Lecture 9

Conservation of Energy and Momentum

Poynting Theorem

Energy Conservation So Far

So far we have seen and used

Electric energy

$$W_{\mathrm{e}} = \frac{1}{2} \iiint_V \mathbf{D} \cdot \mathbf{E} \, dV$$

Poynting vector

$$\mathbf{S} = \frac{1}{2} \mathrm{Re}\left\{\mathbf{E} \times \mathbf{H}^*\right\}$$

Magnetic energy

$$W_{\mathrm{m}} = \frac{1}{2} \iiint_V \mathbf{B} \cdot \mathbf{H} \, dV$$

Where do these expressions come from and how to derive them?

Understanding the Origin of the Poynting Theorem

What is the power done on charges by the fields ?

Lorentz Force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy is force time distance

$$dW = \mathbf{F} \cdot d\mathbf{l}$$

Note that velocity is

$$\mathbf{v} = \frac{d\mathbf{l}}{dt}$$

$$dW = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt = q\mathbf{E} \cdot \mathbf{v}dt$$

$$\frac{dW}{dt} = q\mathbf{E} \cdot \mathbf{v}$$

$$\begin{cases} q = \int_V \rho \; dV \\ \mathbf{J} = \rho \mathbf{v} \end{cases}$$

$$\frac{dW}{dt} = \int_V \mathbf{E} \cdot \mathbf{J} \; dV$$

Rate of work done on charges by an electric field

Griffiths, “Introduction to electrodynamics”, 1999

Derivation of Time-Domain Poynting Theorem

In what follows, the fields are in the time domain!

pre-multiplying by $\mathbf{H}\cdot$ $\quad \nabla \times \mathbf{E} = -\mathbf{K} - \frac{\partial \mathbf{B}}{\partial t}$

pre-multiplying by $\mathbf{E}\cdot$ $\quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

subtracting -

Remember that

$$\frac{dW}{dt} = \int_V \mathbf{E} \cdot \mathbf{J} \; dV$$

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \mathbf{K} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

↓

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \mathbf{K} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

Poynting vector → $(\mathbf{E} \times \mathbf{H})$

↓

$$\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \mathbf{K} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

Consider the identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

Some Mathematical Steps

Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \mathbf{K} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \frac{1}{2}\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \longleftarrow \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \frac{1}{2}\mathbf{E} \cdot \frac{\partial}{\partial t}\left(\epsilon_0 \mathbf{E} + \mathbf{P}\right)$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \frac{\epsilon_0}{2}\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{2}\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t}$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \frac{\epsilon_0}{2}\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{2}\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \frac{1}{2}\left(\mathbf{P} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{P} \cdot \frac{\partial \mathbf{E}}{\partial t}\right)$$

We add this self-cancelling term

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{E} \cdot \mathbf{D}\right) + \frac{1}{2}\left(\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{P} \cdot \frac{\partial \mathbf{E}}{\partial t}\right)$$

Time-Domain Poyting Theorem

Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \mathbf{K} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

where the last two terms are given by

$$\begin{cases} \mathbf{E} \cdot \dfrac{\partial \mathbf{D}}{\partial t} = \dfrac{1}{2} \dfrac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D}) + \dfrac{1}{2} \left(\mathbf{E} \cdot \dfrac{\partial \mathbf{P}}{\partial t} - \mathbf{P} \cdot \dfrac{\partial \mathbf{E}}{\partial t} \right) \\ \mathbf{H} \cdot \dfrac{\partial \mathbf{B}}{\partial t} = \dfrac{1}{2} \dfrac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{B}) + \dfrac{\mu_0}{2} \left(\mathbf{H} \cdot \dfrac{\partial \mathbf{M}}{\partial t} - \mathbf{M} \cdot \dfrac{\partial \mathbf{H}}{\partial t} \right) \end{cases}$$

Poynting theorem

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} = -I_{\mathrm{J}} - I_{\mathrm{K}} - I_{\mathrm{P}} - I_{\mathrm{M}}$$

density of EM energy

$$w = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$$

work done on charges

$$I_{\mathrm{J}} = \mathbf{E} \cdot \mathbf{J}$$

$$I_{\mathrm{K}} = \mathbf{H} \cdot \mathbf{K}$$

work done on polarizations

$$I_{\mathrm{P}} = \frac{1}{2} \left(\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{P} \cdot \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$I_{\mathrm{M}} = \frac{\mu_0}{2} \left(\mathbf{H} \cdot \frac{\partial \mathbf{M}}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{H}}{\partial t} \right)$$

Interpretation of the Poynting Theorem

Poynting theorem

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} = -I_{\mathrm{J}} - I_{\mathrm{K}} - I_{\mathrm{P}} - I_{\mathrm{M}}$$

Lorentz Force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

Density of EM energy inside volume *V*

$$w = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$$

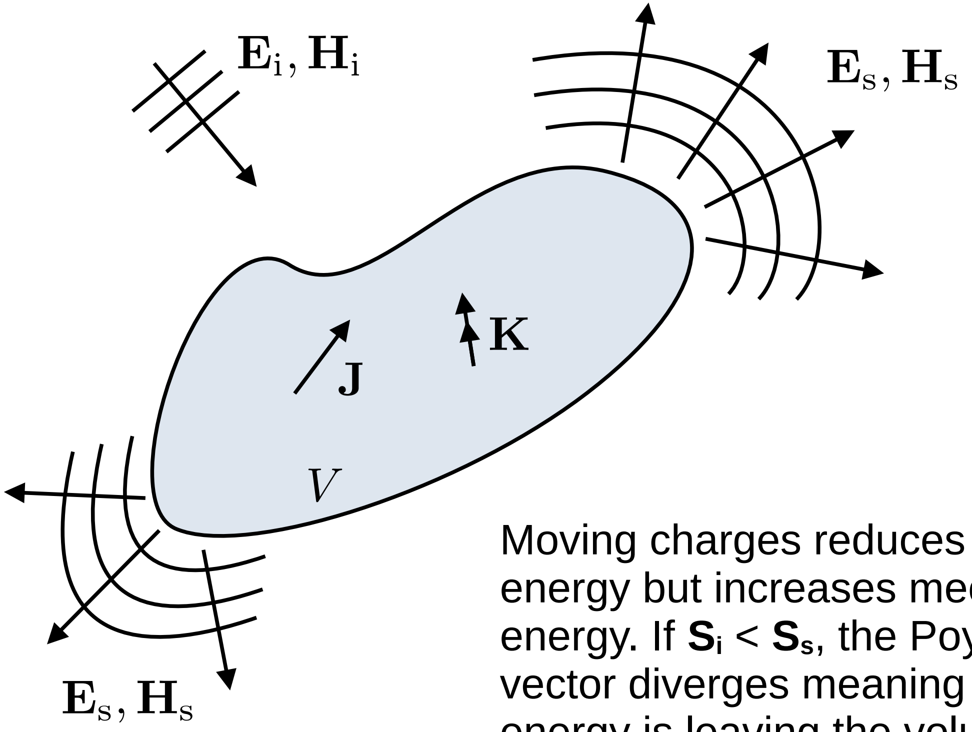
Work done on charges and polarizations

$$I_{\mathrm{J}} = \mathbf{E}\cdot\mathbf{J}$$

$$I_{\mathrm{K}} = \mathbf{H}\cdot\mathbf{K}$$

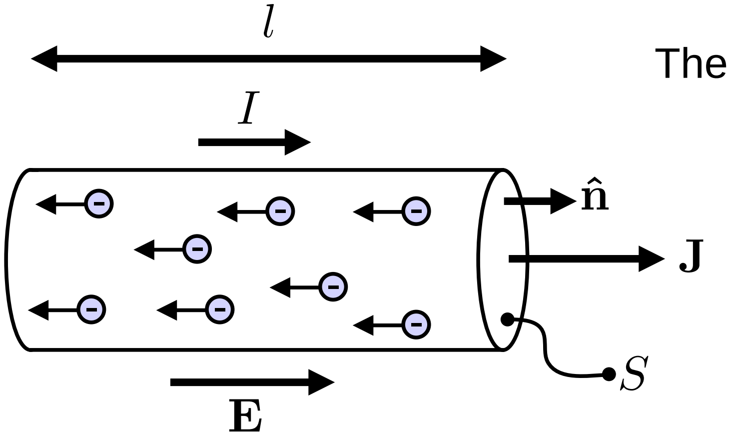
$$I_{\mathrm{P}} = \frac{1}{2}\left(\mathbf{E}\cdot\frac{\partial \mathbf{P}}{\partial t} - \mathbf{P}\cdot\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$I_{\mathrm{M}} = \frac{\mu_0}{2}\left(\mathbf{H}\cdot\frac{\partial \mathbf{M}}{\partial t} - \mathbf{M}\cdot\frac{\partial \mathbf{H}}{\partial t}\right)$$



Moving charges reduces EM energy but increases mechanical energy. If $\mathbf{S}_{\mathrm{i}} < \mathbf{S}_{\mathrm{s}}$, the Poynting vector diverges meaning that EM energy is leaving the volume.

Joule Law



The work done on electric charges is

$$I_{\mathrm{J}} = \mathbf{E} \cdot \mathbf{J}$$

$$P = \iiint_V \mathbf{E} \cdot \mathbf{J} \; dV$$

In a conductive material

$$\mathbf{J} = \sigma \mathbf{E}$$

In the conductor, the electric field is

$$E = \frac{V}{l}$$

Remember that resistance is

$$R = \frac{l}{\sigma S}$$

$$\mathbf{E} \cdot \mathbf{J} = \sigma E^2 = \frac{l}{RS} \left(\frac{V}{l} \right)^2 = \frac{V^2}{RlS}$$

$$P = \iiint_V \mathbf{E} \cdot \mathbf{J} \; dV = \iiint_V \frac{V^2}{RlS} \; dV = \frac{V^2}{R}$$

Ohm law

$$V = RI$$

$$P = \frac{V^2}{R} \qquad P = VI \qquad P = RI^2$$

Time Averaging Operation

Assuming time-harmonic fields, the instantaneous Poynting vector is

$$\begin{aligned}\mathbf{S}(t) &= \mathbf{E}(t) \times \mathbf{H}(t) \\ &= \mathrm{Re}\{\mathbf{E}e^{j\omega t}\} \times \mathrm{Re}\{\mathbf{H}e^{j\omega t}\} \\ &= \frac{1}{2}\left(\mathbf{E}e^{j\omega t} + \mathbf{E}^* e^{-j\omega t}\right) \times \frac{1}{2}\left(\mathbf{H}e^{j\omega t} + \mathbf{H}^* e^{-j\omega t}\right) \\ &= \frac{1}{4}\left(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}e^{2j\omega t} + \mathbf{E}^* \times \mathbf{H}^* e^{-2j\omega t}\right) \\ &= \frac{1}{2}\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\} + \frac{1}{2}\mathrm{Re}\{\mathbf{E} \times \mathbf{H}e^{2j\omega t}\}\end{aligned}$$

Using

$$\mathrm{Re}\{z\} = \frac{1}{2}\left(z + z^*\right)$$

The time-average Poynting vector is

$$\langle \mathbf{S} \rangle = \frac{1}{T}\int_0^T \mathbf{S}(t)dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \left[\frac{1}{2}\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\} + \frac{1}{2}\mathrm{Re}\{\mathbf{E} \times \mathbf{H}e^{2j\omega t}\}\right] dt = \frac{1}{2}\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\}$$

Instantaneous Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

Time-average Poynting vector

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\}$$

Time-Average Poynting Theorem

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} = -I_\mathrm{J} - I_\mathrm{K} - I_\mathrm{P} - I_\mathrm{M} \quad \longrightarrow \quad \nabla \cdot \langle \mathbf{S} \rangle = -\langle I_\mathrm{J} \rangle - \langle I_\mathrm{K} \rangle - \langle I_\mathrm{P} \rangle - \langle I_\mathrm{M} \rangle$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$w = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$$

$$I_\mathrm{J} = \mathbf{E}\cdot\mathbf{J}$$

$$I_\mathrm{K} = \mathbf{H}\cdot\mathbf{K}$$

$$I_\mathrm{P} = \frac{1}{2}\left(\mathbf{E}\cdot\frac{\partial \mathbf{P}}{\partial t} - \mathbf{P}\cdot\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$I_\mathrm{M} = \frac{\mu_0}{2}\left(\mathbf{H}\cdot\frac{\partial \mathbf{M}}{\partial t} - \mathbf{M}\cdot\frac{\partial \mathbf{H}}{\partial t}\right)$$

Note that

$$\left\langle \frac{\partial w}{\partial t} \right\rangle = 0$$

where we have used

$$\frac{\partial}{\partial t} \rightarrow j\omega$$

$$\mathrm{Re}\{jz\} = -\mathrm{Im}\{z\}$$

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\}$$

$$\langle w \rangle = \frac{1}{4}\mathrm{Re}\left\{\mathbf{E}\cdot\mathbf{D}^* + \mathbf{H}\cdot\mathbf{B}^*\right\}$$

$$\langle I_\mathrm{J} \rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{E}\cdot\mathbf{J}^*\}$$

$$\langle I_\mathrm{K} \rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{H}\cdot\mathbf{K}^*\}$$

$$\langle I_\mathrm{P} \rangle = \frac{\omega}{4}\mathrm{Im}\left\{\mathbf{P}^*\cdot\mathbf{E} - \mathbf{E}^*\cdot\mathbf{P}\right\}$$

$$\langle I_\mathrm{M} \rangle = \frac{\omega\mu_0}{4}\mathrm{Im}\left\{\mathbf{M}^*\cdot\mathbf{H} - \mathbf{H}^*\cdot\mathbf{M}\right\}$$

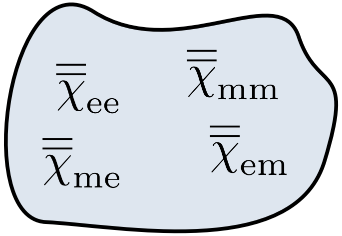
Towards Lossless-Gainless Conditions

Time-average Poynting theorem

$$\nabla \cdot \langle \mathbf{S} \rangle = -\langle I_{\mathrm{J}} \rangle - \langle I_{\mathrm{K}} \rangle - \langle I_{\mathrm{P}} \rangle - \langle I_{\mathrm{M}} \rangle$$

Let's consider a medium without currents (**J** = **K** = 0) but only induced polarizations

Medium described in terms of susceptibilities



Induced polarizations in the medium

$$\mathbf{P} = \epsilon_0 \overline{\overline{\chi}}_{\mathrm{ee}} \cdot \mathbf{E} + \frac{1}{c_0} \overline{\overline{\chi}}_{\mathrm{em}} \cdot \mathbf{H}$$

$$\mathbf{M} = \overline{\overline{\chi}}_{\mathrm{mm}} \cdot \mathbf{H} + \frac{1}{\eta_0} \overline{\overline{\chi}}_{\mathrm{me}} \cdot \mathbf{E}$$

$$\langle I_{\mathrm{P}} \rangle = \frac{\omega}{4} \mathrm{Im}\left\{ \mathbf{P}^* \cdot \mathbf{E} - \mathbf{E}^* \cdot \mathbf{P} \right\}$$

$$\langle I_{\mathrm{M}} \rangle = \frac{\omega \mu_0}{4} \mathrm{Im}\left\{ \mathbf{M}^* \cdot \mathbf{H} - \mathbf{H}^* \cdot \mathbf{M} \right\}$$

example of derivation

$$\mathbf{P}^* \cdot \mathbf{E}$$

$$\epsilon_0 \overline{\overline{\chi}}_{\mathrm{ee}}^* \cdot \mathbf{E}^* \cdot \mathbf{E}$$

$$\epsilon_0 \mathbf{E}^* \cdot \overline{\overline{\chi}}_{\mathrm{ee}}^{\dagger} \cdot \mathbf{E}$$

after substitution and re-arranging the terms

$$\langle I_{\mathrm{P}} \rangle = \frac{\omega \epsilon_0}{4} \mathrm{Im}\left\{ \mathbf{E}^* \cdot \left(\overline{\overline{\chi}}_{\mathrm{ee}}^{\dagger} - \overline{\overline{\chi}}_{\mathrm{ee}} \right) \cdot \mathbf{E} - 2\eta \mathbf{E}^* \cdot \overline{\overline{\chi}}_{\mathrm{em}} \cdot \mathbf{H} \right\}$$

$$\langle I_{\mathrm{M}} \rangle = \frac{\omega \mu_0}{4} \mathrm{Im}\left\{ \mathbf{H}^* \cdot \left(\overline{\overline{\chi}}_{\mathrm{mm}}^{\dagger} - \overline{\overline{\chi}}_{\mathrm{mm}} \right) \cdot \mathbf{H} + \frac{2}{\eta} \mathbf{E}^* \cdot \overline{\overline{\chi}}_{\mathrm{me}}^{\dagger} \cdot \mathbf{H} \right\}$$

† : conjugate transpose

Lossless-Gainless Conditions

$$\nabla \cdot \langle \mathbf{S} \rangle = -\langle I_\mathrm{P} \rangle - \langle I_\mathrm{M} \rangle$$

$$\begin{cases} \langle I_\mathrm{P} \rangle = \dfrac{\omega \epsilon_0}{4} \mathrm{Im}\left\{ \mathbf{E}^* \cdot \left(\overline{\overline{\chi}}_\mathrm{ee}^\dagger - \overline{\overline{\chi}}_\mathrm{ee} \right) \cdot \mathbf{E} - 2\eta \mathbf{E}^* \cdot \overline{\overline{\chi}}_\mathrm{em} \cdot \mathbf{H} \right\} \\ \langle I_\mathrm{M} \rangle = \dfrac{\omega \mu_0}{4} \mathrm{Im}\left\{ \mathbf{H}^* \cdot \left(\overline{\overline{\chi}}_\mathrm{mm}^\dagger - \overline{\overline{\chi}}_\mathrm{mm} \right) \cdot \mathbf{H} + \dfrac{2}{\eta} \mathbf{E}^* \cdot \overline{\overline{\chi}}_\mathrm{me}^\dagger \cdot \mathbf{H} \right\} \end{cases}$$

After substituting and re-arranging the terms

$$\nabla \cdot \langle \mathbf{S} \rangle = \frac{1}{4} \mathrm{Im}\left\{ \omega \epsilon_0 \mathbf{E}^* \cdot \left(\overline{\overline{\chi}}_\mathrm{ee} - \overline{\overline{\chi}}_\mathrm{ee}^\dagger \right) \cdot \mathbf{E} + \omega \mu_0 \mathbf{H}^* \cdot \left(\overline{\overline{\chi}}_\mathrm{mm} - \overline{\overline{\chi}}_\mathrm{mm}^\dagger \right) \cdot \mathbf{H} + 2k \mathbf{E}^* \cdot \left(\overline{\overline{\chi}}_\mathrm{em} - \overline{\overline{\chi}}_\mathrm{me}^\dagger \right) \cdot \mathbf{H} \right\}$$

If a system is lossless/gainless then

$$\nabla \cdot \langle \mathbf{S} \rangle = 0$$

$$\overline{\overline{\chi}}_\mathrm{ee} = \overline{\overline{\chi}}_\mathrm{ee}^\dagger$$

$$\overline{\overline{\chi}}_\mathrm{mm} = \overline{\overline{\chi}}_\mathrm{mm}^\dagger$$

$$\overline{\overline{\chi}}_\mathrm{em} = \overline{\overline{\chi}}_\mathrm{me}^\dagger$$

A system is lossless/gainless if its susceptibilities satisfy these conditions

Time-Average Electric and Magnetic Energy

Time-average EM energy density

$$\langle w \rangle = \frac{1}{4}\mathrm{Re}\{\mathbf{E}\cdot\mathbf{D}^* + \mathbf{H}\cdot\mathbf{B}^*\}$$

$$\begin{cases} W_\mathrm{e} = \dfrac{1}{4}\mathrm{Re}\left\{\iiint_V \mathbf{E}^*\cdot\mathbf{D}\,dV\right\} \\ W_\mathrm{m} = \dfrac{1}{4}\mathrm{Re}\left\{\iiint_V \mathbf{H}^*\cdot\mathbf{B}\,dV\right\} \end{cases}$$

In a static/time-domain case, we have

Electric energy

$$W_\mathrm{e} = \frac{1}{2}\iiint_V \mathbf{D}\cdot\mathbf{E}\;dV$$

Magnetic energy

$$W_\mathrm{m} = \frac{1}{2}\iiint_V \mathbf{B}\cdot\mathbf{H}\;dV$$

Lossless/Gainless and Reciprocal Medium

Lossless/Gainless conditions

$$\overline{\overline{\chi}}_{\mathrm{ee}} = \overline{\overline{\chi}}_{\mathrm{ee}}^{\dagger}$$

$$\overline{\overline{\chi}}_{\mathrm{mm}} = \overline{\overline{\chi}}_{\mathrm{mm}}^{\dagger}$$

$$\overline{\overline{\chi}}_{\mathrm{em}} = \overline{\overline{\chi}}_{\mathrm{me}}^{\dagger}$$

Reciprocity conditions

$$\overline{\overline{\chi}}_{\mathrm{ee}} = \overline{\overline{\chi}}_{\mathrm{ee}}^{T}$$

$$\overline{\overline{\chi}}_{\mathrm{mm}} = \overline{\overline{\chi}}_{\mathrm{mm}}^{T}$$

$$\overline{\overline{\chi}}_{\mathrm{em}} = -\overline{\overline{\chi}}_{\mathrm{me}}^{T}$$

combining both

$$\overline{\overline{\chi}}_{\mathrm{ee}} = \overline{\overline{\chi}}_{\mathrm{ee}}^{*}$$

$$\overline{\overline{\chi}}_{\mathrm{mm}} = \overline{\overline{\chi}}_{\mathrm{mm}}^{*}$$

$$\overline{\overline{\chi}}_{\mathrm{em}} = -\overline{\overline{\chi}}_{\mathrm{me}}^{*}$$

A system is reciprocal lossless and gainless if

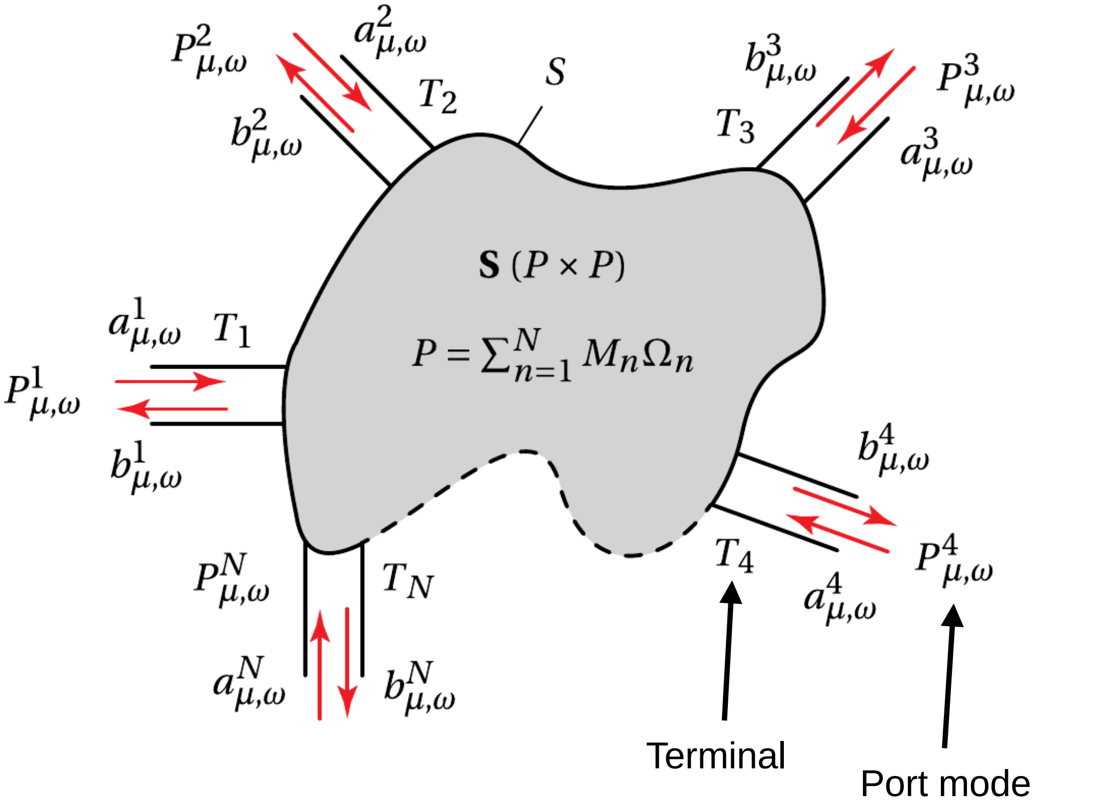
$$\overline{\overline{\chi}}_{\mathrm{ee}} = \overline{\overline{\chi}}_{\mathrm{ee}}^{T} \in \mathbb{R} \qquad \overline{\overline{\chi}}_{\mathrm{mm}} = \overline{\overline{\chi}}_{\mathrm{mm}}^{T} \in \mathbb{R} \qquad \overline{\overline{\chi}}_{\mathrm{em}} = -\overline{\overline{\chi}}_{\mathrm{me}}^{T} \in \mathbb{I}$$

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\xi}} \cdot \mathbf{H}$$

$$\mathbf{B} = \overline{\overline{\zeta}} \cdot \mathbf{E} + \overline{\overline{\mu}} \cdot \mathbf{H}$$

$$\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^{T} \in \mathbb{R} \qquad \overline{\overline{\mu}} = \overline{\overline{\mu}}^{T} \in \mathbb{R} \qquad \overline{\overline{\xi}} = -\overline{\overline{\zeta}}^{T} \in \mathbb{I}$$

Scattering Parameters Formalism



The system total scattering matrix is

$$\mathbf{b} = \overline{\overline{\mathrm{S}}} \cdot \mathbf{a}$$

where

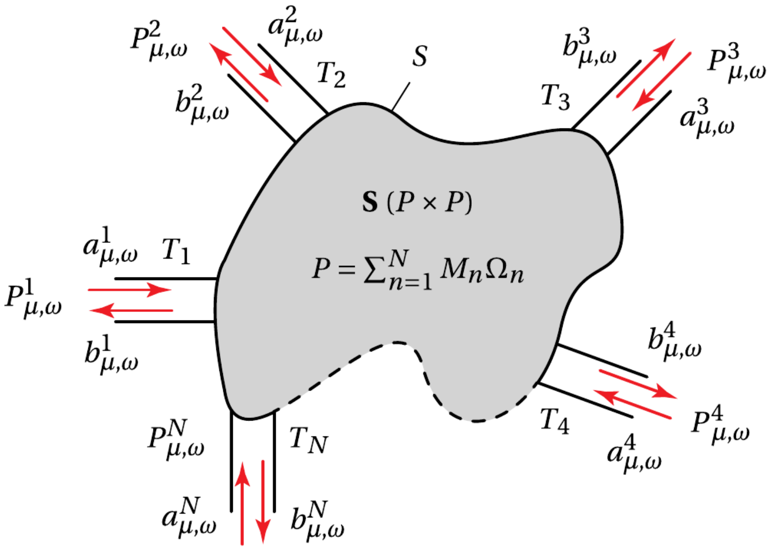
$$\mathbf{a} = [a_1, a_2, \dots]$$
$$\mathbf{b} = [b_1, b_2, \dots]$$

μ: mode (e.g.,TE_{11})
ω: mode frequency

http://dx.doi.org/10.1103/PhysRevApplied.10.047001

Lossless/Gainless Scattering Matrix

Conservation of energy requires that input power equals output power



$$\sum_p |a_p|^2 = \sum_p |b_p|^2$$

$$\mathbf{a} \cdot \mathbf{a}^* = \mathbf{b} \cdot \mathbf{b}^*$$

where $\mathbf{b} = \overline{\overline{\mathrm{S}}} \cdot \mathbf{a}$

$$\mathbf{a} \cdot \mathbf{a}^* = \left(\overline{\overline{\mathrm{S}}} \cdot \mathbf{a}\right) \cdot \left(\overline{\overline{\mathrm{S}}}^* \cdot \mathbf{a}^*\right)$$

$$\mathbf{a} \cdot \mathbf{a}^* = \mathbf{a} \cdot \overline{\overline{\mathrm{S}}}^T \cdot \overline{\overline{\mathrm{S}}}^* \cdot \mathbf{a}^*$$

$$\overline{\overline{\mathrm{S}}}^T \cdot \overline{\overline{\mathrm{S}}}^* = \overline{\overline{\mathrm{I}}}$$

Lossless/Gainless condition

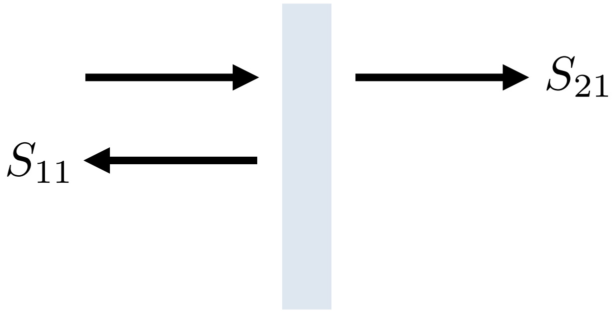
$$\overline{\overline{\mathrm{S}}} \cdot \overline{\overline{\mathrm{S}}}^\dagger = \overline{\overline{\mathrm{I}}}$$

Reciprocity condition

$$\overline{\overline{\mathrm{S}}} = \overline{\overline{\mathrm{S}}}^T$$

Lossless/Gainless Scattering Matrix

Scattering by a thin slab



$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Imposing reciprocity

$$S_{12} = S_{21} = t$$

Lossless/Gainless condition

$$\overline{\overline{\mathrm{S}}} \cdot \overline{\overline{\mathrm{S}}}^{\dagger} = \overline{\overline{\mathrm{I}}}$$

Reciprocity condition

$$\overline{\overline{\mathrm{S}}} = \overline{\overline{\mathrm{S}}}^{T}$$

Imposing passivity and losslessness

$$\overline{\overline{\mathrm{S}}} \cdot \overline{\overline{\mathrm{S}}}^{\dagger} = \begin{bmatrix} S_{11} & t \\ t & S_{22} \end{bmatrix} \cdot \begin{bmatrix} S_{11}^{*} & t^{*} \\ t^{*} & S_{22}^{*} \end{bmatrix} = \begin{bmatrix} |S_{11}|^2 + |t|^2 & S_{11}t^{*} + tS_{22}^{*} \\ tS_{11}^{*} + S_{22}t^{*} & |S_{22}|^2 + |t|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If the slab is symmetric, then $S_{11} = S_{22} = r$

$$|r|^2 + |t|^2 = 1$$

phase of transmission and reflection coefficients are in quadrature

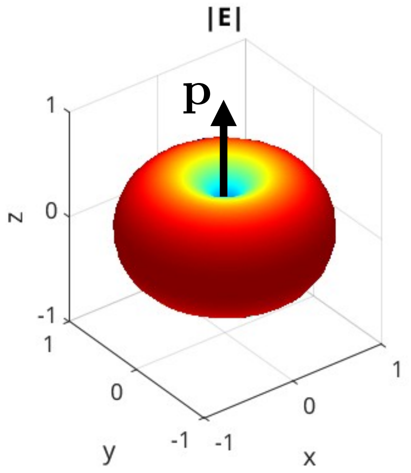
$$tr^{*} + rt^{*} = 0 \longrightarrow \begin{cases} r = |r_0| e^{j\phi_{\mathrm{r}}} \\ t = |t_0| e^{j\phi_{\mathrm{t}}} \end{cases} \longrightarrow \cos(\phi_{\mathrm{t}} - \phi_{\mathrm{r}}) = 0 \longrightarrow \phi_{\mathrm{t}} = \phi_{\mathrm{r}} + \frac{\pi}{2}(2n+1)$$

What Have We Learned So Far….

- The Lorentz force allows us to compute the energy dissipated by the interaction with charges. This is the starting point of the Poynting theorem, which is derived directly from Maxwell equations.
- The time-domain Poynting theorem relates the time variation of energy density in a volume with the divergence of power flow (Poynting vector) and to the work done on “charges” and “polarizations”.
- In the simplest scenario (only electric charges), this leads to the well know Joule law.
- The time-average Poynting theorem allows us to obtain lossless/gainless relationships for the material parameters. You should be able to remember these relations and not confuse them with the reciprocity conditions
- In a lossless/gainless reciprocal system, the permittivity and permeability are real while the magneto-electric coupling terms are imaginary.
- Energy conservation can also be evaluated by looking at the scattering matrix, which should be unitary if the system is gainless/lossless.
- Energy conservation limits our ability to control electromagnetic waves. For instance, we are not free to arbitrarily control the transmission and reflection phase shifts of a slab.

Energy Dissipation of an Electric Dipole

Time-Average Power Scattered by an Electric Dipole



Time-average Poynting theorem

$$\nabla \cdot \langle \mathbf{S} \rangle = -\langle I_{\mathrm{J}} \rangle - \langle I_{\mathrm{K}} \rangle - \langle I_{\mathrm{P}} \rangle - \langle I_{\mathrm{M}} \rangle$$

Let's assume there is only an electric dipole $\mathbf{P} = \mathbf{p}\delta(\mathbf{r}')$

$$\nabla \cdot \langle \mathbf{S} \rangle = \frac{\omega}{4}\mathrm{Im}\left\{\mathbf{P} \cdot \mathbf{E}^* - \mathbf{P}^* \cdot \mathbf{E}\right\} = \frac{\omega}{2}\mathrm{Im}\left\{\mathbf{P} \cdot \mathbf{E}^*\right\}$$

Integrating over a volume and using the divergence theorem

$$\oint \langle \mathbf{S} \rangle \cdot d\mathbf{S} = \frac{\omega}{2}\mathrm{Im}\left\{\int_V \mathbf{P} \cdot \mathbf{E}(\mathbf{r}')^* dV\right\} = \frac{\omega}{2}\mathrm{Im}\left\{\mathbf{p} \cdot \mathbf{E}(\mathbf{r}')^*\right\}$$

$$\langle P \rangle = \frac{\omega}{2}\mathrm{Im}\left\{\mathbf{p} \cdot \mathbf{E}(\mathbf{r}')^*\right\}$$

electric field at the dipole position

consider the identity

$$\mathrm{Im}\{z - z^*\} = 2\mathrm{Im}\{\mathrm{Im}\{z\}j\} = 2\mathrm{Im}\{z\}$$

Power Scattered by an Electric Dipole

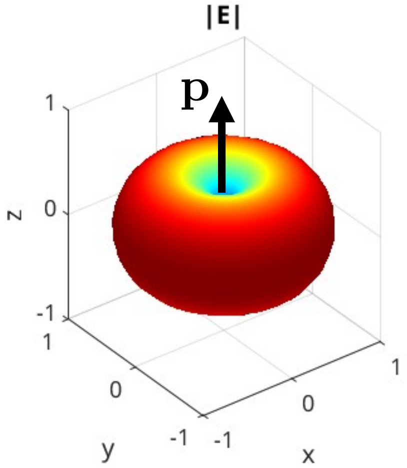
Let's express the electric field as that of the dipole itself

$$\langle P \rangle = \frac{\omega}{2} \mathrm{Im}\left\{ \mathbf{p} \cdot \mathbf{E}(\mathbf{r}')^* \right\}$$

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int \overline{\overline{\mathrm{G}}}_{\mathrm{e}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, dV'$$

$$\begin{cases} \mathbf{P} = \mathbf{p}\delta(\mathbf{r}') \\ \mathbf{J} = j\omega\mathbf{P} = j\omega\mathbf{p}\delta(\mathbf{r}') \end{cases}$$

$$\mathbf{E}(\mathbf{r}) = \omega^2 \mu \overline{\overline{\mathrm{G}}}_{\mathrm{e}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{p}$$



$$\langle P \rangle = \frac{\omega^3 \mu}{2} \mathrm{Im}\left\{ \mathbf{p} \cdot \overline{\overline{\mathrm{G}}}_{\mathrm{e}}(\mathbf{r}',\mathbf{r}')^* \cdot \mathbf{p}^* \right\}$$

$$\langle P \rangle = \frac{\omega^3 \mu}{2} |\mathbf{p}|^2 \hat{\mathbf{n}}_{\mathrm{p}} \cdot \mathrm{Im}\left\{ \overline{\overline{\mathrm{G}}}_{\mathrm{e}}(\mathbf{r}',\mathbf{r}')^* \right\} \cdot \hat{\mathbf{n}}_{\mathrm{p}}$$

where $\mathbf{n}_p$ is a unit vector in the direction of **p**

Diverges !!

$$R = |\mathbf{r}' - \mathbf{r}'| = 0$$

Dyadic Green function

$$\overline{\overline{\mathrm{G}}}_{\mathrm{e}}(\mathbf{r},\mathbf{r}') = \left[\left(1 - \frac{j}{kR} - \frac{1}{k^2 R^2} \right) \overline{\overline{\mathrm{I}}} - \left(1 - \frac{3j}{kR} - \frac{3}{k^2 R^2} \right) \hat{\mathbf{R}}\hat{\mathbf{R}} \right] \frac{e^{-jkR}}{4\pi R}$$

Power Scattered by an Electric Dipole

$$\langle P\rangle = \frac{\omega^3\mu}{2}|\mathbf{p}|^2 \mathrm{Im}\left\{\hat{\mathbf{n}}_\mathrm{p}\cdot\left[\left(1+\frac{j}{kR}-\frac{1}{k^2R^2}\right)\bar{\bar{\mathrm{I}}}-\left(1+\frac{3j}{kR}-\frac{3}{k^2R^2}\right)\hat{\mathbf{R}}\hat{\mathbf{R}}\right]\cdot\hat{\mathbf{n}}_\mathrm{p}\frac{e^{jkR}}{4\pi R}\right\}$$

$$\langle P\rangle = \frac{\omega^3\mu}{2}|\mathbf{p}|^2 \mathrm{Im}\left\{\left[\left(1-\hat{R}_\mathrm{p}^2\right)-\frac{j}{kR}\left(3\hat{R}_\mathrm{p}^2-1\right)+\frac{1}{k^2R^2}\left(3\hat{R}_\mathrm{p}^2-1\right)\right]\frac{e^{jkR}}{4\pi R}\right\}$$

$$\langle P\rangle = \frac{\omega}{8\pi\epsilon}|\mathbf{p}|^2 \mathrm{Im}\left\{\lim_{R\to 0}\left[\frac{k^2}{R}\left(1-\hat{R}_\mathrm{p}^2\right)-\frac{jk}{R^2}\left(3\hat{R}_\mathrm{p}^2-1\right)+\frac{1}{R^3}\left(3\hat{R}_\mathrm{p}^2-1\right)\right]e^{jkR}\right\}$$

Taylor expansion

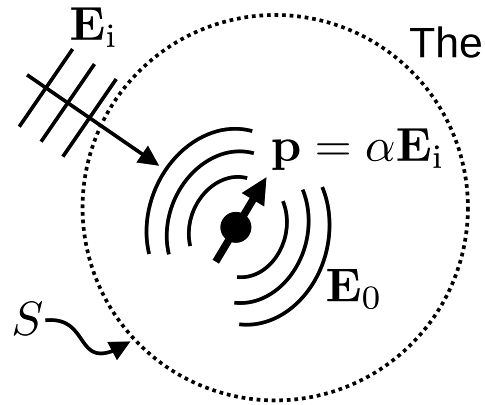
$$\langle P\rangle = \frac{\omega}{8\pi\epsilon}|\mathbf{p}|^2 \mathrm{Im}\left\{\lim_{R\to 0}\left(\frac{k^2}{R}+\frac{jk}{R^2}-\frac{1}{R^3}\right)\left[1+jkR+\frac{1}{2}(jkR)^2+\frac{1}{6}(jkR)^3+\dots\right]\right\}$$

$$\langle P\rangle = \frac{\omega}{8\pi\epsilon}|\mathbf{p}|^2 \lim_{R\to 0}\left[\frac{2}{3}k^3-\frac{k^5R^2}{8}+\dots\right]$$

$$\boxed{\langle P\rangle = \frac{ck^4}{12\pi\epsilon}|\mathbf{p}|^2}$$

Even though the Green function diverges when R→0, its imaginary part does not!

Scattering condition for an Electric Dipole Polarizability



The total field is the sum of the dipole field and the incident field

$$\mathbf{E}(\mathbf{r}') = \mathbf{E}_0(\mathbf{r}') + \mathbf{E}_\mathrm{i}(\mathbf{r}')$$

$$\langle P \rangle = \oint \langle \mathbf{S} \rangle \cdot d\mathbf{S} = \frac{\omega}{2} \mathrm{Im}\left\{ \mathbf{p} \cdot \mathbf{E}(\mathbf{r}')^* \right\} = 0$$

if lossless, all power entering *S* must leave it

$$\langle P \rangle = \frac{\omega}{2} \mathrm{Im}\left\{ \mathbf{p} \cdot (\mathbf{E}_0(\mathbf{r}')^* + \mathbf{E}_\mathrm{i}(\mathbf{r}')^*) \right\} = \frac{ck^4}{12\pi\epsilon} |\mathbf{p}|^2 + \frac{\omega}{2} \mathrm{Im}\left\{ \mathbf{p} \cdot \mathbf{E}_\mathrm{i}(\mathbf{r}')^* \right\} = \underbrace{\frac{ck^4}{12\pi\epsilon} |\alpha|^2 |\mathbf{E}_\mathrm{i}|^2 + \frac{\omega}{2} \mathrm{Im}\left\{ \alpha \right\} |\mathbf{E}_\mathrm{i}|^2}_{=0}$$

$$\longrightarrow \quad -\frac{\omega}{2} \mathrm{Im}\left\{ \alpha \right\} = \frac{ck^4}{12\pi\epsilon} |\alpha|^2 \quad \longrightarrow \quad \frac{\alpha^* - \alpha}{2j} = \frac{k^3}{6\pi\epsilon} \alpha \alpha^* \quad \longrightarrow \quad \frac{1}{2j} \left(\frac{1}{\alpha} - \frac{1}{\alpha^*} \right) = \frac{k^3}{6\pi\epsilon}$$

$$\boxed{\mathrm{Im}\left\{ \frac{1}{\alpha} \right\} = \frac{k^3}{6\pi\epsilon}}$$

Scattering loss for an isotropic electric polarizability

For more details, see section 5.2.2 in
Tretyakov, “Analytical modeling in applied electromagnetics”, 2003

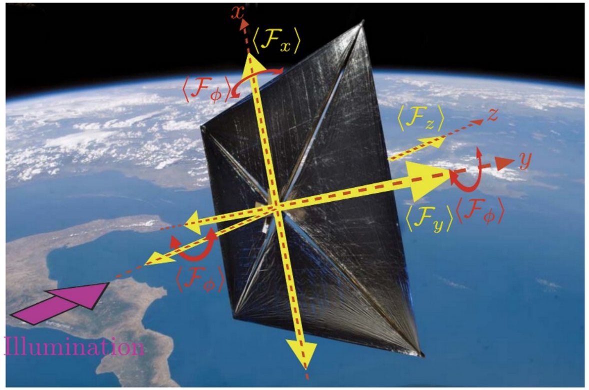
What Have We Learned So Far….

- We can express the power radiated by a dipole as the imaginary part of the dot product between the dipole moment and the electric field at the dipole position. This may then be expressed in terms of the dyadic Green function.
- Even though the dyadic Green function is evaluated at R = 0, it does not lead to a divergence of the radiated power as its imaginary part does not diverge (only its real part).
- This formalism allows us to find that the polarizability of a particle (even a lossless one) is complex. This is due to scattering loss, which is really defined as how much energy is removed from the incident wave.
- If billions of small particles are put together in close proximity ($<<\lambda$), then the interference between their scattering, when illuminated by a plane wave, is canceled out except in the direction of the plane wave itself. This means that, in this case, there is no more scattering loss, i.e., we are not removing energy in the direction of propagation of the incident wave. This leads to an effective material parameter (permittivity or refractive index) being purely real (assuming of course that the particles are themselves lossless => no absorption).

Electromagnetic Force and Torque

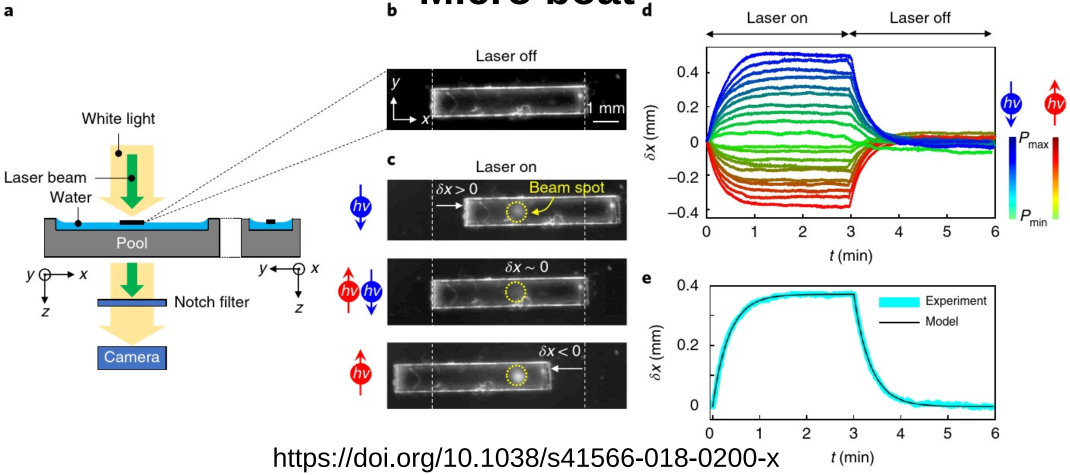
Examples

Metasurface solar sail



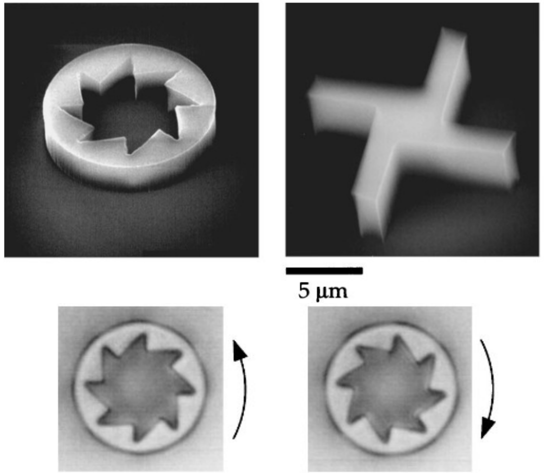
https://doi.org/10.1109/TAP.2019.2925279

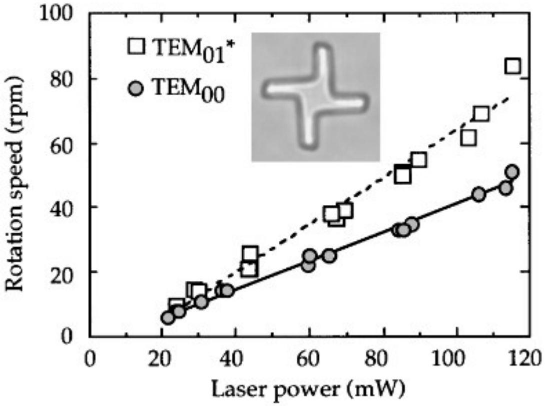
Micro boat



https://doi.org/10.1038/s41566-018-0200-x

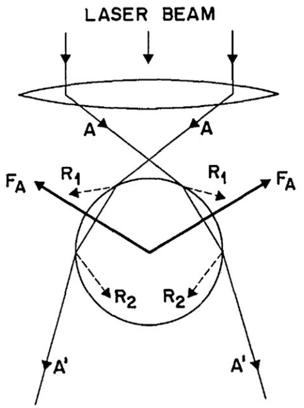
Micro rotor





http://dx.doi.org/10.1063/1.366163

Optical trapping

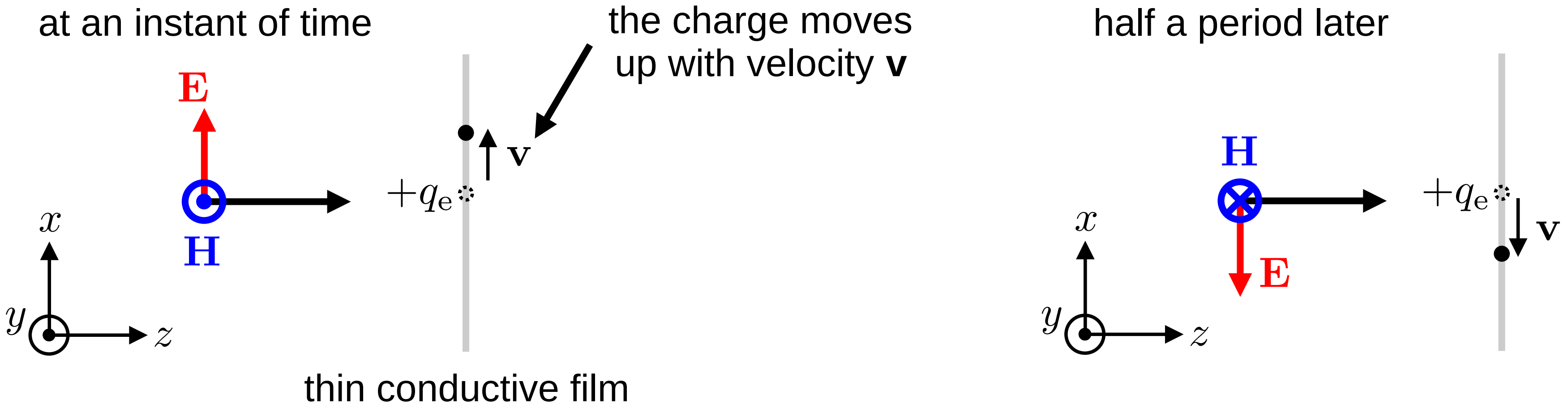


https://doi.org/10.1364/OL.11.000288

Electromagnetic Waves and Forces

only considering the electric field

$$\mathbf{F} = q_{\mathrm{e}}\mathbf{E}$$



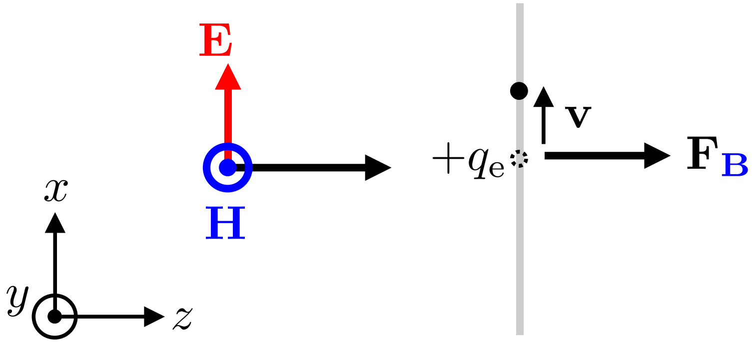
When only considering the electric field effect, we see that the charge is moving up and down. Averaging over a full time period, the net motion of the charge is zero.

Electromagnetic Waves and Forces

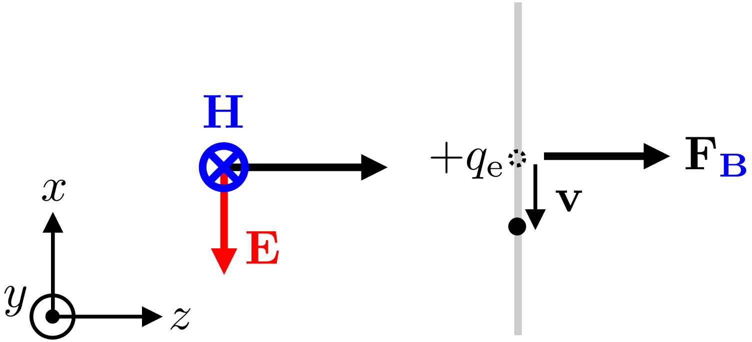
considering both the electric and magnetic fields

$$\mathbf{F} = q_\mathrm{e}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \qquad \text{where} \quad \mathbf{B} = \mu_0 \mathbf{H}$$

at an instant of time



half a period later

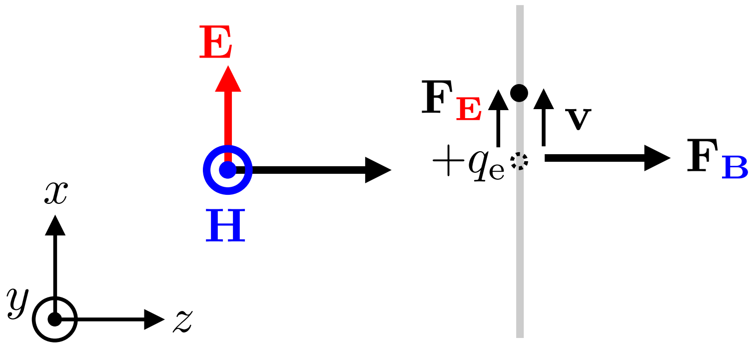


When considering the combined electric and magnetic field effects, the charge experiences a non-zero net force as if pushed by the wave.

Electromagnetic Linear Momentum

Demonstration that a plane wave carries momentum

$$\mathbf{F} = q_e \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$



$$|\mathbf{F}_\mathbf{B}| = q_e |\mathbf{B}||\mathbf{v}| \longleftarrow |\mathbf{B}| = \frac{|\mathbf{E}|}{c}$$

For a plane wave

$$|\mathbf{F}_\mathbf{B}| = q_e \frac{|\mathbf{E}|}{c} |\mathbf{v}| \longleftarrow |\mathbf{F}_\mathbf{E}| = q_e |\mathbf{E}|$$

$$|\mathbf{F}_\mathbf{B}| = \frac{|\mathbf{F}_\mathbf{E}|}{c} |\mathbf{v}|$$

velocity

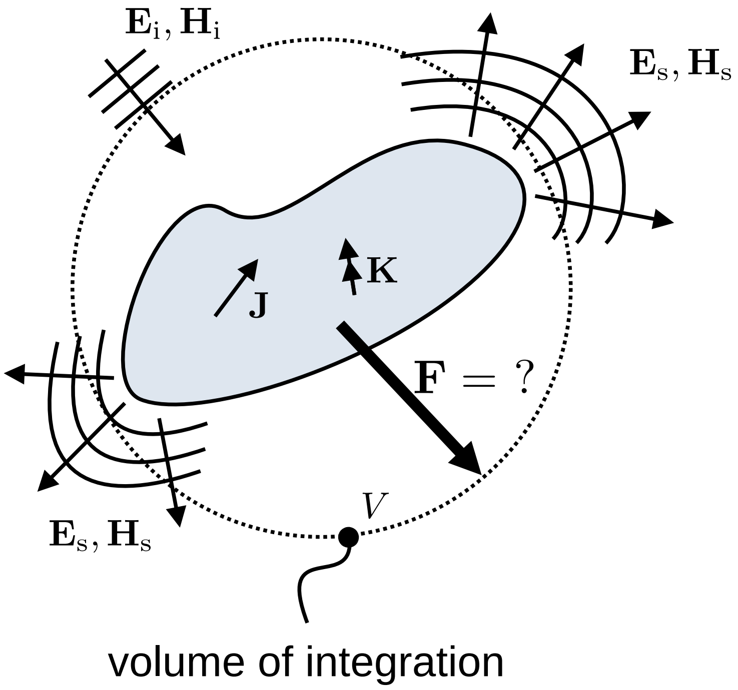
$$|\mathbf{v}| = v = \frac{dr}{dt}$$

$$F_\mathbf{B} = \frac{1}{c} F_\mathbf{E} \frac{dr}{dt}$$

$$p = \frac{E}{c} \longleftarrow \underbrace{\int F_\mathbf{B}\, dt}_{\text{momentum}} = \frac{1}{c} \underbrace{\int F_\mathbf{E}\, dr}_{\text{energy}}$$

E = energy
(not electric field)

Derivation of the Electromagnetic Force



General form of the Lorentz force including magnetic charge

$$\mathbf{F} = q_\mathrm{e}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) + q_\mathrm{m}\left(\frac{1}{\mu_0}\mathbf{B} - \epsilon_0 \mathbf{v} \times \mathbf{E}\right)$$

By considering a density of force, we can use

$$\mathbf{J} = \rho_\mathrm{e}\mathbf{v} \qquad \mathbf{K} = \rho_\mathrm{m}\mathbf{v}$$

$$\mathbf{F} = \int \mathbf{f}\ dV = \int_V \rho_\mathrm{e}\mathbf{E} + \mathbf{J} \times \mathbf{B} + \frac{1}{\mu_0}\rho_\mathrm{m}\mathbf{B} - \epsilon_0 \mathbf{K} \times \mathbf{E}\ dV$$

Step by Step Derivation (1/2)

$$\mathbf{f} = \rho_{\mathrm{e}}\mathbf{E} + \mathbf{J}\times\mathbf{B} + \frac{1}{\mu_0}\rho_{\mathrm{m}}\mathbf{B} - \epsilon_0\mathbf{K}\times\mathbf{E}$$

$$\mathbf{f} = \epsilon_0\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \left(\frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B} + \frac{1}{\mu_0}\left(\nabla\cdot\mathbf{B}\right)\mathbf{B} + \epsilon_0\left(\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{E}$$

$$\mathbf{f} = \epsilon_0\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \frac{1}{\mu_0}\left(\nabla\cdot\mathbf{B}\right)\mathbf{B} + \epsilon_0\left(\nabla\times\mathbf{E}\right)\times\mathbf{E} + \epsilon_0\frac{\partial\mathbf{B}}{\partial t}\times\mathbf{E}$$

$$\mathbf{f} = \epsilon_0\left[\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \left(\nabla\times\mathbf{E}\right)\times\mathbf{E}\right] + \frac{1}{\mu_0}\left[\left(\nabla\cdot\mathbf{B}\right)\mathbf{B} + \left(\nabla\times\mathbf{B}\right)\times\mathbf{B}\right] - \epsilon_0\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\right)$$

$$\mathbf{f} = \epsilon_0\left[\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \left(\nabla\times\mathbf{E}\right)\times\mathbf{E}\right] + \frac{1}{\mu_0}\left[\left(\nabla\cdot\mathbf{B}\right)\mathbf{B} + \left(\nabla\times\mathbf{B}\right)\times\mathbf{B}\right] - \epsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right)$$

where we have used

$$\nabla\cdot\mathbf{D} = \rho_{\mathrm{e}} \quad\longrightarrow\quad \rho_{\mathrm{e}} = \epsilon_0\nabla\cdot\mathbf{E}$$

$$\nabla\cdot\mathbf{B} = \rho_{\mathrm{m}} \quad\longrightarrow\quad \rho_{\mathrm{m}} = \nabla\cdot\mathbf{B}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\mathbf{E} \quad\longrightarrow\quad \mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\times\mathbf{E} = -\mathbf{K} - \frac{\partial\mathbf{B}}{\partial t} \quad\longrightarrow\quad \mathbf{K} = -\nabla\times\mathbf{E} - \frac{\partial\mathbf{B}}{\partial t}$$

Step by Step Derivation (2/2)

$$\mathbf{f} = \epsilon_0 \left[(\nabla \cdot \mathbf{E}) \, \mathbf{E} + (\nabla \times \mathbf{E}) \times \mathbf{E} \right] + \frac{1}{\mu_0} \left[(\nabla \cdot \mathbf{B}) \, \mathbf{B} + (\nabla \times \mathbf{B}) \times \mathbf{B} \right] - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})$$

↓

We now use the following identities

$$\nabla \cdot (\mathbf{A}\mathbf{A}) = (\nabla \cdot \mathbf{A}) \, \mathbf{A} + (\mathbf{A} \cdot \nabla) \, \mathbf{A}$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla) \, \mathbf{A} - \frac{1}{2} \nabla (\mathbf{A} \cdot \mathbf{A})$$

↓

$$\mathbf{f} = \epsilon_0 \left[(\nabla \cdot \mathbf{E}) \, \mathbf{E} + (\mathbf{E} \cdot \nabla) \, \mathbf{E} \right] + \frac{1}{\mu_0} \left[(\nabla \cdot \mathbf{B}) \, \mathbf{B} + (\mathbf{B} \cdot \nabla) \, \mathbf{B} \right] - \frac{1}{2} \nabla \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right) - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{f} = \epsilon_0 \nabla \cdot (\mathbf{E}\mathbf{E}) + \frac{1}{\mu_0} \nabla \cdot (\mathbf{B}\mathbf{B}) - \frac{1}{2} \nabla \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right) - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{f} = \nabla \cdot \left[\epsilon_0 \mathbf{E}\mathbf{E} + \frac{1}{\mu_0} \mathbf{B}\mathbf{B} - \frac{1}{2} \left(\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right) \bar{\bar{\mathrm{I}}} \right] - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})$$

Time-Domain Electromagnetic Force

Assuming that the background medium is vacuum

$$\mathbf{f} = \nabla \cdot \underbrace{\left[\epsilon_0 \mathbf{E}\mathbf{E} + \mu_0 \mathbf{H}\mathbf{H} - \frac{1}{2}\left(\epsilon_0 \mathbf{E}\cdot\mathbf{E} + \mu_0 \mathbf{H}\cdot\mathbf{H}\right)\bar{\bar{\mathrm{I}}}\right]}_{\text{Maxwell stress tensor: T}} - \mu_0\epsilon_0 \frac{\partial}{\partial t}\underbrace{\left(\mathbf{E}\times\mathbf{H}\right)}_{\text{Poynting vector}}$$

Maxwell stress tensor: T

Poynting vector

These are outer products !

This expression is re-written as

$$\mathbf{f} = \nabla \cdot \bar{\bar{\mathrm{T}}} - \mu_0\epsilon_0 \frac{\partial \mathbf{S}}{\partial t}$$

The force is now expressed as the volume integral of **f**

$$\mathbf{F} = \int \mathbf{f}\; dV = \int_V \left(\nabla \cdot \bar{\bar{\mathrm{T}}} - \mu_0\epsilon_0 \frac{\partial \mathbf{S}}{\partial t}\right)\; dV$$

Using the divergence theorem

$$\mathbf{F} = \int_S \bar{\bar{\mathrm{T}}} \cdot \mathbf{n}\; dS - \mu_0\epsilon_0 \frac{\partial}{\partial t}\int_V \mathbf{S}\; dV$$

Time-Average Electromagnetic Force

In the time domain, we have

$$\overline{\overline{\mathrm{T}}} = \epsilon_0 \mathbf{E}\mathbf{E} + \mu_0 \mathbf{H}\mathbf{H} - \frac{1}{2}\left(\epsilon_0 \mathbf{E}\cdot\mathbf{E} + \mu_0 \mathbf{H}\cdot\mathbf{H}\right)\overline{\overline{\mathrm{I}}} \longrightarrow \mathbf{F} = \int_S \overline{\overline{\mathrm{T}}}\cdot\mathbf{n}\, dS - \mu_0\epsilon_0 \frac{\partial}{\partial t}\int_V \mathbf{S}\, dV$$

Applying the time averaging operation, we get

Time-average Maxwell stress tensor

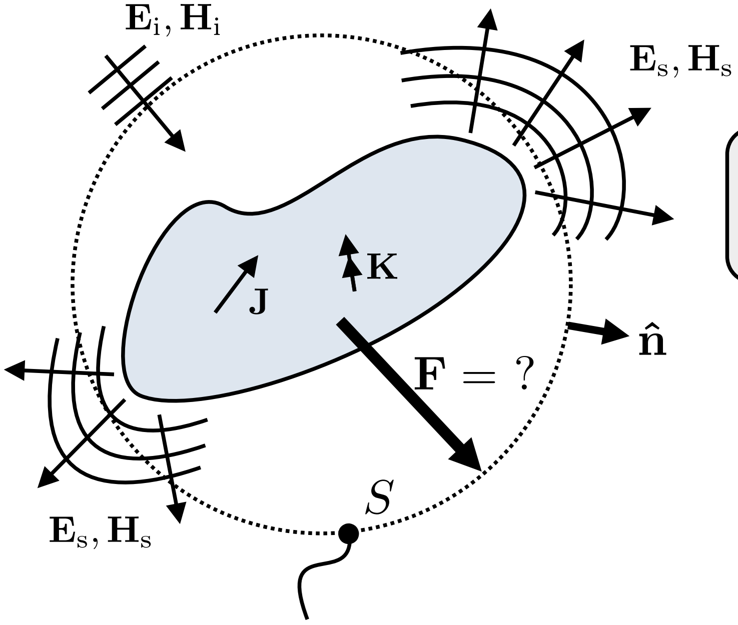
Time-average force

$$\langle\overline{\overline{\mathrm{T}}}\rangle = \frac{1}{2}\mathrm{Re}\left\{\epsilon_0 \mathbf{E}\mathbf{E}^* + \mu_0 \mathbf{H}\mathbf{H}^* - \underbrace{\frac{1}{2}\left(\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2\right)}\overline{\overline{\mathrm{I}}}\right\} \longrightarrow \langle\mathbf{F}\rangle = \int_S \langle\overline{\overline{\mathrm{T}}}\rangle\cdot\hat{\mathbf{n}}\, dS$$

Notice that it corresponds to the density of EM energy

$$\langle w\rangle = \frac{1}{4}\mathrm{Re}\left\{\mathbf{E}\cdot\mathbf{D}^* + \mathbf{H}\cdot\mathbf{B}^*\right\}$$

Time-Average Electromagnetic Force



Time-average Maxwell stress tensor

$$\langle \overline{\overline{\mathrm{T}}} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \epsilon_0 \mathbf{E}\mathbf{E}^* + \mu_0 \mathbf{H}\mathbf{H}^* - \frac{1}{2}\left(\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2 \right) \overline{\overline{\mathrm{I}}} \right\}$$

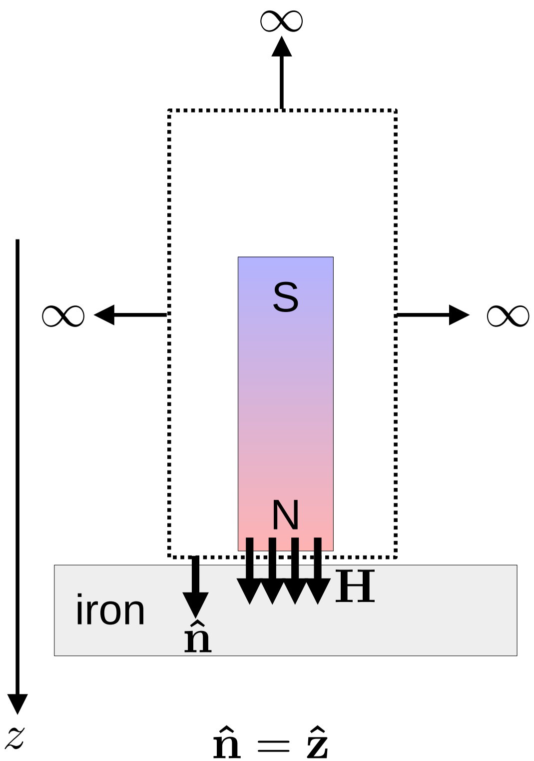
Time-average force

$$\langle \mathbf{F} \rangle = \int_S \langle \overline{\overline{\mathrm{T}}} \rangle \cdot \hat{\mathbf{n}} \; dS$$

The force is computed by integrating
the MST on a surface around the object

Note: the MST is a symmetric matrix

Example: Force of a Magnet



$$\hat{\mathbf{n}} = \hat{\mathbf{z}}$$

$$\mathbf{H} = H_0 \hat{\mathbf{z}}$$

The magnet has a surface area *S* over which we assume that the **H** field is uniform and normal to *S*

Time-average Maxwell stress tensor

$$\langle \overline{\overline{\mathrm{T}}} \rangle = \frac{1}{2} \mathrm{Re} \left\{ \epsilon_0 \mathbf{E}\mathbf{E}^* + \mu_0 \mathbf{H}\mathbf{H}^* - \frac{1}{2} \left(\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2 \right) \overline{\overline{\mathrm{I}}} \right\}$$

where $\mathbf{E} = 0$ $\quad |\mathbf{H}|^2 = H_0^2$

$$\mathbf{H}\mathbf{H}^* = \begin{bmatrix} |H_x|^2 & H_x H_y^* & H_x H_z^* \\ H_y H_x^* & |H_y|^2 & H_y H_z^* \\ H_z H_x^* & H_z H_y^* & |H_z|^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & H_0^2 \end{bmatrix}$$

The MST reduces to $\langle \overline{\overline{\mathrm{T}}} \rangle = \frac{\mu_0}{4} H_0^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

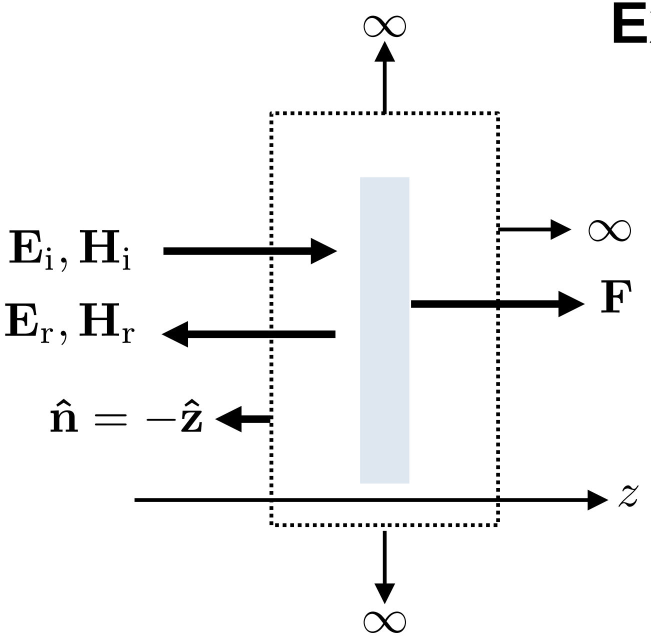
The force is $\langle \mathbf{F} \rangle = \int_S \langle \overline{\overline{\mathrm{T}}} \rangle \cdot \hat{\mathbf{n}} \, dS = \frac{\mu_0}{4} H_0^2 S$

Correct for an <u>oscillating</u> H field

For a static H field, we have

$$\mathbf{F} = \frac{\mu_0}{2} H_0^2 S$$

Example: Radiation Pressure



Slab of absorbing material (no transmission) of reflectance *r*

Time-average Maxwell stress tensor

$$\langle \overline{\overline{\mathrm{T}}} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \epsilon_0 \mathbf{E}\mathbf{E}^* + \mu_0 \mathbf{H}\mathbf{H}^* - \frac{1}{2}\left(\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2\right)\overline{\overline{\mathrm{I}}} \right\}$$

The fields on the left integration surface are

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \left(e^{-jkz} + r e^{+jkz}\right)$$

$$\mathbf{H} = \hat{\mathbf{y}} \frac{E_0}{\eta}\left(e^{-jkz} - r e^{+jkz}\right)$$

Since the fields are along *x* and *y* and **n** is along *z*, we have that

$$\mathbf{E}\mathbf{E}^* \cdot \hat{\mathbf{n}} = 0 \qquad \mathbf{H}\mathbf{H}^* \cdot \hat{\mathbf{n}} = 0$$

It follows that $\langle \overline{\overline{\mathrm{T}}} \rangle \cdot \hat{\mathbf{n}} = \frac{\epsilon_0}{2} E_0^2 (1 + |r|^2)\hat{\mathbf{z}}$

Time-average force

$$\langle \mathbf{F} \rangle = \int_S \langle \overline{\overline{\mathrm{T}}} \rangle \cdot \hat{\mathbf{n}} \, dS$$

Radiation pressure [N/m²]

$$P = \frac{\epsilon_0}{2} E_0^2 (1 + |r|^2)$$

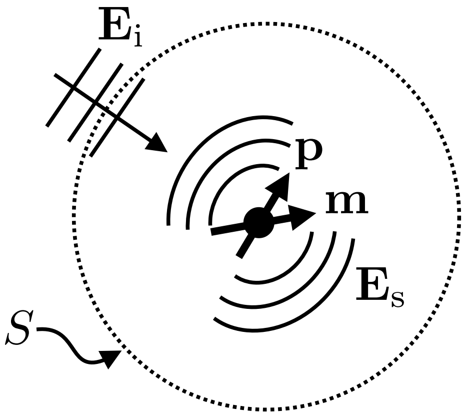
or

$$P = \frac{I_0}{c}(1 + |r|^2)$$

Electromagnetic Force on Small Particles

Time-average Maxwell stress tensor

$$\langle \overline{\overline{\mathrm{T}}} \rangle = \frac{1}{2}\mathrm{Re}\left[\epsilon_0 \mathbf{E}\mathbf{E}^* + \mu_0 \mathbf{H}\mathbf{H}^* - \frac{1}{2}\left(\epsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2\right)\overline{\overline{\mathrm{I}}}\right]$$



Let's consider a small particle modeled as an electric and magnetic dipoles. The total field is the sum of the incident field and the field scattered by the dipoles

$$\mathbf{E} = \mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{s} \longrightarrow \mathbf{E}^2 = \mathbf{E}_\mathrm{i}^2 + \mathbf{E}_\mathrm{s}\mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{s}^2$$

the force of the incident field alone is zero $\langle \mathbf{F}_\mathrm{i} \rangle = 0$

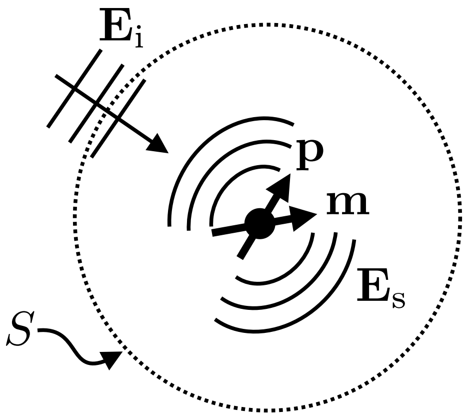
$\langle \mathbf{F}_\mathrm{mix} \rangle$ $\langle \mathbf{F}_\mathrm{s} \rangle$

The force on the particle is then given by

$$\langle \mathbf{F} \rangle = \frac{1}{2}\mathrm{Re}\left\{(\nabla \mathbf{E}_\mathrm{i}^*)\cdot \mathbf{p} + (\nabla \mathbf{B}_\mathrm{i}^*)\cdot \mathbf{m}\right\} - \frac{k^4}{12\pi\epsilon_0 c}\mathrm{Re}\left\{\mathbf{p}\times\mathbf{m}^*\right\}$$

The derivation of this equation is complicated. Refer to the supplementary information of https://doi.org/10.1038/nphoton.2011.153

Electromagnetic Force From Poynting Vectors



Time-average force

$$\langle \mathbf{F} \rangle = \int_S \langle \overline{\overline{\mathrm{T}}} \rangle \cdot \hat{\mathbf{n}} \; dS$$

Force in terms of the Stress tensor

Alternatively

$$\langle \mathbf{F} \rangle = -\frac{1}{c} \int_{S_\infty} \left[\langle \mathbf{S} \rangle - \langle \mathbf{S}_\mathrm{i} \rangle \right] \; dS$$

Force in terms of the Poynting vectors (only valid for particles)

Integration surface taken to infinity

Poynting vector of total field

Poynting vector of incident field

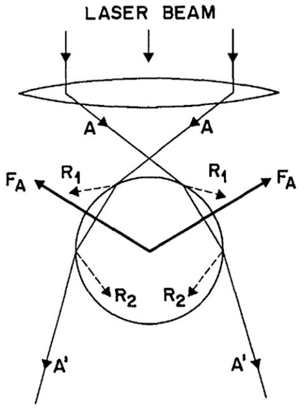
https://doi.org/10.1038/nphoton.2011.153

Electromagnetic Force on Small Particles

$$\langle \mathbf{F} \rangle = \frac{1}{2}\mathrm{Re}\left\{ (\nabla \mathbf{E}_\mathrm{i}^*) \cdot \mathbf{p} + (\nabla \mathbf{B}_\mathrm{i}^*) \cdot \mathbf{m} \right\} - \frac{k^4}{12\pi\epsilon_0 c}\mathrm{Re}\left\{ \mathbf{p} \times \mathbf{m}^* \right\}$$

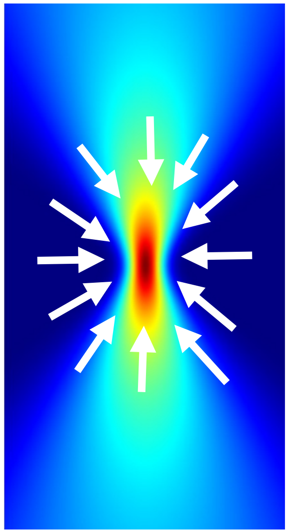
$$\langle \mathbf{F} \rangle = -\frac{1}{c}\int_{S_\infty} \left[\langle \mathbf{S} \rangle - \langle \mathbf{S}_\mathrm{i} \rangle \right] \, dS$$

Optical trapping

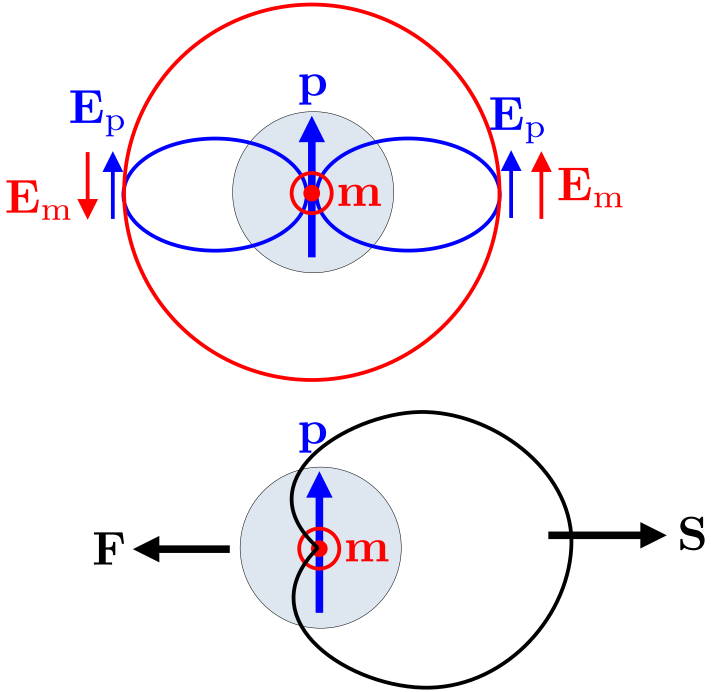


https://doi.org/10.1364/OL.11.000288

Electric field amplitude of a focused Gaussian beam

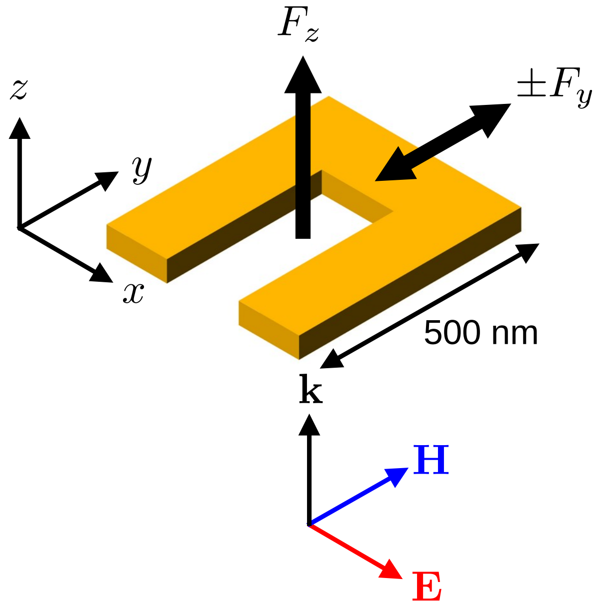


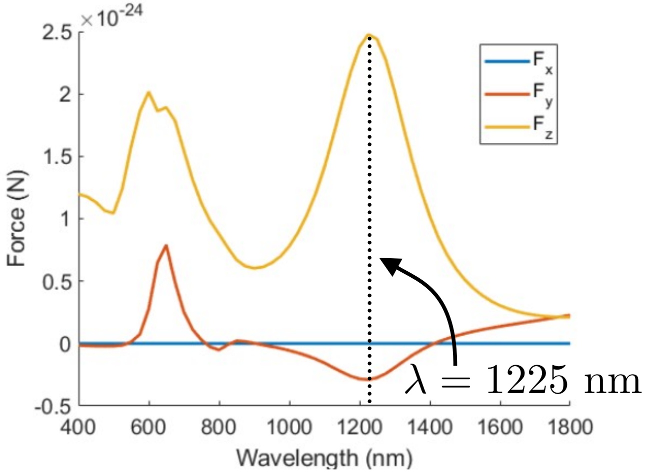
The gradient of the field leads to a force on a dipolar particle that traps it in the region of high-field intensity.



Interefence between **p** and **m** dipoles lead to asymmetric scattering and, by reaction, a force.

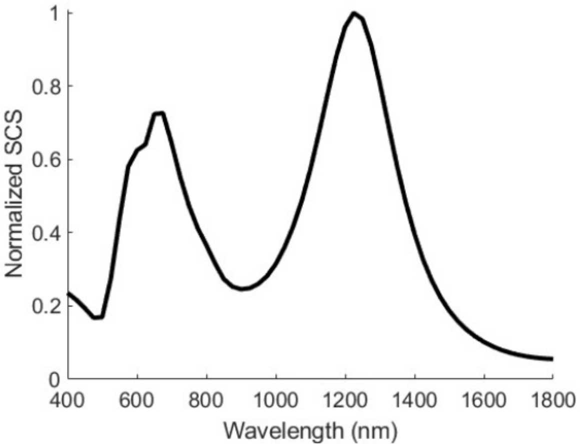
Force on a U-Shaped Particle



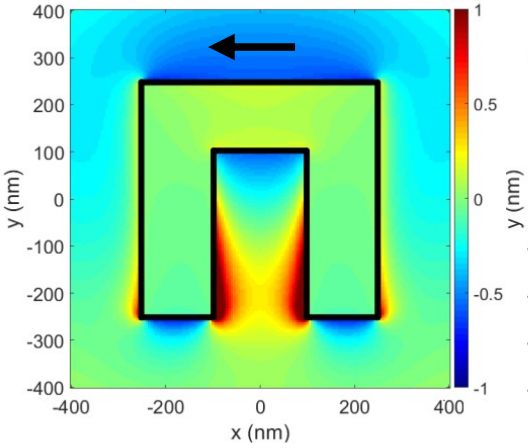


Force

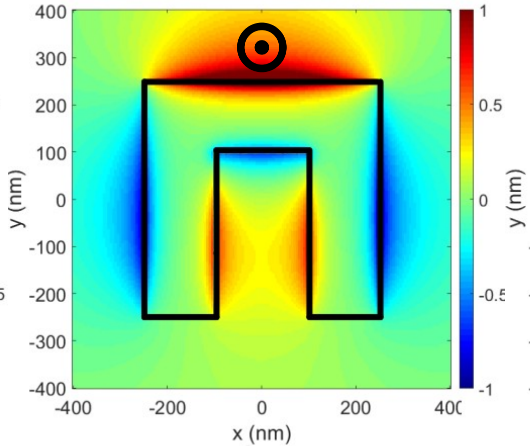
$$\langle \mathbf{F} \rangle = -\frac{1}{c} \int_{S_\infty} [\langle \mathbf{S} \rangle - \langle \mathbf{S}_{\mathrm{i}} \rangle] \; dS$$



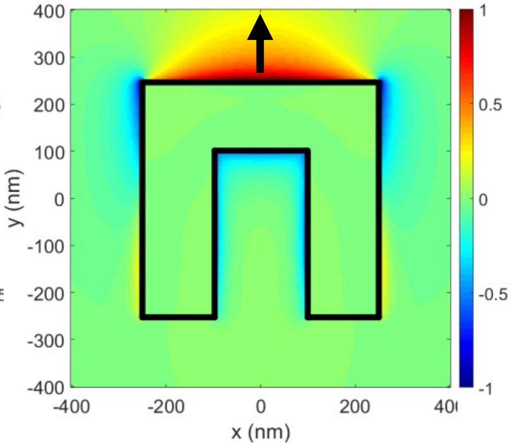
Electric field, E_x



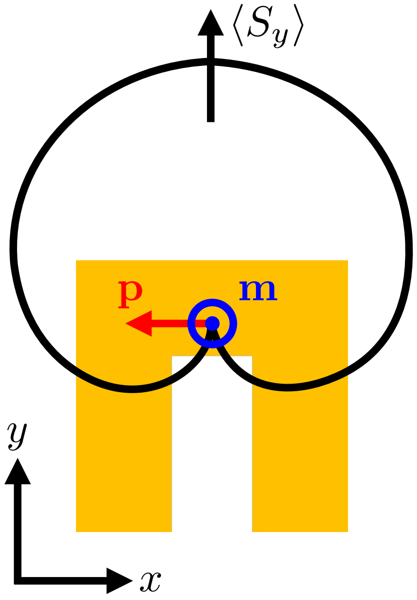
Magnetic field, H_z



Poynting Vector, S_y



Force on a U-Shaped Particle: Kerker Effect



The incident wave is propagating only along z, so

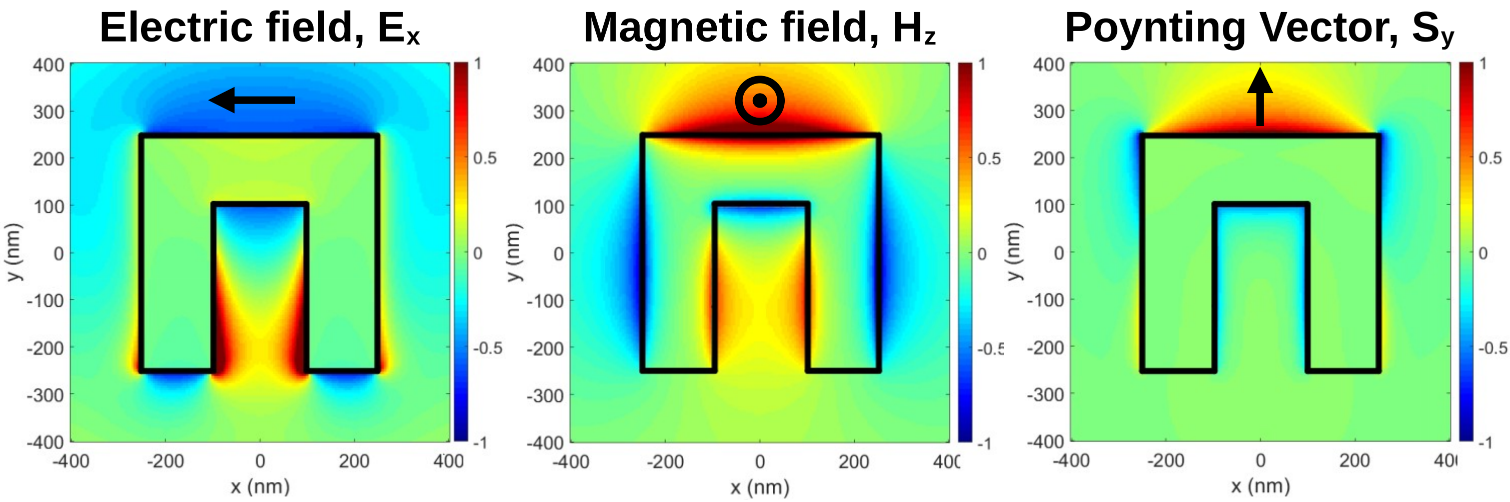
$$\langle S_{y,\mathrm{i}} \rangle = 0$$

$$\boxed{\langle \mathbf{F} \rangle = -\frac{1}{c}\int_{S_\infty} \left[\langle \mathbf{S} \rangle - \langle \mathbf{S}_\mathrm{i} \rangle\right]\, dS} \rightarrow \langle F_y \rangle = -\frac{1}{c}\int_{S_\infty} \langle S_y \rangle \, dS = -\frac{P_y}{c}$$

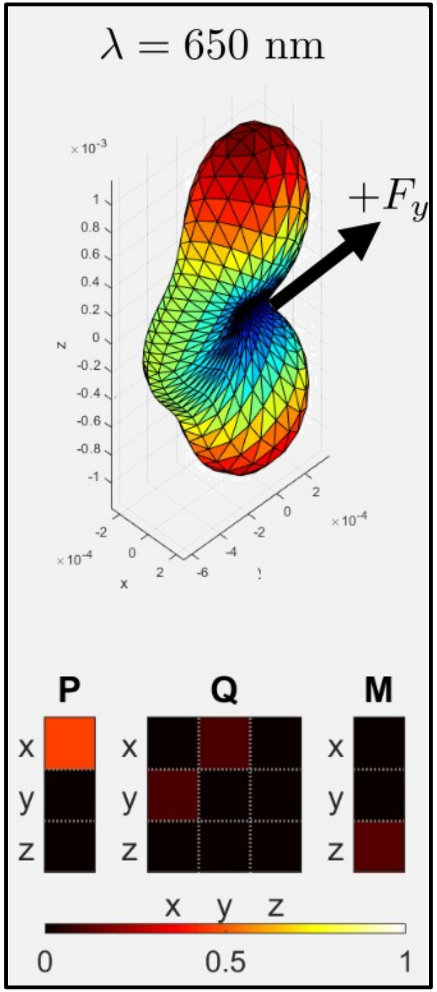
Power radiated in the y-direction

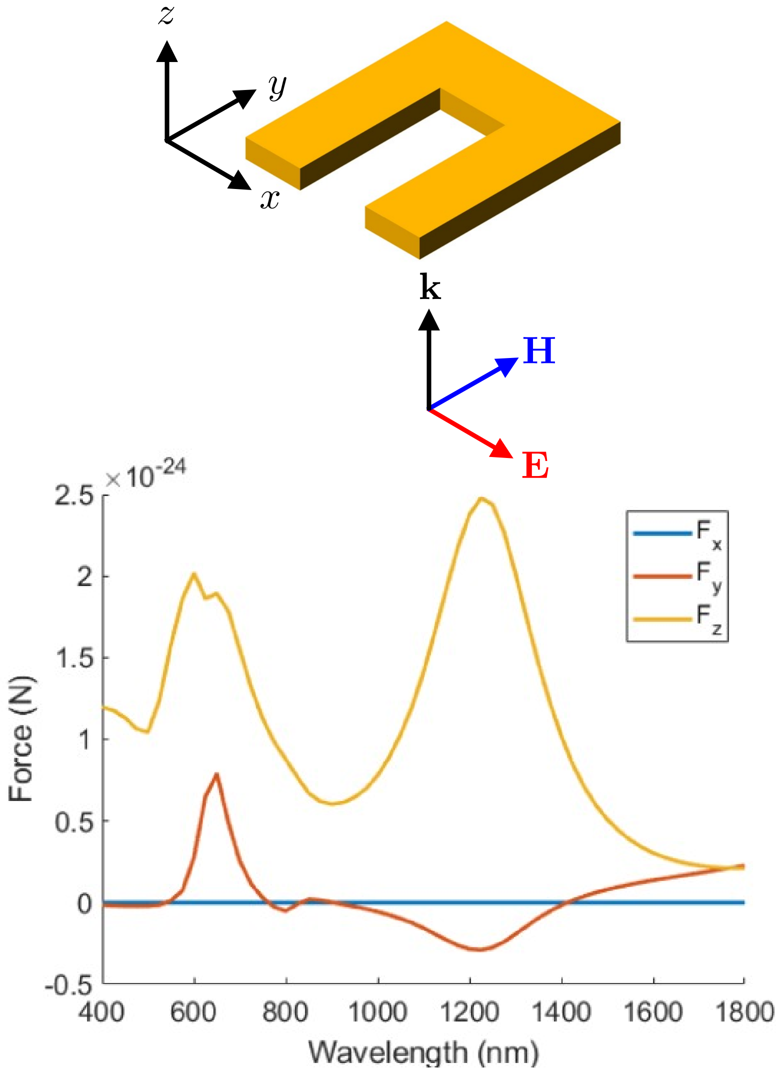
Force due to Kerker effect

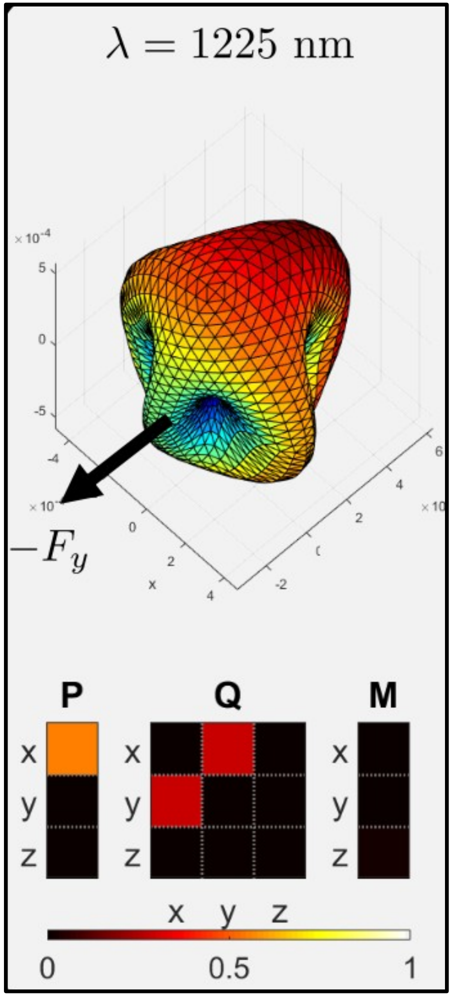
$$\left\{\begin{aligned} \mathbf{p} &= A_\mathrm{p} e^{j\phi_\mathrm{p}} \hat{\mathbf{x}} \\ \mathbf{m} &= A_\mathrm{m} e^{j\phi_\mathrm{m}} \hat{\mathbf{z}} \end{aligned}\right\} \rightarrow \langle F_y \rangle = A_\mathrm{p} A_\mathrm{m} \frac{k^3 \mu \omega}{60\pi} \cos\left(\phi_\mathrm{p} - \phi_\mathrm{m}\right)$$



The Actual Response Involves the Electric Quadrupole







Electromagnetic Torque

Angular momentum (mechanics)

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where **p** is linear momentum

Angular momentum (electromagnetics)

$$\langle \mathbf{L} \rangle = \frac{1}{c^2} \int \mathbf{r} \times \langle \mathbf{S} \rangle \; dV$$

The torque is the time variation of angular momentum

$$\boldsymbol{\Gamma} = \frac{\partial \mathbf{L}}{\partial t}$$

The time-domain electromagnetic torque is defined as

$$\boldsymbol{\Gamma} = \frac{\partial}{\partial t}\mathbf{L} = \frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{\partial \mathbf{p}}{\partial t} = \mathbf{r} \times \mathbf{F} = \int_V \mathbf{r} \times \mathbf{f} \; dV = \int_V \mathbf{r} \times \left(\nabla \cdot \overline{\overline{\mathrm{T}}} - \mu_0 \epsilon_0 \frac{\partial \mathbf{S}}{\partial t} \right) \; dV$$

The time-average electromagnetic torque is defined as

$$\langle \boldsymbol{\Gamma} \rangle = \int_V \mathbf{r} \times \left(\nabla \cdot \langle \overline{\overline{\mathrm{T}}} \rangle \right) \; dV \longrightarrow \langle \boldsymbol{\Gamma} \rangle = \int_S \mathbf{r} \times \left(\langle \overline{\overline{\mathrm{T}}} \rangle \cdot \hat{\mathbf{n}} \right) \; dS$$

if the integration surface is a sphere

$$\mathbf{r} = \hat{\mathbf{n}}$$

$$\langle \boldsymbol{\Gamma} \rangle = \frac{1}{2} \mathrm{Re} \left\{ \int_S \hat{\mathbf{n}} \times \left[(\epsilon_0 \mathbf{E}\mathbf{E}^* + \mu_0 \mathbf{H}\mathbf{H}^*) \cdot \hat{\mathbf{n}} \right] \; dS \right\}$$

demonstration

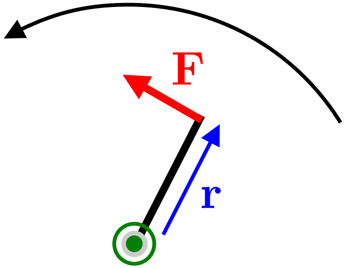
$$\mathbf{r} \times \left(\nabla \cdot \langle \overline{\overline{\mathrm{T}}} \rangle \right) = \mathbf{e}_i \varepsilon_{ijk} r_j \partial_l \langle T_{kl} \rangle$$

$$= \mathbf{e}_i \varepsilon_{ijk} \partial_l r_j \langle T_{kl} \rangle - \mathbf{e}_i \varepsilon_{ijk} \partial_l r_j \langle T_{kl} \rangle$$

$$= \mathbf{e}_i \varepsilon_{ijk} \partial_l r_j \langle T_{kl} \rangle - \mathbf{e}_i \varepsilon_{ijk} \delta_{jl} \langle T_{kl} \rangle$$

$$= \mathbf{e}_i \varepsilon_{ijk} \partial_l r_j \langle T_{kl} \rangle - \mathbf{e}_i \varepsilon_{ijk} \langle T_{kj} \rangle$$

$$= \mathbf{e}_i \varepsilon_{ijk} \partial_l r_j \langle T_{kl} \rangle = \nabla \cdot \mathbf{r} \times \langle \overline{\overline{\mathrm{T}}} \rangle$$

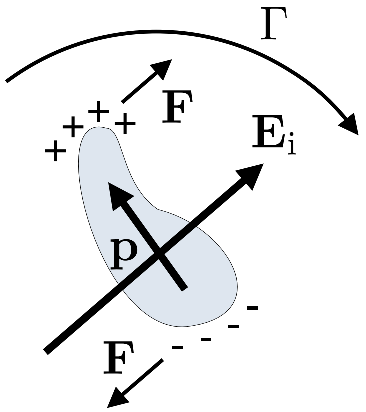


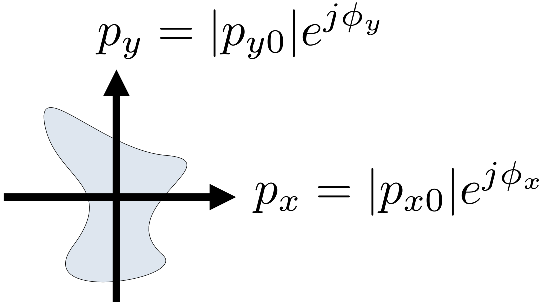
$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$$

Electromagnetic Torque on Small Particles

Valid in the case of dipolar particles

$$\langle \mathbf{\Gamma} \rangle = \frac{1}{2}\mathrm{Re}\left\{\mathbf{p} \times \mathbf{E}_\mathrm{i}^* + \mathbf{m} \times \mathbf{B}_\mathrm{i}^*\right\} - \frac{k^3}{3}\mathrm{Im}\left\{\frac{1}{\epsilon}\mathbf{p}^* \times \mathbf{p} + \mu \mathbf{m}^* \times \mathbf{m}\right\}$$





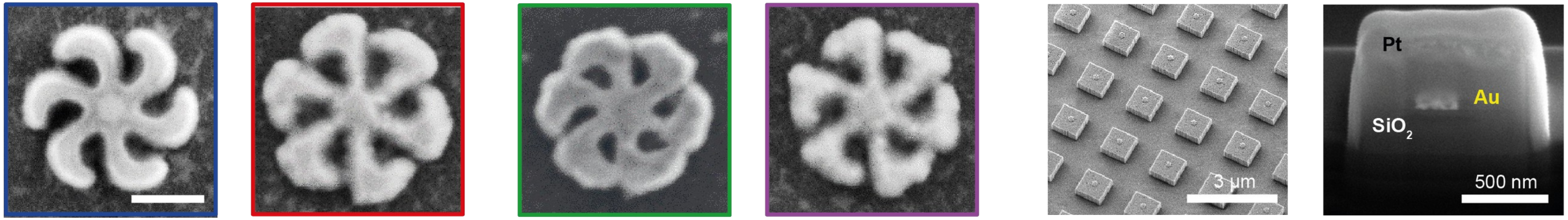
$\phi_x \neq \phi_y$

the particle radiates with out-of-phase p_x and p_y dipoles generating elliptical polarization. By conservation of angular momentum, the particle must rotate.

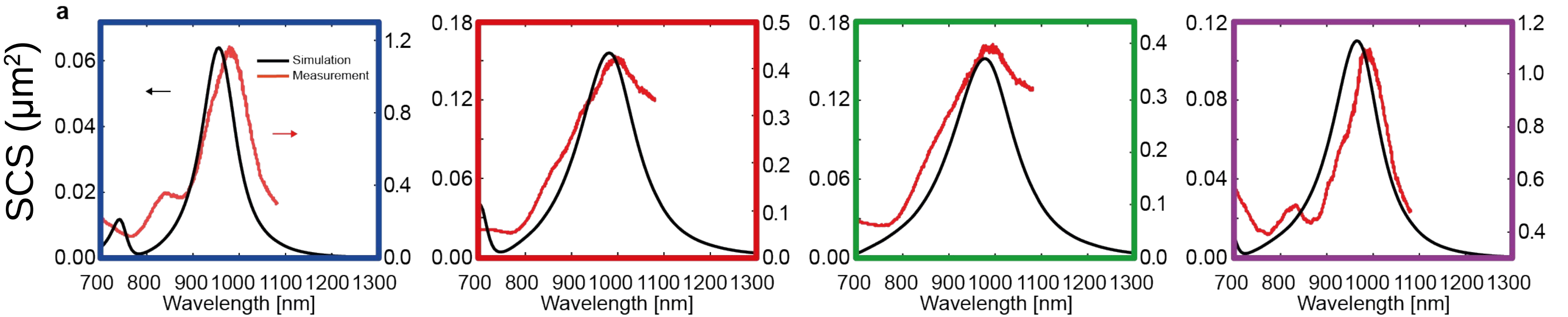
The derivation of this equation is complicated. Refer to http://dx.doi.org/10.1103/PhysRevA.92.043843

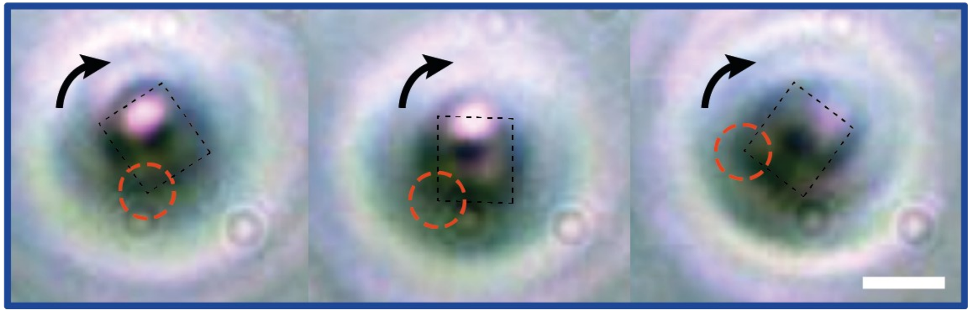
Torque on Flat Spiral-Shaped Particles

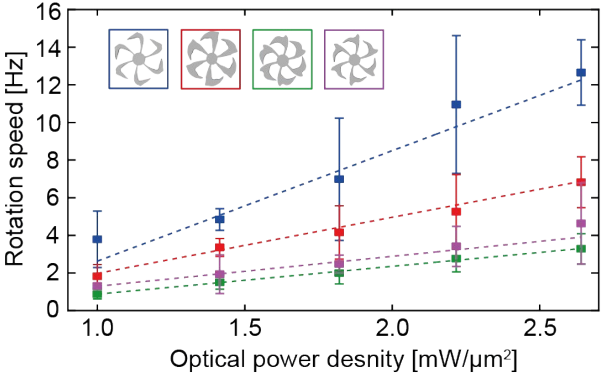
Fabricated samples (Gold spirals)



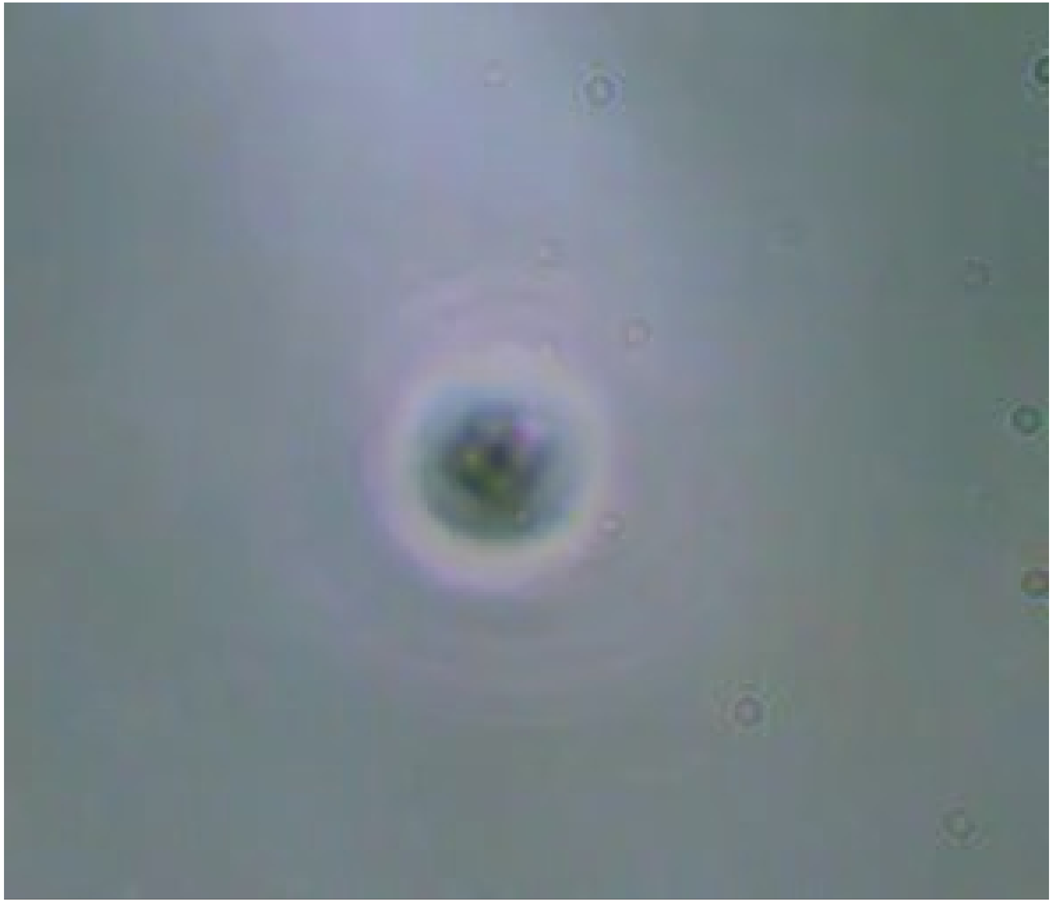
Measured scattering cross-sections and rotation speeds



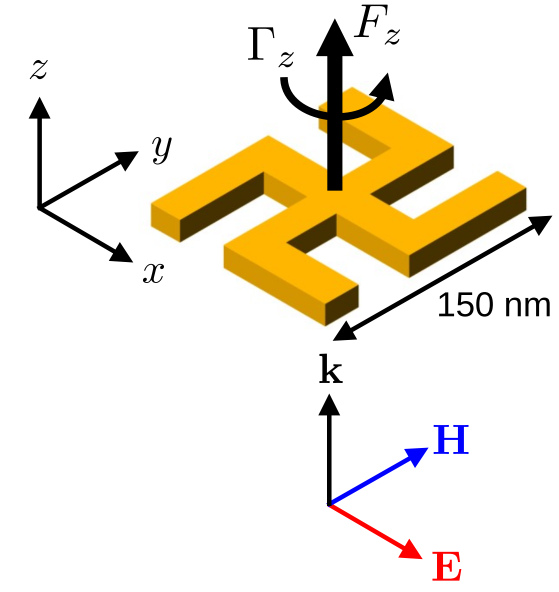


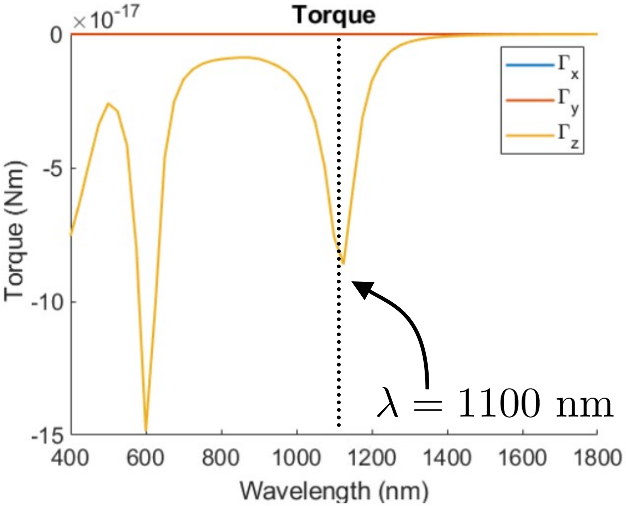


Experimental Demonstration



The Actual Response Involves the Electric Quadrupole

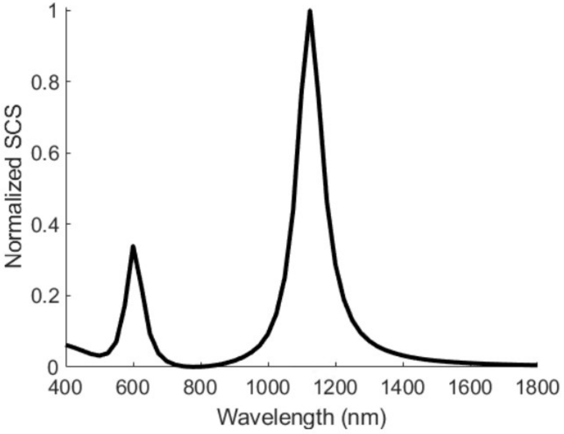




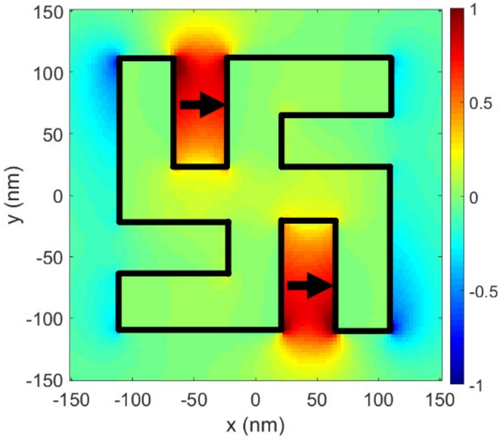
Angular momentum of the field

$$\langle \mathbf{L} \rangle = \frac{1}{c^2} \int \mathbf{r} \times \langle \mathbf{S} \rangle \, dV$$

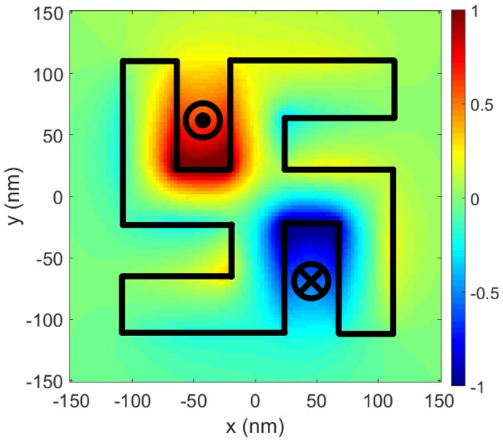
The particle rotates in the opposite direction than that of the field



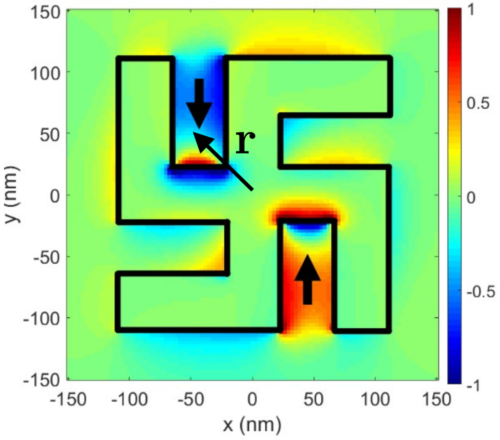
Electric field, E_x



Magnetic field, H_z



Poynting Vector, S_y



What Have We Learned So Far….

- Electromagnetic waves carry momentum (linear and angular) and can thus induce forces and torques
- Starting from the Lorentz force and using Maxwell equations, we can obtain the Maxwell stress tensor which directly allows to compute the forces and torques by surface integration
- Radiation pressure is twice as strong on reflective structures than absorbing ones
- For small particles, the electromagnetic force may also be computed using the different between the total and the incident Poynting vectors
- This allows us to understand forces simply by looking at the radiation pattern of the particle
- Interestingly, forces that are transverse to the direction of propagation of the exciting wave are possible
- Similarly, the Poynting vector can be used to assess the angular momentum of the field scattered by the particle, which allows us to deduce the torque