Kinematics – Part 2

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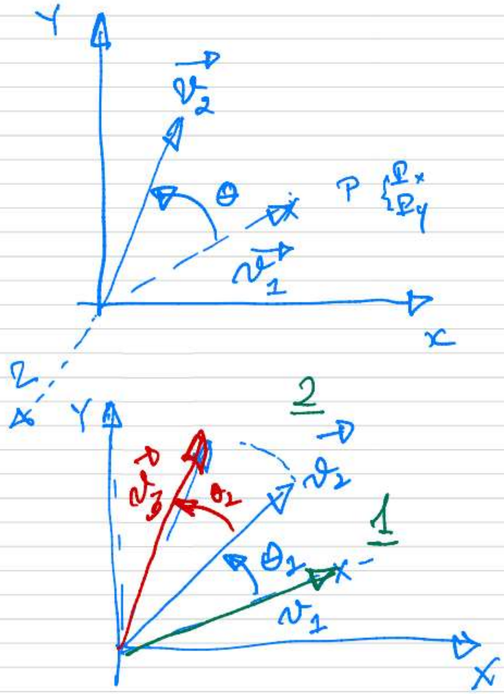
- Rotation around an arbitrary point in 2D
- Successive rotations and rotation matices in 3D
- Homogenous matrix in 3D
- Rotation around an arbitrary axis in 3D
- Geometric modeling

A succession of rotations



$$\underbrace{\vec{v}_2}_{} = \underbrace{R_z(\theta)}_{}\cdot\underbrace{\vec{v}_1}_{}$$

$$\vec{v}_3 = R_z(\theta_2)\,\vec{v}_2$$

$$\downarrow$$

$$R_z(\theta_1)\cdot\vec{v}_1$$

$$\vec{v}_3 = \underbrace{R_z(\theta_2)\cdot R_z(\theta_1)}_{R_z(\theta_1,\theta_2)}\,\vec{v}_1$$

$$R_z(\theta_1,\theta_2) = \underline{R_z(\theta_2)}\cdot R_z(\theta_1)$$

$$R_1 \rightsquigarrow R_2 \rightsquigarrow R_3 \;\dots\; R_n$$

$$\boxed{R = R_n\cdot R_{n-1}\cdot R_{n-2}\cdots R_1}$$

Valid in 2D as well as in 3D

A succession of 2 rotations

$$\xrightarrow{R_1}\quad\xrightarrow{R_2}$$

$$\underline{R} = \underline{R_2} \cdot \underline{R_1} \qquad \text{around axis } \underline{z}$$

$$R_2 = \begin{vmatrix} \cos\theta_2 & -\sin(\theta_2) \\ \sin\theta_2 & \cos(\theta_2) \end{vmatrix} = \begin{vmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{vmatrix}$$

$$R_1 = \begin{vmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{vmatrix}$$

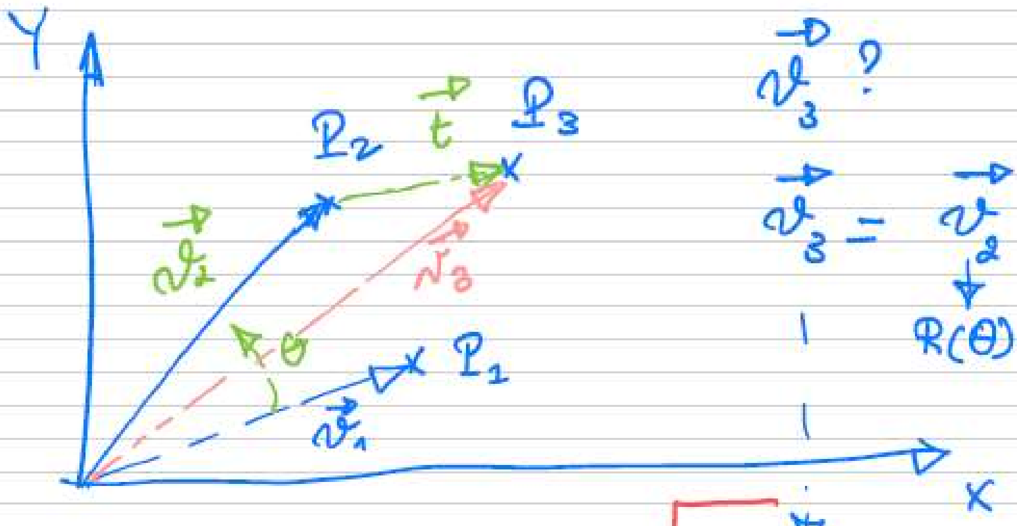
$$R = R_2 \cdot R_1 = \begin{vmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{vmatrix} \begin{vmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{vmatrix} = \begin{vmatrix} c_1 c_2 - s_1 s_2 & -s_1 c_2 - s_2 c_1 \\ c_1 s_2 + c_2 s_1 & -s_1 s_2 + c_1 c_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{vmatrix}$$

$$= R_z(\theta_1 + \theta_2)$$

Rotation + Translation

$\vec{t}$ combined with a Rotation



$\vec{v}_3$?

$$\vec{v}_3 = \underset{\underset{R(\theta)\cdot\vec{v}_1}{\downarrow}}{\vec{v}_2} + \vec{t}$$

$$\boxed{\vec{v}_3 = \underbrace{R(\theta)\cdot\vec{v}_1}_{\text{Vector}} + \underset{\downarrow\ \text{Vector}}{\vec{t}}}$$

Rotation + Translation

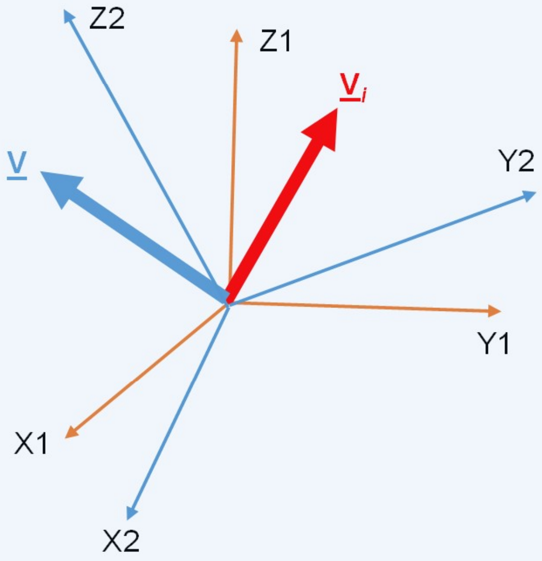
$\boxed{?}$ use a matriciel representation to combine - translation operation & - Rotation operator.

△ yes - Using Homogenous Matrices.

$$\begin{Bmatrix} x \\ y \end{Bmatrix} \leftarrow v_3 = R(\theta)\, \vec{v}_1 + \vec{t}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} R & \vec{t} \\ 0 & 1 \end{bmatrix}}_{M_H(\theta, t)} \cdot \begin{Bmatrix} x_1 \\ y_1 \\ 1 \end{Bmatrix} = \begin{bmatrix} R\vec{v}_1 + \vec{t} \\ 1 \end{bmatrix}$$

Generalized rotation matrix



Let us consider a vector V expressed in the base frame {1} as follows

$\underline{V} = V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1$

$\underline{V} = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$

We can write:

$\underline{V} = V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1 = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$

$\underline{\mathbf{V}}$ has in {2} the same coordinates as $\underline{\mathbf{V}}_i$ in {1}, that are $[V_{x2}, V_{y2}, V_{z2}]$

We search to :
Express the coordinates of V in {1} , that are $[V_{x1}, V_{y1}, V_{z1}]$
function of the initial vector Vi , ie. $[V_{x2}, V_{y2}, V_{z2}]$

$$\begin{bmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{bmatrix} = [?].\begin{bmatrix} V_{x2} \\ V_{y2} \\ V_{z2} \end{bmatrix}$$

Consideration of the **active transformation** wrt the basic referential.

Rotation matrices

Direction cosine matrix

Active and Passive Transformation

$$\boldsymbol{R} = \begin{bmatrix} \underline{x}_2\underline{x}_1 & \underline{y}_2\underline{x}_1 & \underline{z}_2\underline{x}_1 \\ \underline{x}_2\underline{y}_1 & \underline{y}_1\underline{y}_2 & \underline{z}_2\underline{y}_1 \\ \underline{x}_2\underline{z}_1 & \underline{y}_2\underline{z}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

$$\boldsymbol{R_p} = \begin{bmatrix} \underline{x}_1\underline{x}_2 & \underline{x}_2\underline{y}_1 & \underline{x}_2\underline{z}_1 \\ \underline{y}_2\underline{x}_1 & \underline{y}_1\underline{y}_2 & \underline{y}_2\underline{z}_1 \\ \underline{z}_2\underline{x}_1 & \underline{z}_2\underline{y}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

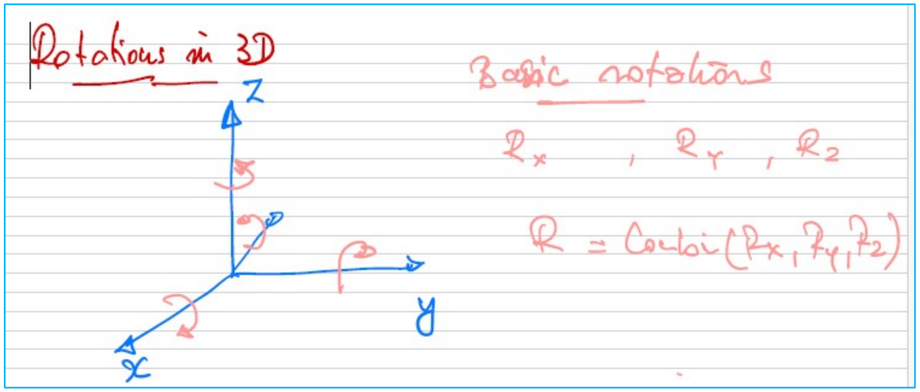
$$\boldsymbol{R} = \boldsymbol{R_p^T} = \begin{bmatrix} \underline{x}_2\underline{x}_1 & \underline{y}_2\underline{x}_1 & \underline{z}_2\underline{x}_1 \\ \underline{x}_2\underline{y}_1 & \underline{y}_1\underline{y}_2 & \underline{z}_2\underline{y}_1 \\ \underline{x}_2\underline{z}_1 & \underline{y}_2\underline{z}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

Rotations around X, Y, Z : Basic matrices of rotations

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$


Rotations in 3D

Basic rotations

R_x , R_y , R_z

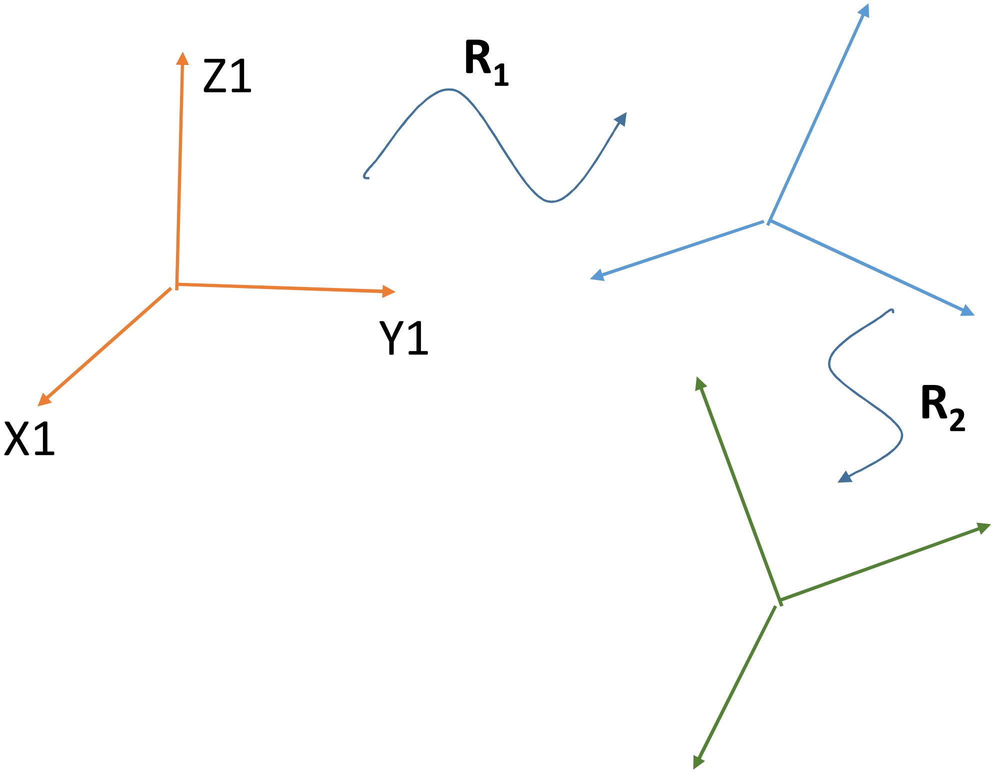
$R = \text{Combi}(R_x, R_y, R_z)$

Successive rotations

Rule :

The combined rotation matrix of several rotations is the product of the corresponding rotation matrices, always starting with that associated with the last rotation.

$$\mathbf{R = R_n \cdot R_{n-1} \cdot R_{n-2} \cdot \ldots \cdot R_1}$$



The rotation of order n corresponds to the last operation performed.

The rotation matrix R1 is associated with the first rotation performed.

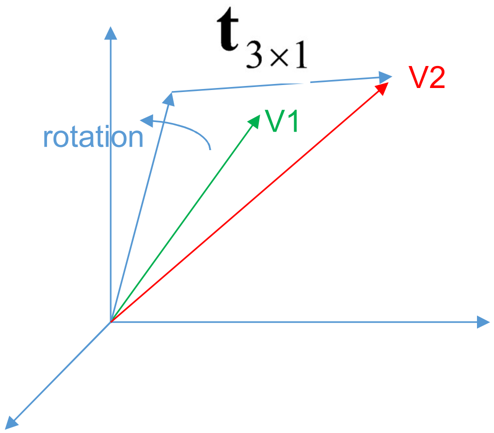
Exercise 3.2

Deduce the rotation matrix obtained in the following cases :

a) rotation by 90° around z, **then** rotation by 90° around y

b) rotation by 90° around y, **then** rotation by 90° around z

3D- homogeneous matrices



$$\begin{bmatrix} R_{3\times3} & t_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix}$$

The homogenous matrix that combines

- The rotation R

and

- The translation t

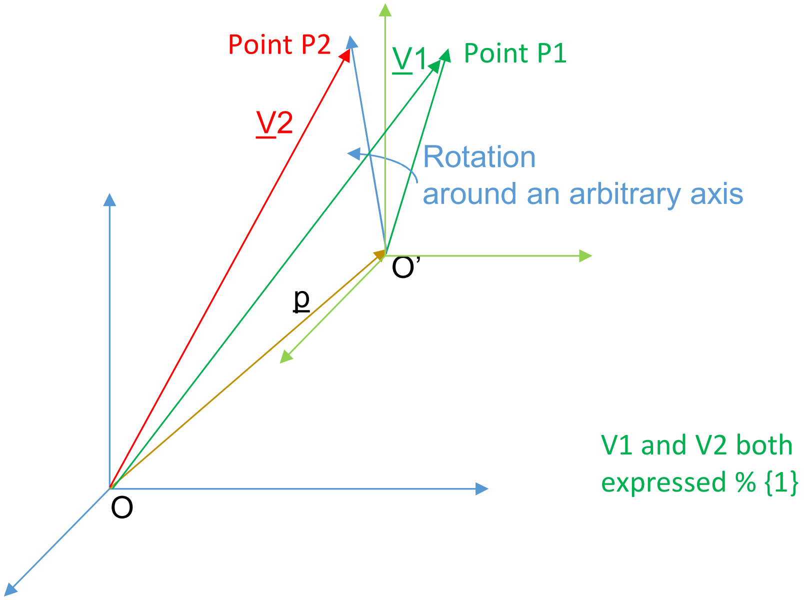
The vector **V2** obtained after a rotation and a translation of the vector V1 is then expressed as :

$$\begin{bmatrix} V2 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{3\times3} & t_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} V1 \\ 1 \end{bmatrix}$$

3D- homogeneous matrices –

Rotation around an arbitrary axis

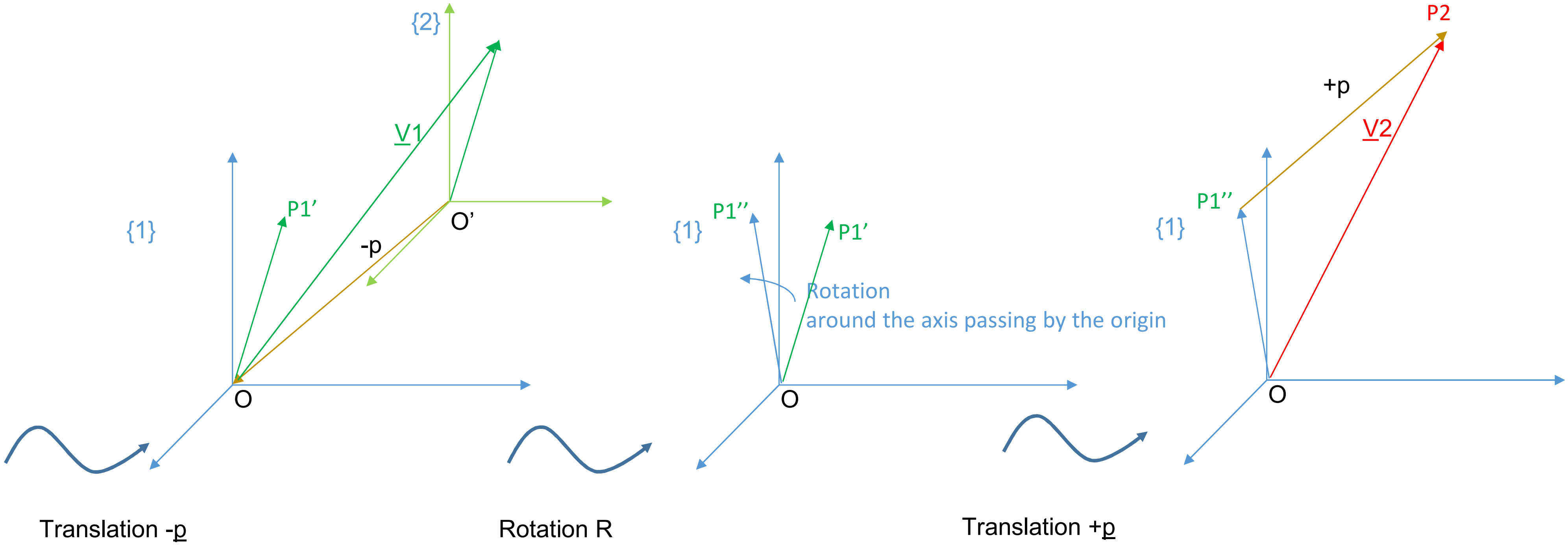
The rotation around an axis not passing through the origin is obtained as follows:



3D- homogeneous matrices –

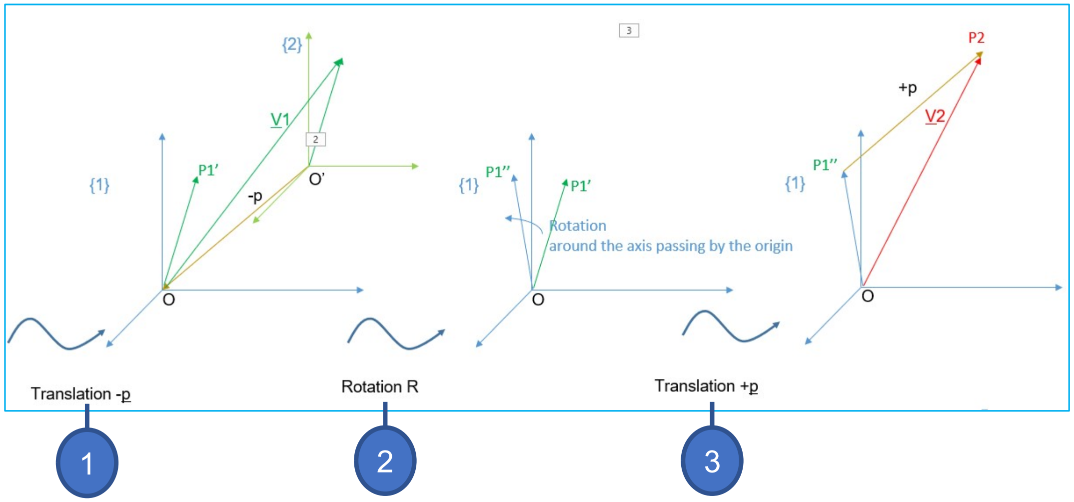
Rotation around an arbitrary axis

The rotation around an axis not passing through the origin is obtained as follows:



3D- homogeneous matrices –
Rotation around an arbitrary axis

$$\underbrace{\begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}}_{③}\underbrace{\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}}_{②}\underbrace{\begin{bmatrix} I & -p \\ 0 & 1 \end{bmatrix}}_{①} = \begin{bmatrix} R & p-Rp \\ 0 & 1 \end{bmatrix}$$

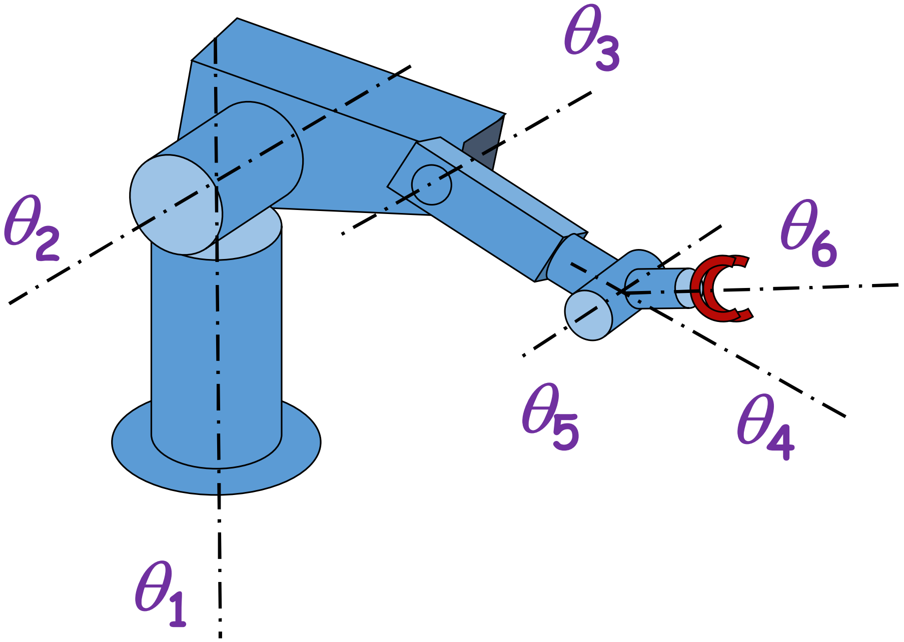


The vector **V2** obtained after a rotation and a translation of the vector V1 is then expressed as :

$$\boxed{\begin{bmatrix} V2 \\ 1 \end{bmatrix} = \begin{bmatrix} R & p - Rp \\ 0 & 1 \end{bmatrix}\begin{bmatrix} V1 \\ 1 \end{bmatrix}}$$

with the vector p of the origin O, at any point on the axis of rotation

Geometric models : Direct and inverse



Direct Gemoetric Model DGM (Forward Kinematics)

The DGMD gives the operational coordinates according to joint variables (robot variables)

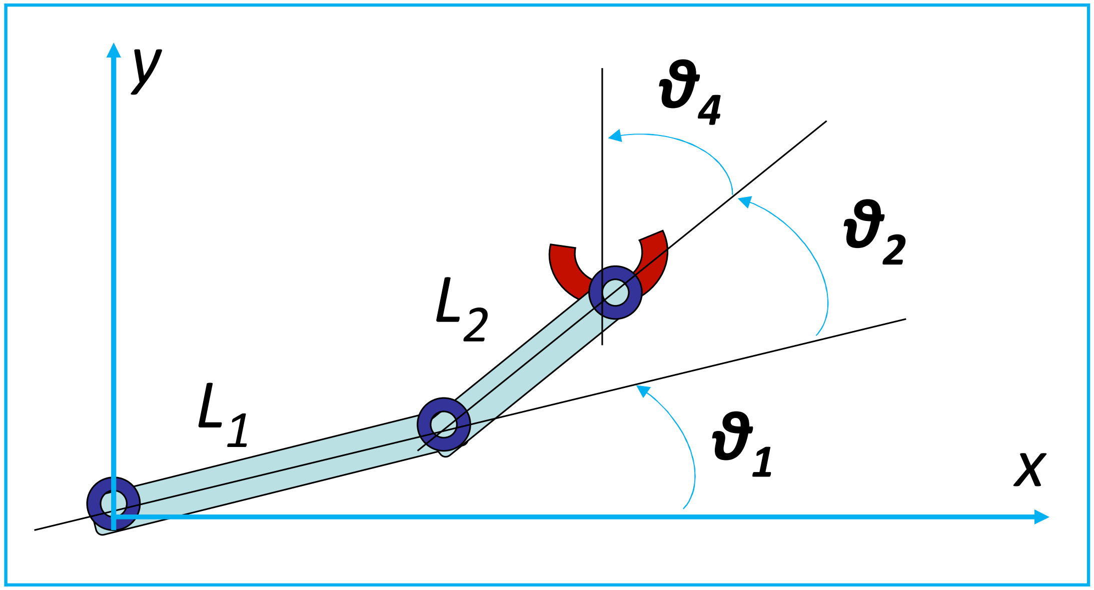
$$\{x, y, z, \Theta\} = F\,(q1, q2, \ldots\ qi, \ldots.\ qn)$$

$$= F\ (\vartheta 1, \vartheta 2, \ldots\ \vartheta i, \ldots.\ \vartheta n\ )$$

Position of a point (Tool Center Point TCP)
and orientation (e.g. by quaternion)

Example: Direct GM of the SCARA

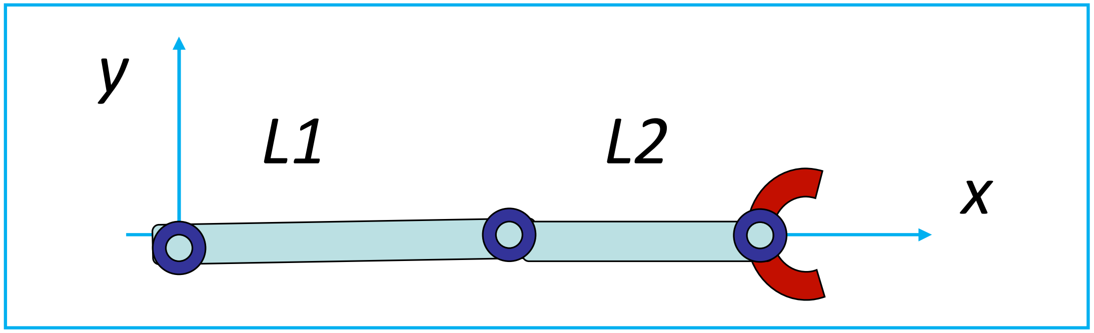
SCARA
(ϑ_3= d = vertical coordinate = **z**)



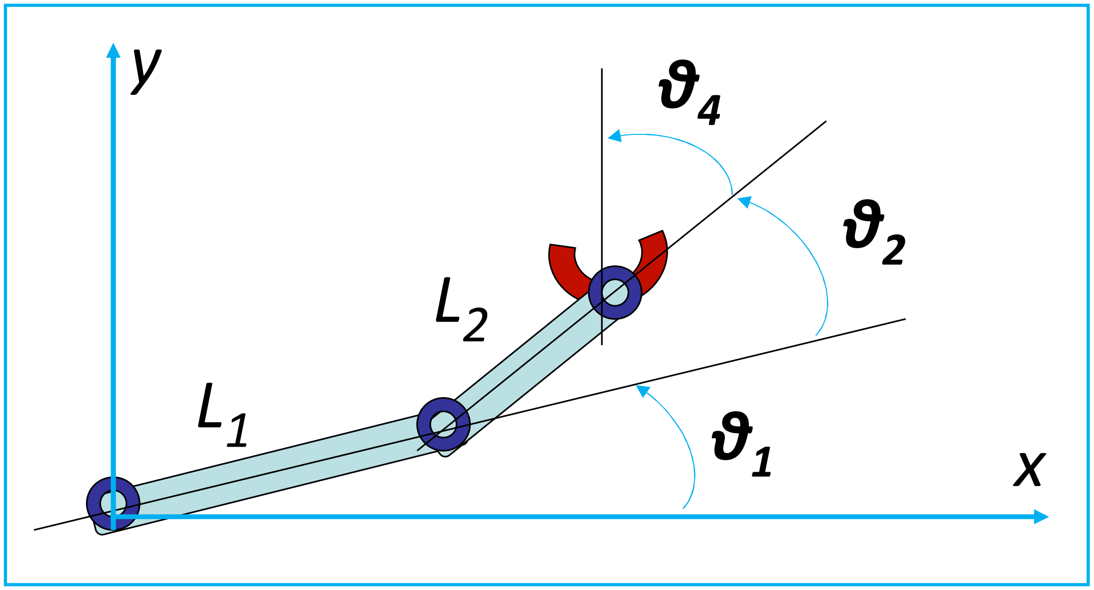
1. *Definition of the **joint variables ϑi***
2. *Define the **reference positions ϑi = 0***

   Note the definition of ϑ2 = 0 with respect to the arm L1 and not with respect to x)
3. *Define the parameters (**Li**) of the robot*

Reference position $\vartheta i = 0$



The DGM (MGD) gives orientation and wrist position



$x = \ldots ?$

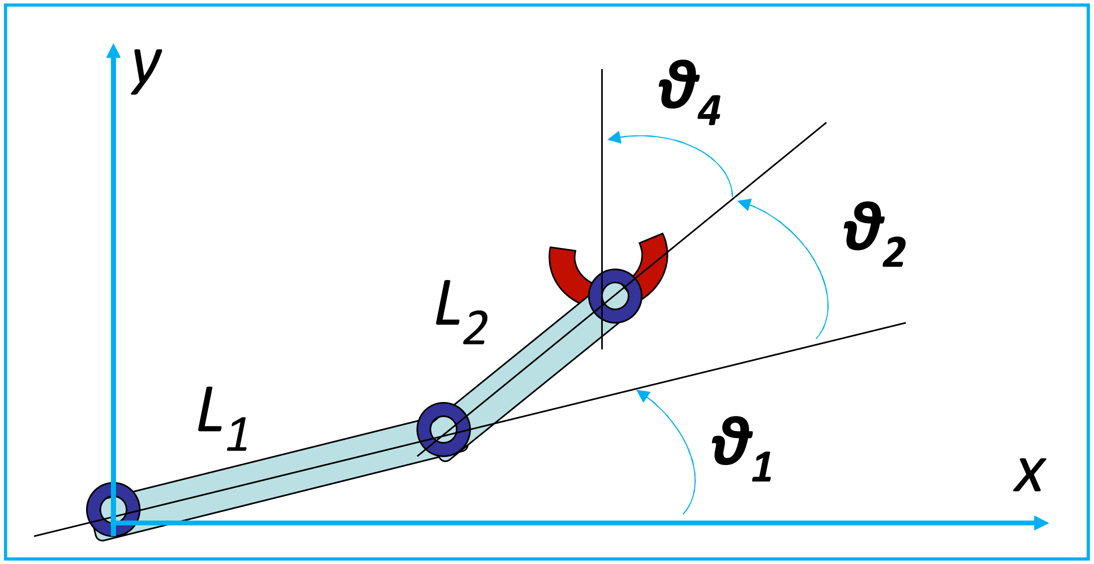
$y = \ldots ?$

$z = \vartheta 3 = d$ (vertical position)

$\varphi = \vartheta 1 + \vartheta 2 + \vartheta 4$

Analytic approach:

Position of the tool center point (TCP) & orientation of the wrist



Notation:
$\cos(\theta_1 + \theta_2) = c_{12}$
$\sin(\theta_1 + \theta_2) = s_{12}$

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) = L_1 c_1 + L_2 c_{12}$$

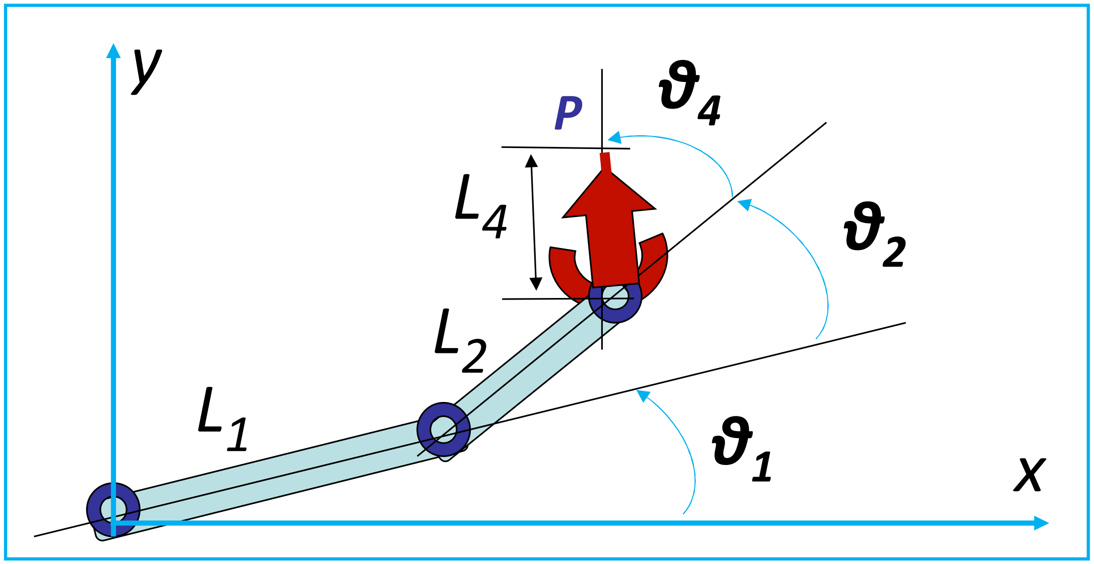
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) = L_1 s_1 + L_2 s_{12}$$

$$z = \theta_3 = d$$

$$\varphi = \theta_1 + \theta_2 + \theta_4$$

Analytic approach:

Position of the tool center point (TCP) & orientation of the wrist



$$x = L_1 c_1 + L_2 c_{12} + L_4 c_{124}$$

$$y = L_1 s_1 + L_2 s_{12} + L_4 s_{124}$$

$$z = \theta_3 = d$$

$$\varphi = \theta_1 + \theta_2 + \theta_4$$

L4 concerns the tool is not a robot parameter

DGM of a 6 DOF robot?

This analytic approach will become difficult to apply for a 6 DOF robot

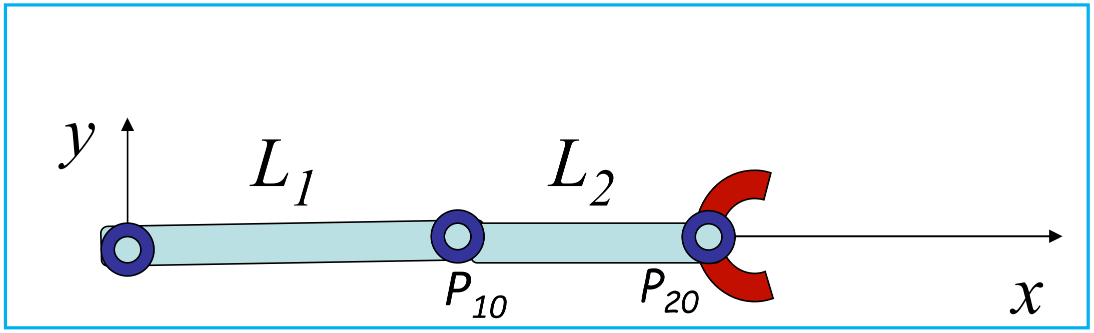
Use the homogeneous matrices associated to serially linked joints !

We will apply the previopus findings to the SCARA robot.

First, it suffices to consider the problem in two dimensions.

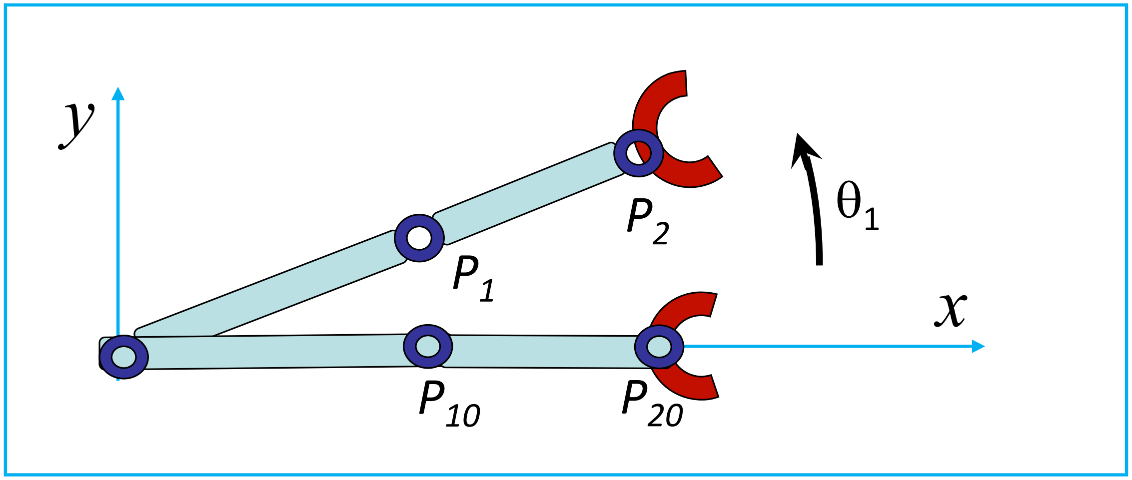
We therefore use 2D homogeneous matrices

Tool Center Point as an output



$$L_{12} = L_1 + L_2$$

1) Rotation of θ1 around the origin

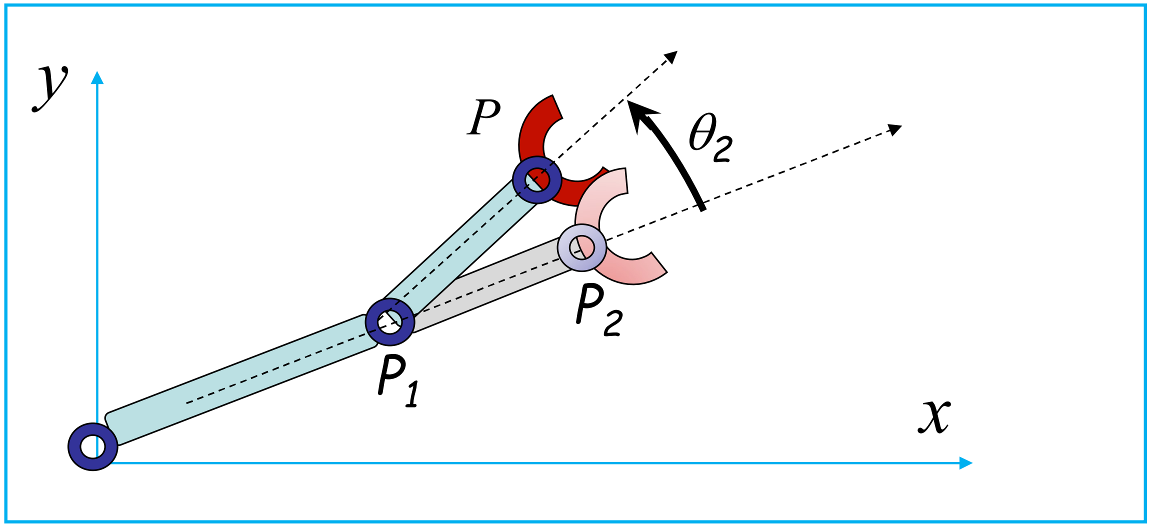


$$P_{20} = \underline{p}(\theta_i = 0) = \begin{bmatrix} L_1 + L_2 \\ 0 \\ 1 \end{bmatrix}$$

$$L_{12} = L_1 + L_2$$

$$P_2 = P(\theta_1, \theta_2 = \theta_4 = 0) = \underbrace{\begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R\ (\theta_1)} \underbrace{\begin{bmatrix} L_{12} \\ 0 \\ 1 \end{bmatrix}}_{P_{20}}$$

2.) Rotation of θ_2 around p_1



Center of rotation

$$p_1 = \begin{bmatrix} L_1 c_1 \\ L_1 s_1 \end{bmatrix}$$

$$P(\theta_1,\theta_2)=\underbrace{\begin{bmatrix}\mathbf{R}_2 & & p_1-\mathbf{R}_2 p_1\\ 0 & 0 & 1\end{bmatrix}}_{R\,(\theta_2,P_1)}\underbrace{\begin{bmatrix}c_1&-s_1&0\\s_1&c_1&0\\0&0&1\end{bmatrix}\begin{bmatrix}L_{12}\\0\\1\end{bmatrix}}_{P_2}$$

Direct geometric model (for the TCP)

$$P(\theta_1,\theta_2)=\begin{bmatrix} \mathbf{R}_2 & p_1-\mathbf{R}_2 p_1 \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} L_{12} \\ 0 \\ 1 \end{bmatrix}$$

Intermediate result

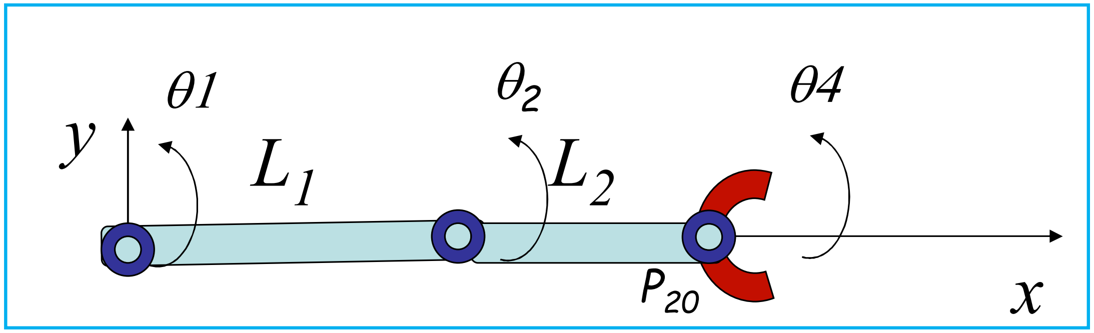
$$\mathbf{R}_2 p_1=\begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix}\begin{bmatrix} c_1 \\ s_1 \end{bmatrix}L_1=\begin{bmatrix} c_2c_1-s_2s_1 \\ c_1s_2+c_2s_1 \end{bmatrix}L_1=\begin{bmatrix} c_{12} \\ s_{12} \end{bmatrix}L_1$$

$$\boxed{P(\theta_1,\theta_2)=\begin{bmatrix} \mathbf{R}_{12} & \begin{bmatrix} (c_1-c_{12})L_1 \\ (s_1-s_{12})L_1 \end{bmatrix} \\ 0 \quad 0 & 1 \end{bmatrix}\begin{bmatrix} L_{12} \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} c_{12}L_{12}+(c_1-c_{12})L_1 \\ s_{12}L_{12}+(s_1-s_{12})L_1 \\ 1 \end{bmatrix}=\begin{bmatrix} c_1L_1+c_{12}L_2 \\ s_1L_1+s_{12}L_2 \\ 1 \end{bmatrix}}$$

Additional rotation θ_4

Indication, consider the following consecutive rotations

1. Rotation of θ_4 around the center point $[L_1 + L_2 , 0]^T$
2. Rotation of θ_2 around the center point $[L_1 , 0]^T$
3. Rotation of θ_1 around the center point $[0 , 0]^T$



Additional rotation θ_4

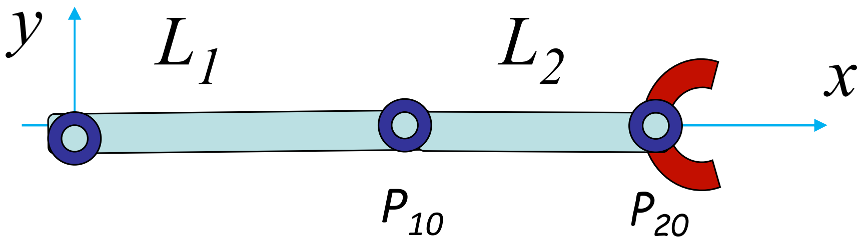
Consider the consecutive centers of rotation at each joint .

Perform the rotations in the reverse order:

θ_4 then θ_2 then θ_1

$$K_{DGM} = \begin{bmatrix} \mathbf{R}_1 & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2 & \mathbf{p}_1 - \mathbf{R}_2\mathbf{p}_1 \\ 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_4 & \mathbf{p}_2 - \mathbf{R}_4\mathbf{p}_2 \\ 0 \quad 0 & 1 \end{bmatrix}$$

K_{DGM} is the equivalent Kinematic homogenous matrix



With $\underline{\mathbf{p}}_{10} = [L_1 , 0]'$

and $\underline{\mathbf{p}}_{20} = [L_{12} , 0]'$

Ex.

1. Rot. de θ_4 de autour de $[\, L_1 + L_2 \,, 0 \,]^T$

$$\begin{bmatrix} \mathbf{R} & \mathbf{p}-\mathbf{Rp} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{vmatrix} c_4 & -s_4 & ? \\ s_4 & c_4 & ? \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{p}-\mathbf{Rp} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{vmatrix} c_4 & -s_4 & L_{12} v_4 \\ s_4 & c_4 & -L_{12} s_4 \\ 0 & 0 & 1 \end{vmatrix}$$

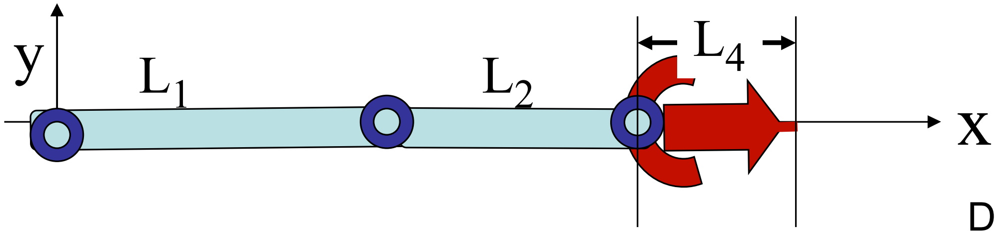
versine(θ) = $1 - \cos\theta$

$v_4 = 1 - \cos \theta_4$

$L_{12} = L_1 + L_2$

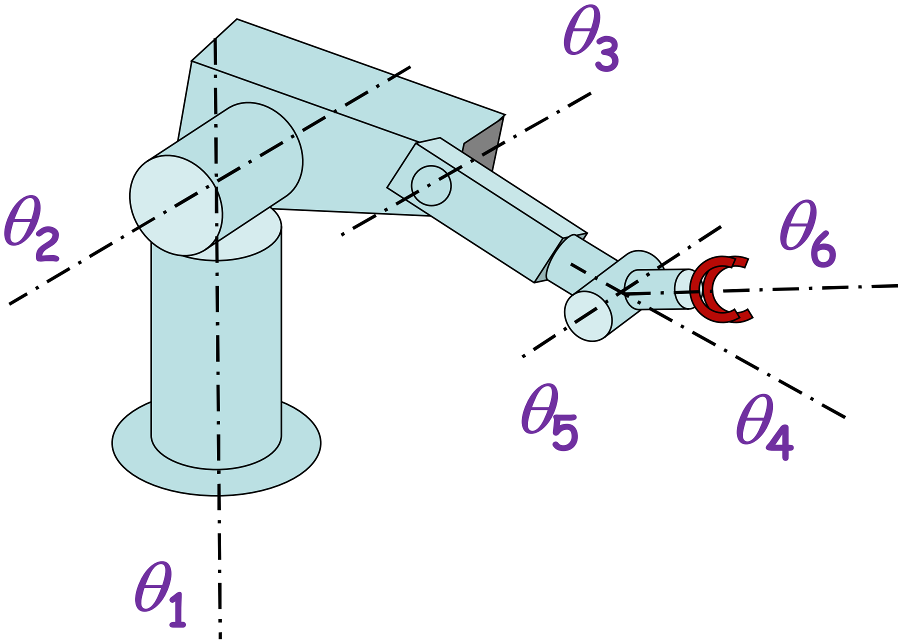
Methodology to establis DGM in 4 steps

1. Define the **joint variables**
2. Define their **positions of reference**
3. Define the **geometric parameters of the robot**
4. **Chain** the serially-linked successive movements (multiplication of the consecutive homogenous matrices) **starting from the end to the base**



DGM of a 6 DOF robot

1. Joint variables (robot variables)



Convention:

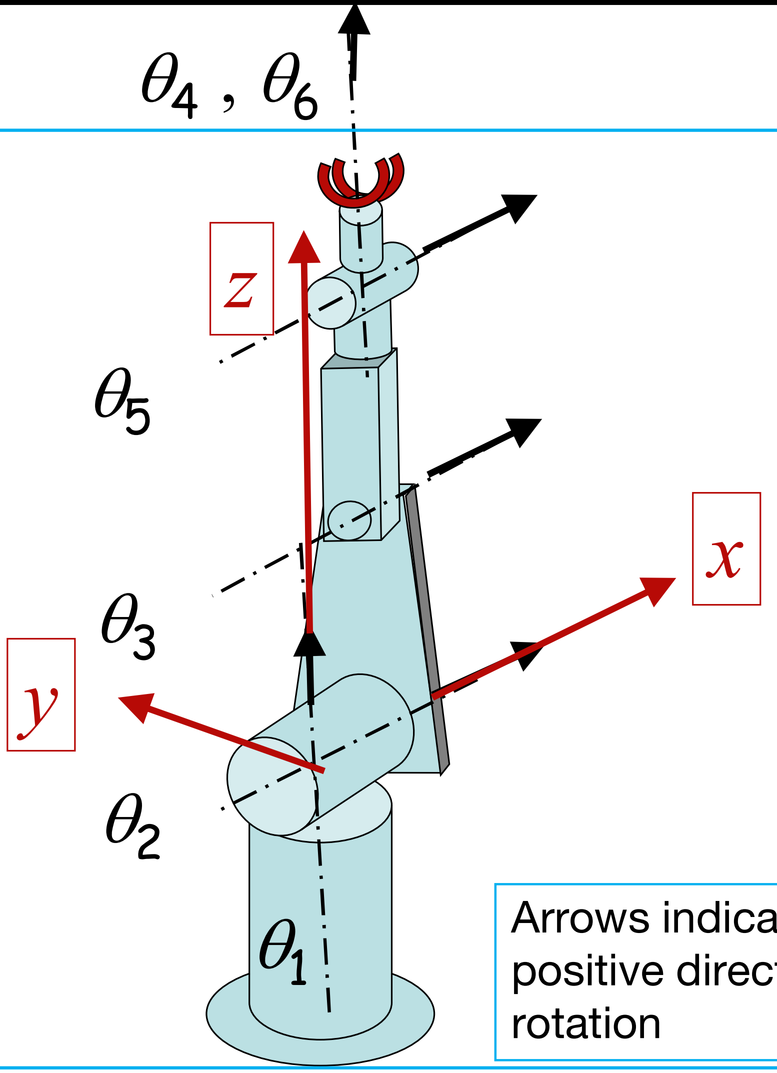
Start indexing from the base (frame) to the wrist

DGM of a 6 DOF robot

2. Positions of reference

$$\theta_i = 0$$

The referential {x, y, z} is fixed on the frame (operational coordinates)



Arrows indicate positive direction of rotation

DGM of a 6 DOF robot

3. Robot parameters

The axes of θ_1 and θ_2 intersect => $\mathbf{L_1 = 0}$

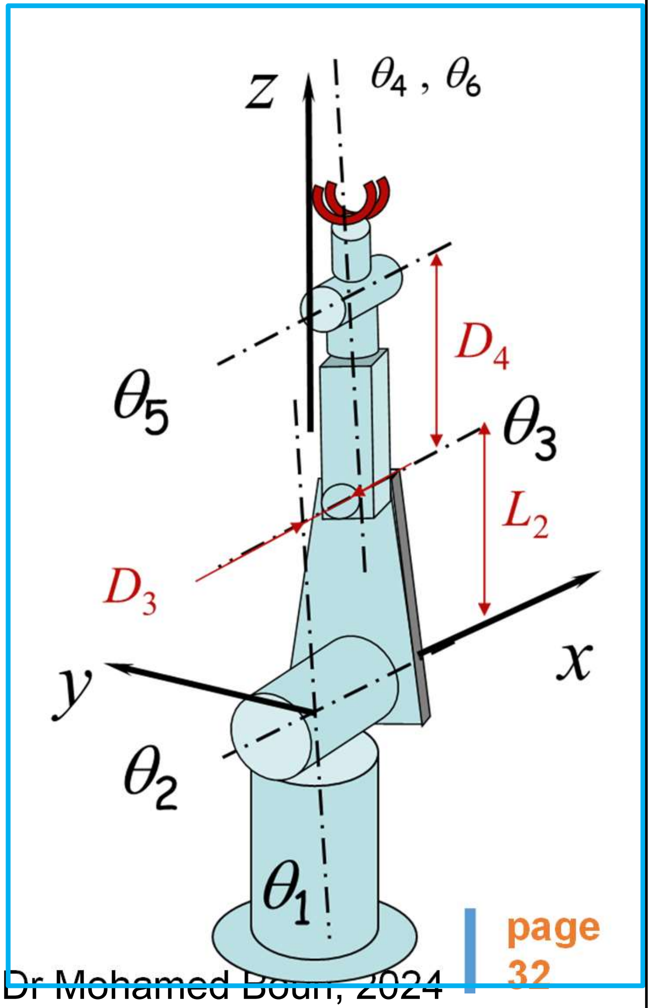
The axes of θ_2 and θ_3 are parallel, dist. $\mathbf{L_2}$

The axes of θ_3 and θ_4 intersect => $\mathbf{L_3 = 0}$

The axes of θ_1 and θ_4 are offset on the axis θ_3 by a distance $\mathbf{D_3}$

The axes of θ_4 and θ_5 intersect => $\mathbf{L_4 = 0}$

The axes of θ_3 and θ_5 are offset on the axis θ_4 by a distance $\mathbf{D_4}$



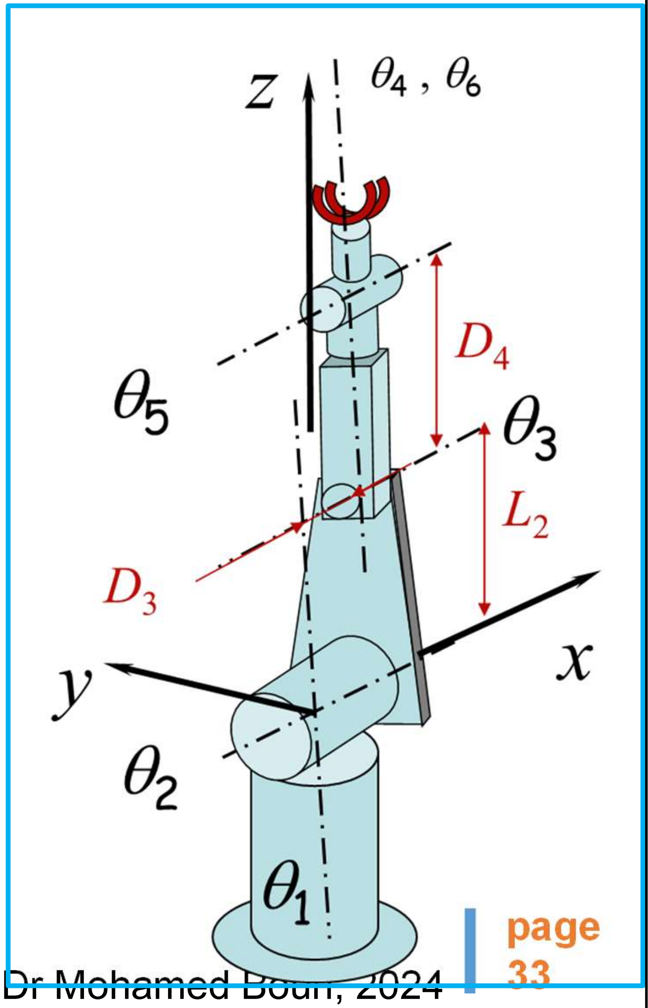
DGM of a 6 DOF robot

4. Sequence of movements

4.1. Rot. of θ_6 around the axis z, distant at $\mathbf{p} = [D_3, 0, 0]'$

$$\mathbf{p}-\mathbf{R}\mathbf{p}=\begin{bmatrix} D_3 \\ 0 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} c_6 & -s_6 & 0 \\ s_6 & c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z} \begin{bmatrix} D_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D_3 v_6 \\ -D_3 s_6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{p}-\mathbf{R}\mathbf{p} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} c_6 & -s_6 & 0 & D_3 v_6 \\ s_6 & c_6 & 0 & -D_3 s_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{K}_6$$



DGM of a 6 DOF robot

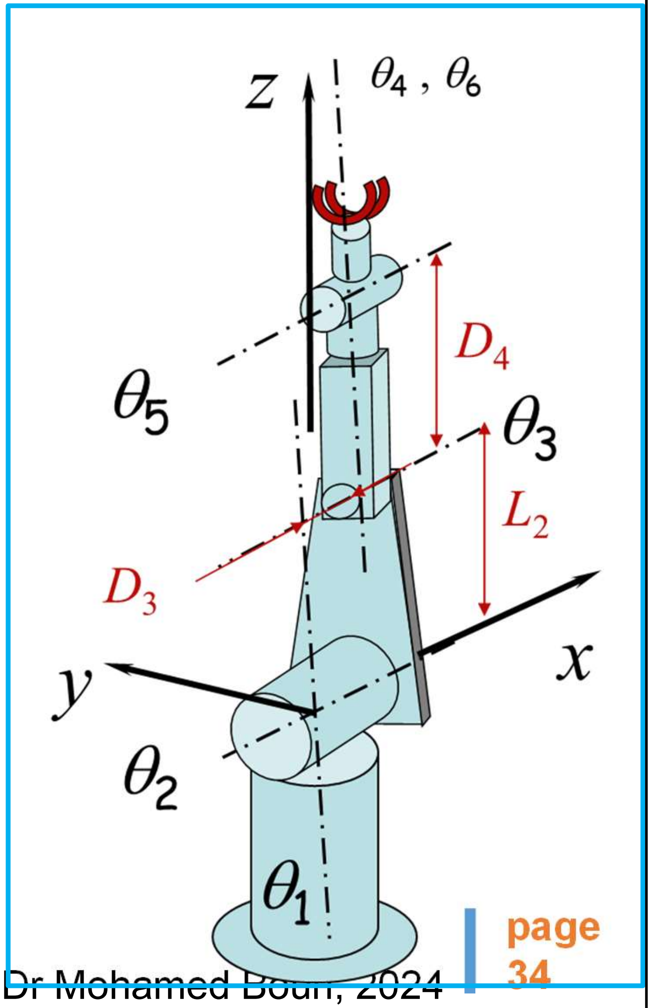
4. Sequence of movements

4.2. Rot. of θ_5 around the axis x, distant at $\mathbf{p} = [0,\ 0,\ L_2+D_4]'$

$$\mathbf{p}-\mathbf{R}\mathbf{p}=\left[\begin{bmatrix}0\\0\\L_{24}\end{bmatrix}\right]-\underbrace{\begin{bmatrix}1&0&0\\0&c_5&-s_5\\0&s_5&c_5\end{bmatrix}}_{\text{R}_x}\begin{bmatrix}0\\0\\L_{24}\end{bmatrix}=\begin{bmatrix}0\\L_{24}s_5\\L_{24}v_5\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{R} & & & \mathbf{p}-\mathbf{R}\mathbf{p}\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&c_5&-s_5&L_{24}s_5\\0&s_5&c_5&L_{24}v_5\\0&0&0&1\end{bmatrix}=\mathbf{K}_5$$

and so on for K4, K3, K2, K1



DGM of a 6 DOF robot

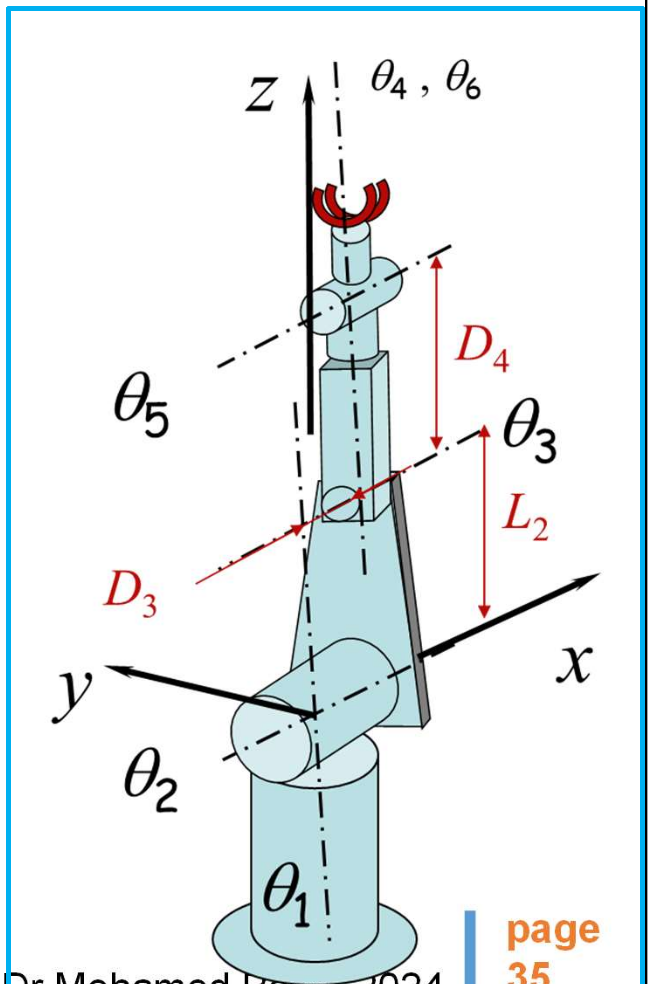
4. Sequence of movements

4.2. Rot. of θ_5 around the axis x, distant at $\mathbf{p} = [0, 0, L_2 + D_4]'$

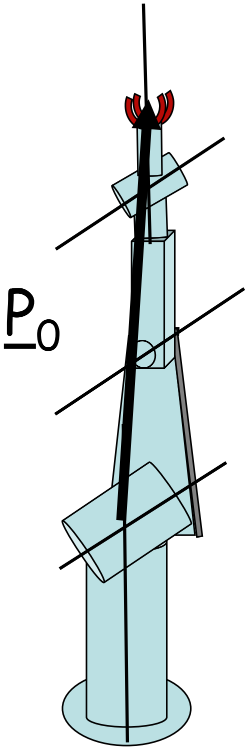
$$\mathbf{p} - \mathbf{Rp} = \left[\begin{bmatrix} 0 \\ 0 \\ L_{24} \end{bmatrix}\right] - \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_5 & -s_5 \\ 0 & s_5 & c_5 \end{bmatrix}}_{R_x} \begin{bmatrix} 0 \\ 0 \\ L_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ L_{24} s_5 \\ L_{24} v_5 \end{bmatrix}$$

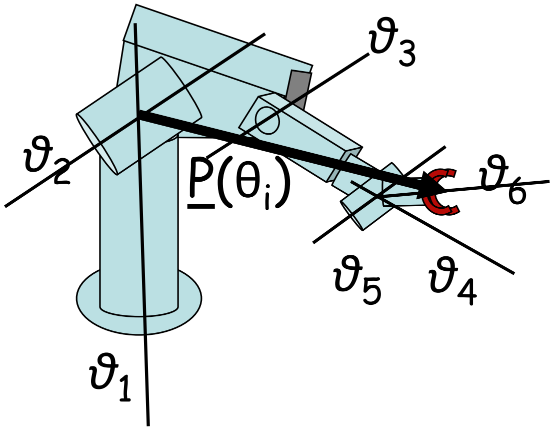
$$\begin{bmatrix} \mathbf{R} & \mathbf{p} - \mathbf{Rp} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_5 & -s_5 & L_{24} s_5 \\ 0 & s_5 & c_5 & L_{24} v_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{K}_5$$

and so on for K4, K3, K2, K1



DGM of a 6 DOF robot



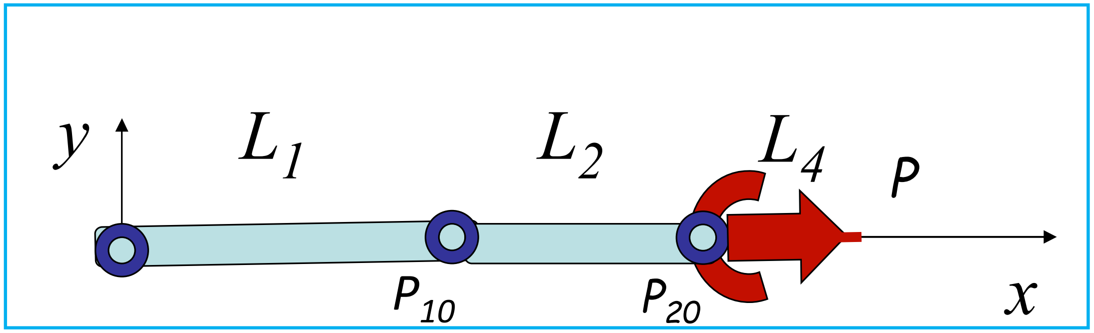


Direct geometric model of the robot:

$$P(\theta i) = (K1\ K2\ K3\ K4\ K5\ K6\ )\ P_0$$

Develop all the 6 homogeneous matrices Ki

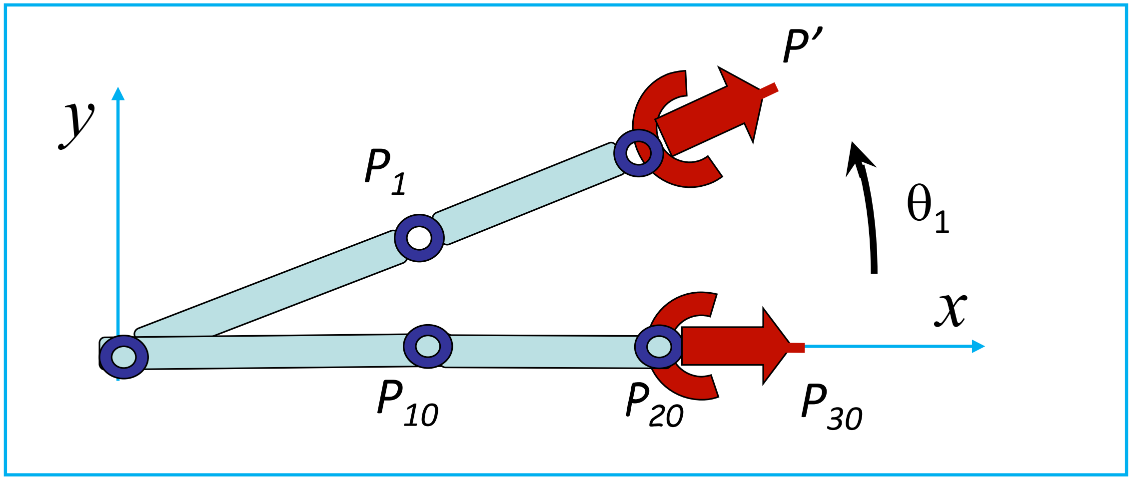
Reference position of any point P of the tool



$$L_{124} = L_1 + L_2 + L_4$$

appendix

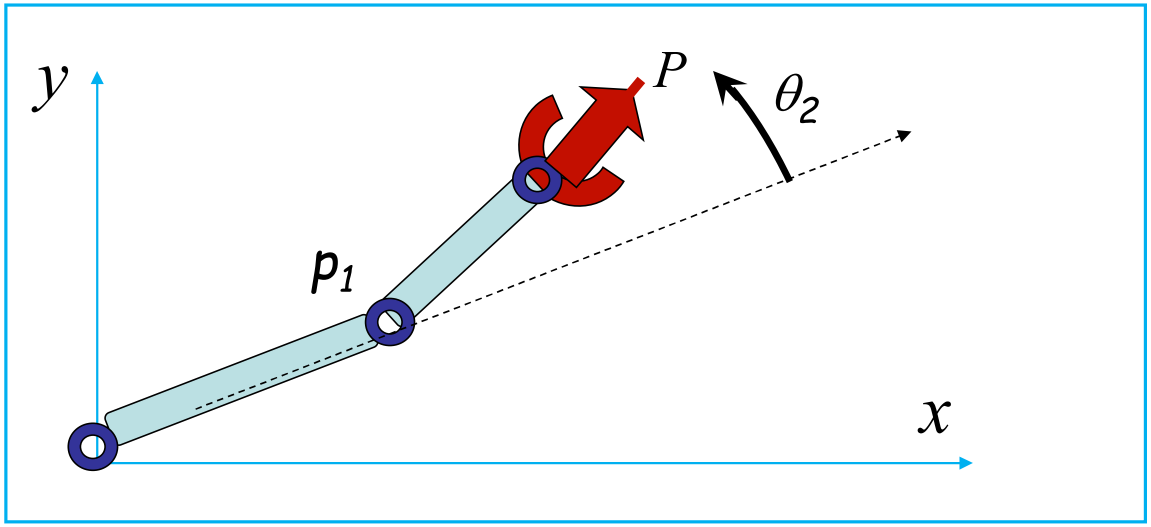
1) Rotation of θ1 around the origin



$$P' = P(\theta_1, \theta_2 = \theta_4 = 0) = \begin{bmatrix} \boxed{\begin{matrix} c_1 & -s_1 \\ s_1 & c_1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} L_{124} \\ 0 \\ 1 \end{bmatrix}$$

$$L_{124} = L_1 + L_2 + L_4$$

2.) Rotation of θ_2 around p_1



<u>Center of rotation</u>

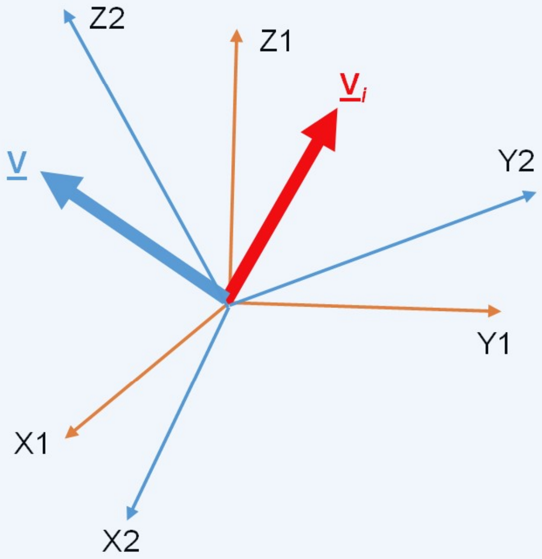
$$\underline{p}_1 = \begin{bmatrix} L_1 c_1 \\ L_1 s_1 \\ 1 \end{bmatrix}$$

$$P'(\theta_1, \theta_2) = \begin{bmatrix} \mathbf{R}_2 & p_1 - \mathbf{R}_2 p_1 \\ 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_{124} \\ 0 \\ 1 \end{bmatrix}$$

Appendix

Computing a generalized rotation matrix

Generalized rotation matrix



Let us consider a vector V expressed in the base frame {1} as follows

$\underline{V} = V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1$

$\underline{V} = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$

We can write:

$\underline{V} = V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1 = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$

V has in {2} the same coordinates as $\underline{V}_i$ in {1}, that are $[V_{x2}, V_{y2}, V_{z2}]$

We search to :
Express the coordinates of V in {1} , that are $[V_{x1}, V_{y1}, V_{z1}]$ function of the initial vector Vi , ie. $[V_{x2}, V_{y2}, V_{z2}]$

$$\begin{bmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{bmatrix} = [?].\begin{bmatrix} V_{x2} \\ V_{y2} \\ V_{z2} \end{bmatrix}$$

Consideration of the **active transformation** wrt the basic referential.

Rotation matrices

irection cosine matrix

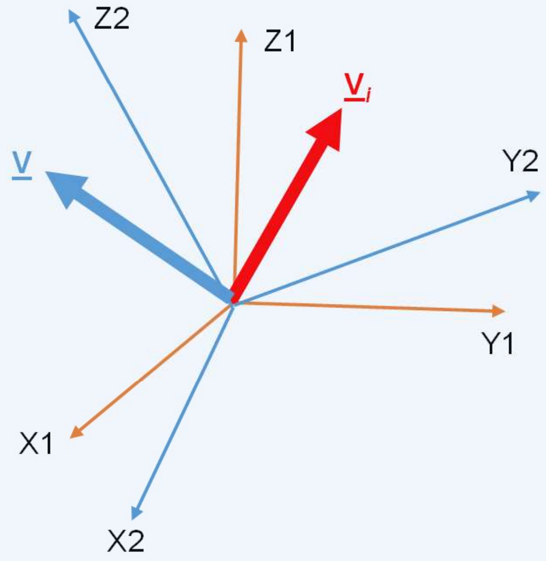
Active and Passive Transformation

$$R = \begin{bmatrix} \underline{x}_2\underline{x}_1 & \underline{y}_2\underline{x}_1 & \underline{z}_2\underline{x}_1 \\ \underline{x}_2\underline{y}_1 & \underline{y}_1\underline{y}_2 & \underline{z}_2\underline{y}_1 \\ \underline{x}_2\underline{z}_1 & \underline{y}_2\underline{z}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

$$R_p = \begin{bmatrix} \underline{x}_1\underline{x}_2 & \underline{x}_2\underline{y}_1 & \underline{x}_2\underline{z}_1 \\ \underline{y}_2\underline{x}_1 & \underline{y}_1\underline{y}_2 & \underline{y}_2\underline{z}_1 \\ \underline{z}_2\underline{x}_1 & \underline{z}_2\underline{y}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

$$R = R_p^T = \begin{bmatrix} \underline{x}_2\underline{x}_1 & \underline{y}_2\underline{x}_1 & \underline{z}_2\underline{x}_1 \\ \underline{x}_2\underline{y}_1 & \underline{y}_1\underline{y}_2 & \underline{z}_2\underline{y}_1 \\ \underline{x}_2\underline{z}_1 & \underline{y}_2\underline{z}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

Generalized rotation matrix



Let us consider a vector V expressed in the base frame {1} as follows

$$\underline{V} = V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1$$

$$\underline{V} = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$$

$$\begin{bmatrix}V_{x1}\\V_{y1}\\V_{z1}\end{bmatrix} = [?].\begin{bmatrix}V_{x2}\\V_{y2}\\V_{z2}\end{bmatrix}$$

We can write:

$$\underline{V} = V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1 = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$$

$$V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1 = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$$

$$V_{x1}.\underline{x}_1.\underline{x}_1 + V_{y1}.\underline{y}_1.\cancel{\underline{x}_1} + V_{z1}.\underline{z}_1.\cancel{\underline{x}_1} = V_{x2}.\underline{x}_2.\underline{x}_1 + V_{y2}.\underline{y}_2.\underline{x}_1 + V_{z2}.\underline{z}_2.\underline{x}_1$$

$$\mathbf{V_{x1} = V_{x2}.\underline{x}_2.\underline{x}_1 + V_{y2}.\underline{y}_2.\underline{x}_1 + V_{z2}.\underline{z}_2.\underline{x}_1}$$

$$V_{x1}.\underline{x}_1.\underline{y}_1 + V_{y1}.\underline{y}_1.\underline{y}_1 + V_{z1}.\underline{z}_1.\underline{y}_1 = V_{x2}.\underline{x}_2.\underline{y}_1 + V_{y2}.\underline{y}_2.\underline{y}_1 + V_{z2}.\underline{z}_2.\underline{y}_1$$

$$V_{x1}.\cancel{\underline{x}_1.\underline{y}_1} + V_{y1}.\underline{y}_1.\underline{y}_1 + V_{z1}.\cancel{\underline{z}_1.\underline{y}_1} = V_{x2}.\underline{x}_2.\underline{y}_1 + V_{y2}.\underline{y}_2.\underline{y}_1 + V_{z2}.\underline{z}_2.\underline{y}_1$$

$$\mathbf{V_{y1} = V_{x2}.\underline{x}_2.\underline{y}_1 + V_{y2}.\underline{y}_2.\underline{y}_1 + V_{z2}.\underline{z}_2.\underline{y}_1}$$

$$\mathbf{V_{z1} = V_{x2}.\underline{x}_2.\underline{z}_1\ V_{y2}.\underline{y}_2.\underline{z}_1 + V_{z2}.\underline{z}_2.\ .\underline{z}_1}$$

Consideration of the **active transformation** wrt the basic referential.

Rotation matrix
Direction cosine matrix

Vx1 = Vx2.x2 . x1 + Vy2.y2 . x1 + Vz2.z2 . x1

Vy1 = Vx2.x2 . y1 + Vy2.y2 . y1 + Vz2.z2 . y1

Vz1 = Vx2.x2 . z1 + Vy2.y2 . z1 + Vz2.z2 . z1

$$\begin{bmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{bmatrix} = \begin{bmatrix} \underline{x}2.\,\underline{x}1 & \underline{y}2.\,\underline{x}1 & \underline{z}2.\,\underline{x}1 \\ \underline{x}2.\,\underline{y}1 & \underline{y}2.\,\underline{y}1 & \underline{z}2.\,\underline{y}1 \\ \underline{x}2.\,\underline{z}1 & \underline{y}2.\,\underline{z}1 & \underline{z}2.\,\underline{z}1 \end{bmatrix} \begin{bmatrix} V_{x2} \\ V_{y2} \\ V_{z2} \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} \underline{x}2.\,\underline{x}1 & \underline{y}2.\,\underline{x}1 & \underline{z}2.\,\underline{x}1 \\ \underline{x}2.\,\underline{y}1 & \underline{y}2.\,\underline{y}1 & \underline{z}2.\,\underline{y}1 \\ \underline{x}2.\,\underline{z}1 & \underline{y}2.\,\underline{z}1 & \underline{z}2.\,\underline{z}1 \end{bmatrix}$$

Rotation matrix
Direction cosine matrix
Passive Transformation

We can write :

$\underline{\mathbf{V}} = V_{x1}.\underline{x}_1 + V_{y1}.\underline{y}_1 + V_{z1}.\underline{z}_1 = V_{x2}.\underline{x}_2 + V_{y2}.\underline{y}_2 + V_{z2}.\underline{z}_2$

$$\begin{bmatrix} V_{x2} \\ V_{y2} \\ V_{z2} \end{bmatrix} = \begin{bmatrix} \underline{x}_1\underline{x}_2 & \underline{x}_2\underline{y}_1 & \underline{x}_2\underline{z}_1 \\ \underline{y}_2\underline{x}_1 & \underline{y}_1\underline{y}_2 & \underline{y}_2\underline{z}_1 \\ \underline{z}_2\underline{x}_1 & \underline{z}_2\underline{y}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix} \begin{bmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{bmatrix}$$

$$R_p = \begin{bmatrix} \underline{x}_1\underline{x}_2 & \underline{x}_2\underline{y}_1 & \underline{x}_2\underline{z}_1 \\ \underline{y}_2\underline{x}_1 & \underline{y}_1\underline{y}_2 & \underline{y}_2\underline{z}_1 \\ \underline{z}_2\underline{x}_1 & \underline{z}_2\underline{y}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

$$\boldsymbol{R} = \boldsymbol{R}_p^T = \begin{bmatrix} \underline{x}2.\,\underline{x}1 & \underline{y}2.\,\underline{x}1 & \underline{z}2.\,\underline{x}1 \\ \underline{x}2.\,\underline{y}1 & \underline{y}2.\,\underline{y}1 & \underline{z}2.\,\underline{y}1 \\ \underline{x}2.\,\underline{z}1 & \underline{y}2.\,\underline{z}1 & \underline{z}2.\,\underline{z}1 \end{bmatrix}$$

Rotation matrices

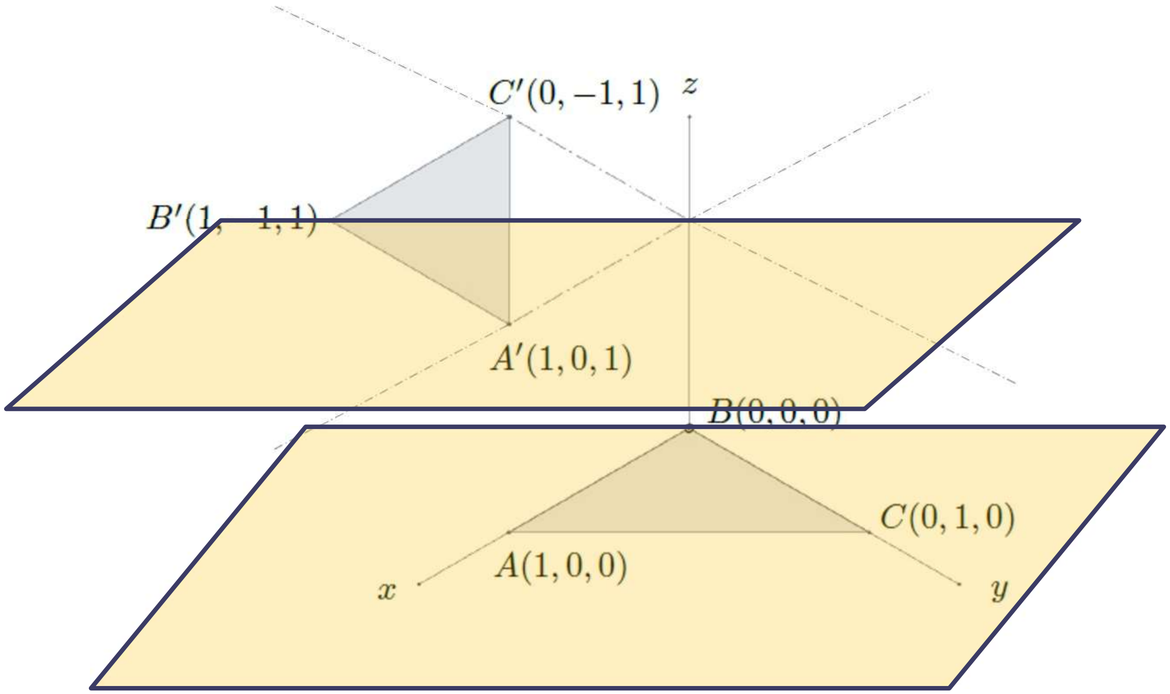
irection cosine matrix

Active and Passive Transformation

$$\boldsymbol{R} = \begin{bmatrix} \underline{x}_2\underline{x}_1 & \underline{y}_2\underline{x}_1 & \underline{z}_2\underline{x}_1 \\ \underline{x}_2\underline{y}_1 & \underline{y}_1\underline{y}_2 & \underline{z}_2\underline{y}_1 \\ \underline{x}_2\underline{z}_1 & \underline{y}_2\underline{z}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

$$\boldsymbol{R_p} = \begin{bmatrix} \underline{x}_1\underline{x}_2 & \underline{x}_2\underline{y}_1 & \underline{x}_2\underline{z}_1 \\ \underline{y}_2\underline{x}_1 & \underline{y}_1\underline{y}_2 & \underline{y}_2\underline{z}_1 \\ \underline{z}_2\underline{x}_1 & \underline{z}_2\underline{y}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$

$$\boldsymbol{R} = \boldsymbol{R_p^T} = \begin{bmatrix} \underline{x}_2\underline{x}_1 & \underline{y}_2\underline{x}_1 & \underline{z}_2\underline{x}_1 \\ \underline{x}_2\underline{y}_1 & \underline{y}_1\underline{y}_2 & \underline{z}_2\underline{y}_1 \\ \underline{x}_2\underline{z}_1 & \underline{y}_2\underline{z}_1 & \underline{z}_1\underline{z}_2 \end{bmatrix}$$


$C'(0,-1,1)$
z
$B'(1,-1,1)$
$A'(1,0,1)$
$B(0,0,0)$
$C(0,1,0)$
$A(1,0,0)$
x
y