Supplement – Dynamic Exercises

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Example 1:

Deduce the forward and inverse dynamic equations of a DC motor.
***i*,** is the motor control current.
***J_m*,** is the inertia (moment of inertia) of the motor.
J_L, is the inertia (moment of inertia) of the load.
***k_c*,** is the torque constant of the motor.

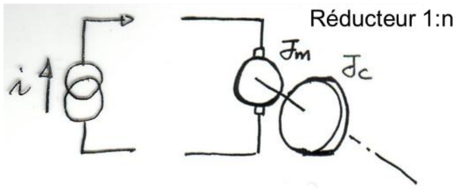

Fig. 1 DC motorized shaft, gearbox and load

Solution

Most robot axes use DC motors, often brushless, with a reducer* to increase the torque capacity of the actuation.

*) A reducer is primarily a torque multiplier.

To write the equation of motion dynamics of a rotating axis, we must start with Euler's equation, which I propose to you in the following format:

$$\sum \ \Gamma_X = J_{t\_rX}\ddot{\theta}_X$$

- $\sum \ \Gamma_X$ is the sum of the moments, all taken with respect to a chosen axis of rotation $\boldsymbol{X}$.
- J_{t_rX} is the total moment of inertia referred to the axis of rotation $\boldsymbol{X}$**.** This total moment of inertia is the moment of inertia corresponding to a single load rotating around the axis $\boldsymbol{X}$.
- $\ddot{\theta}_X$ is the acceleration relative to the axis $\boldsymbol{X}$.

In the case of the axis in Figure 1, by referring all the moments to the axis of rotation of the motor, we write the total moment of inertia as seen by the motor as follows:

$$J_{t\_rX} = J_m + \frac{J_L}{n^2}$$

The only active torque is the motor torque Γ_m. There is no resistive torque and the motor torque is proportional to the current by the torque constant k_c. Current control corresponds to torque control. It is**« kif-kif »,** both terms exist and are used identically: *current control* or *torque control*.

$$\Gamma_m = k_c i$$

The equation of dynamics can then be written simply as: $\sum \ \Gamma_X = \Gamma_m = \left(J_m + \frac{J_L}{n^2}\right)\ddot{\theta}_m = k_c i$

The **inverse** dynamic model may then one the following expressions:

$\boxed{\Gamma_m = \left(J_m + \frac{J_L}{n^2}\right)\ddot{\theta}_m}$ **or** $\boxed{i = \frac{1}{k_c}\left(J_m + \frac{J_L}{n^2}\right)\ddot{\theta}_m}$, depending on whether we consider the current or the torque as the input quantity.

The **direct** dynamic model represented by the transfer function is: $\boxed{\frac{\theta_m}{\Gamma_m} = \frac{1}{\left(J_m + \frac{J_C}{n^2}\right)s^2}}$

Parentheses 1 – Combination *Motor + Gearbox* - Calculation of the reported moment of inertia

Consider a rotating axis constructed by a combination **motor + gearbox + rotating load**.

J

J_m, is the moment of inertia of the motor.

J_L, is the moment of inertia of the load.

The objective is to determine the total moment of inertia (or **inertia through misuse of language**) as seen by the motor. This inertia is that which the motor is supposed to perceive from its axis of rotation.

$$E_{KT} = \frac{1}{2} J_{tot\_m} \omega_m^2 = \frac{1}{2} J_m \omega_m^2 + \frac{1}{2} J_L \omega_L^2$$

E_{KT} is the total kinetic energy.

J_{tot_m} is the total inertia as seen by the motor, it is our unknown.

By expressing the speed of the load as a function of that of the motor, we obtain:

$$E_{KT} = \frac{1}{2} J_{tot\_m} \omega_m^2 = \frac{1}{2} J_m \omega_m^2 + \frac{1}{2} J_L \left(\frac{\omega_m}{n}\right)^2 = \frac{1}{2}\left(J_m + \frac{J_L}{n^2}\right) \omega_m^2$$

So, finally:

$$J_{tot\_m} = \left(J_m + \frac{J_L}{n^2}\right)$$

Considering the transmission efficiency η, the kinetic energy of the load is reduced by a coefficient η - **It's as if the energy required should be higher.**

$$E_{KT} = \frac{1}{2} J_{tot\_m}\, \omega_m^2 = \frac{1}{2} J_m \omega_m^2 + \frac{1}{\eta}\left\{\frac{1}{2} J_L \left(\frac{\omega_m}{n}\right)^2\right\} = \frac{1}{2}\left(J_m + \frac{J_L}{\eta\, n^2}\right) \omega_m^2$$

$$J_{tot\_m} = \left(J_m + \frac{J_L}{\eta\, n^2}\right) \left(J_m + n\frac{J_c}{n^2}\right)$$

Another demonstration. The sum of the moments M_L at the load side is written as follows:

$$\sum M_L = J_{tot\_m} \frac{d\omega_m}{dt} = \left(J_m + J_{L\_m}\right) \frac{d\omega_m}{dt}$$

$$= J_m \frac{d\omega_m}{dt} + \frac{1}{\eta}\left(\frac{1}{n}\left(J_L \frac{d\omega_c}{dt}\right)\right) = \left(J_m + \frac{1}{\eta}\left(\frac{J_L}{n^2}\right)\right) \frac{d\omega_m}{dt}$$

With:

$\left(J_L \frac{d\omega_c}{dt}\right)$ is the dynamic torque at the output, due to the load J_L.

$\left(\frac{1}{n}\left(J_L \frac{d\omega_c}{dt}\right)\right)$ is the dynamic torque due to the load J_L..., brought back to the motor side (excluding efficiency)

$\frac{1}{\eta}\left(\frac{1}{n}\left(J_L \frac{d\omega_c}{dt}\right)\right)$ is the dynamic torque due to the load J_c...brought back to the motor side, taking efficiency into account. ____

Similarly, we write the total moment of inertia **brought back to the load side** :

$$E_{KT} = \frac{1}{2} J_{tot\_c}\, \omega_c^2 = \frac{1}{2} J_m \omega_m^2 + \frac{1}{\eta}\left\{\frac{1}{2} J_L \omega_c^2\right\} = \frac{1}{2}\left(J_m n^2 + \frac{J_L}{\eta}\right)\omega_c^2$$

$J_{tot\_L} = (J_L + J_m n^2)$ / at a unit efficiency (no loss)

Considering the efficiency, we will have $J_{tot\_L} = \left(\frac{J_L}{\eta} + J_m\, n^2\right)$

Parenthesis 2 - Combination –*Motor + gearbox* -

Observations

The total moment of inertia of a motor + gearbox combination is written as follows:

$$J_{tot\_m} = \left(J_m + \frac{J_L}{n^2}\right)$$

- The higher the reduction ratio, the less the motor sees (perceives) the load. This is why servomechanisms with a high reduction ratio are easier to control.

- A multi-stage gearbox has higher viscosity due to lubrication and shear between stages. This inherent mechanical damping also makes servomechanisms with multi-stage gearboxes easier to control.

- The reduction reduces the disturbance and resonant torques from the load side by a ratio n.

These beautiful things, however, come with several drawbacks.

- The increase in the reduction ratio is subject to a reduction in acceleration capacity (see course on actuators and optimal reduction ratio).

- A gearbox costs roughly the same as a motor… so it's expensive , and even more expensive in the case of constraints on 1) positioning accuracy, 2) absence of play, 3) stiffness and 4) dynamics.

- A gearbox requires assembly and ongoing maintenance for high dynamic applications.

Example 2:

Deduce the transfer function of the dynamic system governed by the following differential equation:

$$I\ddot{\theta} = \Gamma_m - mglsin\theta - b\dot{\theta}^2 sign(\theta)$$

Solution

This dynamic system is nonlinear due to the presence of two nonlinear functions, sine and sign. A representation of the dynamic model using a transfer function is only possible if we perform a tangent linearization around an operating point, using a Taylor series expansion.

It is absolutely essential to note that…

The concepts of «***transfer function*** ", of "***vibrational modes***" ", of "***poles*** and of ***eigen values***" are purely linear concepts, and therefore assume linear dynamic behaviors.