the 3 following rotations:

$$R(\theta) = R_x(-\phi) \,.\, R_z(\theta) \,.\, R_x(\phi)$$

Rotating an angle ϑ around the axis w using passive rotations.

It does not change anything ☺



$$R = R_{px}(\phi)\,.\,R_z(\theta)\,.\,R_{px}(-\phi)$$

$$R = R_x(-\phi)\,.\,R_z(\theta)\,.\,R_x(\phi)$$

Passage Axis / Angle => Matrix of cos. dir.

Rotation of an angle ϑ around an axis [x, y, z] T with || [x, y, z] ^{T} || = 1

$$\mathbf{R} = (1-\cos\vartheta)\begin{bmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{bmatrix} + \cos\vartheta\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\vartheta\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

(10)

Benjamin Olinde Rodrigues
1795 – 1851

Transformation

Matrix of cos. dir. => Axis/Angle

$$\mathbf{R} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

axis direction vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2\sin(\vartheta)} \begin{bmatrix} f-h \\ g-c \\ b-d \end{bmatrix}$$

$$\cos(\vartheta) = \frac{1}{2}(tr(\mathbf{R}) - 1)$$

$$\sin(\vartheta) = \pm\frac{1}{2}\sqrt{(f-h)^2 + (g-c)^2 + (b-d)^2} \qquad (11)$$

Sign ambiguity (discriminating the angle)

Here's the key part of your question:

> How do we know whether the rotation is by $+\theta$ around $\hat{\mathbf{n}}$, or by $-\theta$ around $-\hat{\mathbf{n}}$?

Because both of these produce the same rotation matrix:

$$R(\hat{\mathbf{n}}, \theta) = R(-\hat{\mathbf{n}}, -\theta)$$

So, by convention:

- You can choose $\theta \in [0, \pi]$ and let the sign of the axis encode direction.
- Or, you can fix $\hat{\mathbf{n}}$ (for example, normalized to point upward or with a specific sign convention) and allow $\theta \in [-\pi, \pi]$.

In practice:

- To discriminate the direction (i.e., determine the signed angle), you can check the skew-symmetric part of R:

$$R - R^T = 2\sin\theta\,[\hat{\mathbf{n}}]_\times$$

where $[\hat{\mathbf{n}}]_\times$ is the cross-product matrix.
The sign of $\sin\theta$ determines the rotation direction.

So you can compute:

$$\sin\theta = \frac{1}{2}\sqrt{(R_{32} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2}$$

and then use the elements of $R - R^T$ to decide the sign of $\sin\theta$, giving a signed angle.

Python code

```
import numpy as np

def rotation_matrix_to_axis_angle(R):
   angle = np.arccos((np.trace(R) - 1) / 2)
   if np.isclose(angle, 0):
      axis = np.array([0, 0, 0])
   else:
      axis = np.array([
         R[2,1] - R[1,2],
         R[0,2] - R[2,0],
         R[1,0] - R[0,1]
      ]) / (2*np.sin(angle))
   # To get signed angle:
   sin_theta = 0.5 * ((R[2,1] - R[1,2])**2 + (R[0,2] - R[2,0])**2 + (R[1,0]
- R[0,1])**2)**0.5
   if np.sign(sin_theta) < 0:
      angle = -angle
   return axis, angle
```

Exercise

Application of (11)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2\sin(\vartheta)} \begin{bmatrix} f-h \\ g-c \\ b-d \end{bmatrix}$$

Use of relation (11) Limitations

The axis/angle expression in equation (11) has two major flaws in computer processing:

1.) The solution is not unique (root pos. Or neg.)

2.) sin (ϑ) = 0 leads to a singularity (undefined axis)

In practice, there will be a bad numerical condition for any angle close to 0 or 180 °

and a blocking of the algorithms for these two cases! "Cardan blocked"

Solution: Quaternions!

These two disadvantages disappear in an elegant way by using the **parameters of Euler** *) or parameters of **Olinde-Rodrigues**.

Also called « **Quaternions** » and are used in industrial robotics.

*) Not to be confused with Euler angles

Quaternions are a generalization of complex numbers.

After long and unsuccessful attempts to extend the geometric interpretation of Complex nb in the 3-d (Argand, 1768-1822, Geneva mathematician), **Hamilton (1843) found the two necessary tricks**:

1. There won't be two, but **three imaginary parts**, in addition to the real part.
2. We must abandon the commutativity of multiplicatio

merci

Hamilton

Real part, imaginary parts

These new "**hypercomplex**" numbers, contain

- a real scalar part λ_0
- Three imaginary components $[\lambda_1, \lambda_2, \lambda_3]^T$ which are

… ..Interpreted as a vector part $\underline{\lambda}$.

the quaternion Q is therefore the quadruple

$$Q = \{ \lambda_0 , \lambda_1, \lambda_2, \lambda_3\} = \{ \lambda_0 , \underline{\lambda}\} \qquad (11a)$$

Imaginary parts:
Generalization of i = √ (-1)

$$Q = \{ \lambda_0 , \lambda_1, \lambda_2, \lambda_3\} = \lambda_0 + i\,\lambda_1 + j\,\lambda_2 + k\,\lambda_3 \qquad (11d)$$

$$i^2 = j^2 = k^2 = ijk = -1 \qquad (11e)$$

$$ij = k = -ji$$

Non-Commutativity!

$$jk = i = -kj$$

$$ki = j = -ik \qquad (11f)$$

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How is the rotation expressed in the quaternion?

- **The axis of rotation** is given by the vector $\underline{\lambda}$

$$\underline{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^T$$

$$Q = \{ \lambda_0 , \lambda_1, \lambda_2, \lambda_3\} = \{ \lambda_0 , \underline{\lambda}\}$$

Angle of rotation ϑ:

$$\lambda_0 = \cos(\vartheta/2)$$

$$|\underline{\lambda}| = \sin(\vartheta/2) \qquad (11b)$$

The axis of rotation disappears for angles of rotation 0 °, 360 °, 720 ° ...

The angle of rotation ϑ is introduced in the following way in the quaternion Q:

$$\lambda_0 = \cos(\vartheta/2) \quad \text{and} \quad \underline{\lambda} = \sin(\vartheta/2)\,[x, y, z]^T, \quad ||x, y, z|| = 1 \qquad (11b')$$

Angle and quaternions:

$$\lambda_0 = \cos(\vartheta/2)$$
$$|\underline{\lambda}| = \sin(\vartheta/2) \qquad (11b)$$

So, all rotational quaternions are unitary:

$$\lambda_0{}^2 + \lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2 = 1 \qquad (11c)$$

Also known as Euler's parameters or Rodrigues parameters

Combining rotations

rules

$$\left\{ \begin{array}{l} i^2 = j^2 = k^2 = -1 \\ ij \;= k = -ji \\ jk = i \;= -kj \\ ki = j \;= -ik \end{array} \right\}$$

Leads to the following product

$$Q_M Q_L = \{ \mu_0 , \underline{\mu} \} \{ \lambda_0 , \underline{\lambda} \} =$$

$$\{ \mu_0 \lambda_0 - \underline{\mu}^T \underline{\lambda} \;,\; \mu_0 \underline{\lambda} + \lambda_0 \underline{\mu} + \underline{\mu} \times \underline{\lambda} \} \qquad (11g)$$

This product defines **the sequence of rotations Q_L then Q_M**

Exercises, using quaternions.

Passage between quaternions and matrix of direction cosines

$$R=\begin{bmatrix} 2(\lambda_0^2+\lambda_1^2)-1 & 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 2(\lambda_1\lambda_3+\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & 2(\lambda_0^2+\lambda_2^2)-1 & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3-\lambda_0\lambda_2) & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) & 2(\lambda_0^2+\lambda_3^2)-1 \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \underline{\lambda}=\frac{1}{2}\begin{bmatrix} \mathrm{sgn}(r_{32}-r_{23})\sqrt{r_{11}-r_{22}-r_{33}+1} \\ \mathrm{sgn}(r_{13}-r_{31})\sqrt{r_{22}-r_{11}-r_{33}+1} \\ \mathrm{sgn}(r_{21}-r_{12})\sqrt{r_{33}-r_{22}-r_{11}+1} \end{bmatrix}$$

$$\lambda_0=\frac{1}{2}\sqrt{r_{11}+r_{22}+r_{33}+1} \qquad =\cos(\vartheta/2)\geq 0$$

Calculation effort
Composition of rotations

	Mul.	Add. & soustr.	total
Matrices of rot.	27	18	45
Quaternions	16	12	28

For the rotation of vectors, it is necessary to use the matrices