

Lecture 6

Temporal and Spatial Dispersion

Temporal Dispersion

Temporal Dispersion

The response of a medium at a given time depends upon past excitations

Convolution in time

$$\mathbf{D}(t) = \epsilon_0 \mathbf{E}(t) + \mathbf{P}(t) \quad \longrightarrow \quad \mathbf{P}(t) = \int_{-\infty}^T \epsilon_0 \chi_{ee}(t - t') \mathbf{E}(t') dt'$$

Causality (only previous times, no future times) Electric response of the system

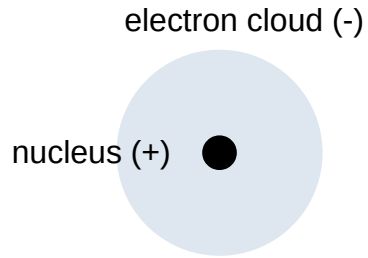
$$\mathbf{P}(t) = \epsilon_0 \chi_{ee}(t) * \mathbf{E}(t)$$

Time-domain Fourier transform

$$\mathbf{P}(\omega) = \epsilon_0 \chi_{ee}(\omega) \mathbf{E}(\omega)$$

“Nonlocality” in time + causality

Lorentz Oscillator Model



Force on the electron cloud

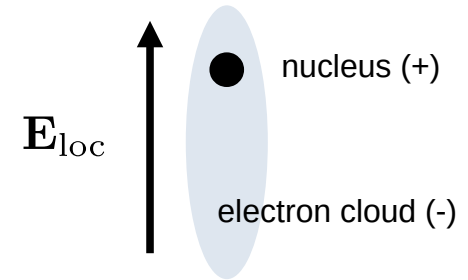
$$\mathbf{F}_e(\mathbf{r}, t) = -q_e \mathbf{E}_{\text{loc}}(\mathbf{r}, t)$$

Restoring force

$$\mathbf{F}_r(\mathbf{r}, t) = -m_e \omega_r^2 \mathbf{d}(\mathbf{r}, t)$$

Frictional force

$$\mathbf{F}_f(\mathbf{r}, t) = -\Gamma m_e \mathbf{v}(\mathbf{r}, t)$$



\mathbf{d} : electron cloud displacement

Γ : dampening factor

$$\sum \mathbf{F} = m\mathbf{a} \longrightarrow$$

$$m_e \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) = -q_e \mathbf{E}_{\text{loc}}(\mathbf{r}, t) - m_e \omega_r^2 \mathbf{d}(\mathbf{r}, t) - \Gamma m_e \mathbf{v}(\mathbf{r}, t)$$

$$q_e = 1.60217646 \times 10^{-19} \text{ C}$$

$$m_e = 9.10938188 \times 10^{-31} \text{ kg}$$

Lorentz Oscillator Model

$$m_e \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) = -q_e \mathbf{E}_{\text{loc}}(\mathbf{r}, t) - m_e \omega_r^2 \mathbf{d}(\mathbf{r}, t) - \Gamma m_e \mathbf{v}(\mathbf{r}, t)$$



$$\mathbf{v}(\mathbf{r}, t) = \frac{\partial}{\partial t} \mathbf{d}(\mathbf{r}, t)$$

$$\frac{\partial^2}{\partial t^2} \mathbf{d}(\mathbf{r}, t) + \Gamma \frac{\partial}{\partial t} \mathbf{d}(\mathbf{r}, t) + \omega_r^2 \mathbf{d}(\mathbf{r}, t) = -\frac{q_e}{m_e} \mathbf{E}_{\text{loc}}(\mathbf{r}, t)$$



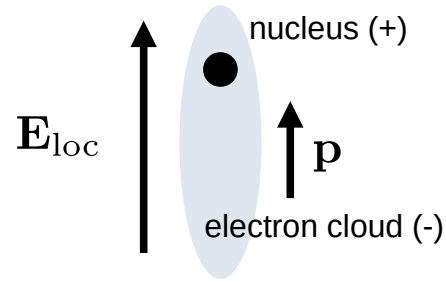
Time-harmonic regime $e^{-j\omega t}$

$$(-\omega^2 - j\omega\Gamma + \omega_r^2) \mathbf{d}(\omega) = -\frac{q_e}{m_e} \mathbf{E}_{\text{loc}}(\omega)$$

Particle Polarizability

Electron cloud displacement

$$(-\omega^2 - j\omega\Gamma + \omega_r^2) \mathbf{d}(\omega) = -\frac{q_e}{m_e} \mathbf{E}_{\text{loc}}(\omega) \quad \longrightarrow \quad \mathbf{d}(\omega) = -\frac{q_e}{m_e} \frac{\mathbf{E}_{\text{loc}}(\omega)}{(\omega_r^2 - \omega^2 - j\omega\Gamma)}$$



The dipole moment is expressed as $\mathbf{p}(\omega) = -q_e \mathbf{d}(\omega)$

$$\mathbf{p}(\omega) = \frac{q_e^2}{m_e} \frac{\mathbf{E}_{\text{loc}}(\omega)}{(\omega_r^2 - \omega^2 - j\omega\Gamma)}$$

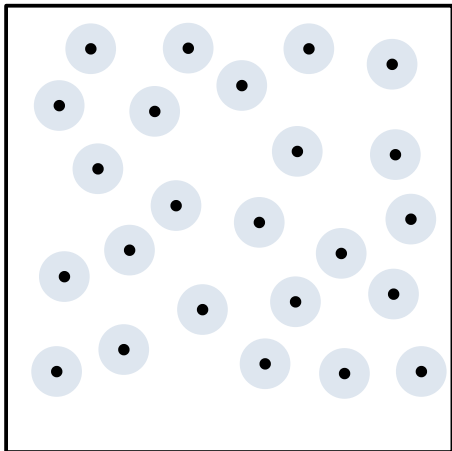
The response of the particle may be expressed in terms of an effective polarizability α

$$\mathbf{p}(\omega) = \alpha(\omega) \mathbf{E}_{\text{loc}}(\omega)$$

$$\alpha(\omega) = \frac{q_e^2}{m_e} \frac{1}{(\omega_r^2 - \omega^2 - j\omega\Gamma)}$$

From Particle to Medium

Unpolarized medium



$$\mathbf{P}(\omega) = \frac{1}{V} \sum_V \mathbf{p}_i(\omega) \longrightarrow \mathbf{P}(\omega) = N \langle \mathbf{p}(\omega) \rangle$$

Statistical average within the volume

N : density of atoms

$$\mathbf{P}(\omega) = \frac{Nq_e^2}{m_e} \frac{\mathbf{E}_{\text{loc}}(\omega)}{(\omega_r^2 - \omega^2 - j\omega\Gamma)}$$

Clausius-Mosotti $\mathbf{E}_{\text{loc}}(\omega) = \mathbf{E}(\omega) + \frac{\mathbf{P}(\omega)}{3\epsilon_0}$

$$\mathbf{P}(\omega) = \frac{Nq_e^2}{m_e} \frac{\mathbf{E}(\omega)}{(\omega_0^2 - \omega^2 - j\omega\Gamma)}$$

where $\omega_0 = \sqrt{\frac{\omega_r^2 - Nq_e^2}{3m_e\epsilon_0}}$

Using an effective susceptibility $\mathbf{P} = \epsilon_0 \chi_{ee} \mathbf{E}$

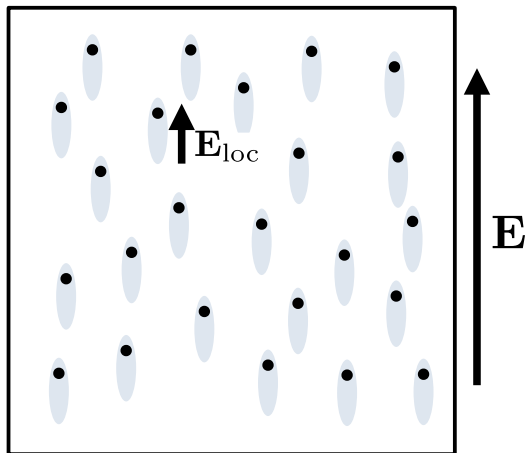
$$\chi_{ee}(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$

where $\omega_p = \sqrt{\frac{Nq_e^2}{\epsilon_0 m_e}}$

is the plasma frequency

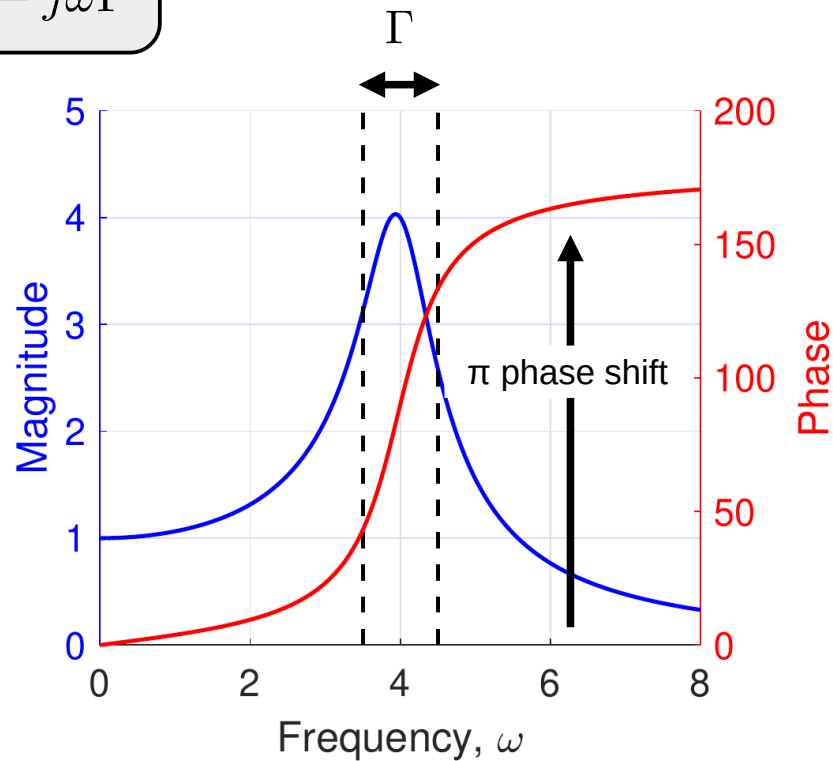
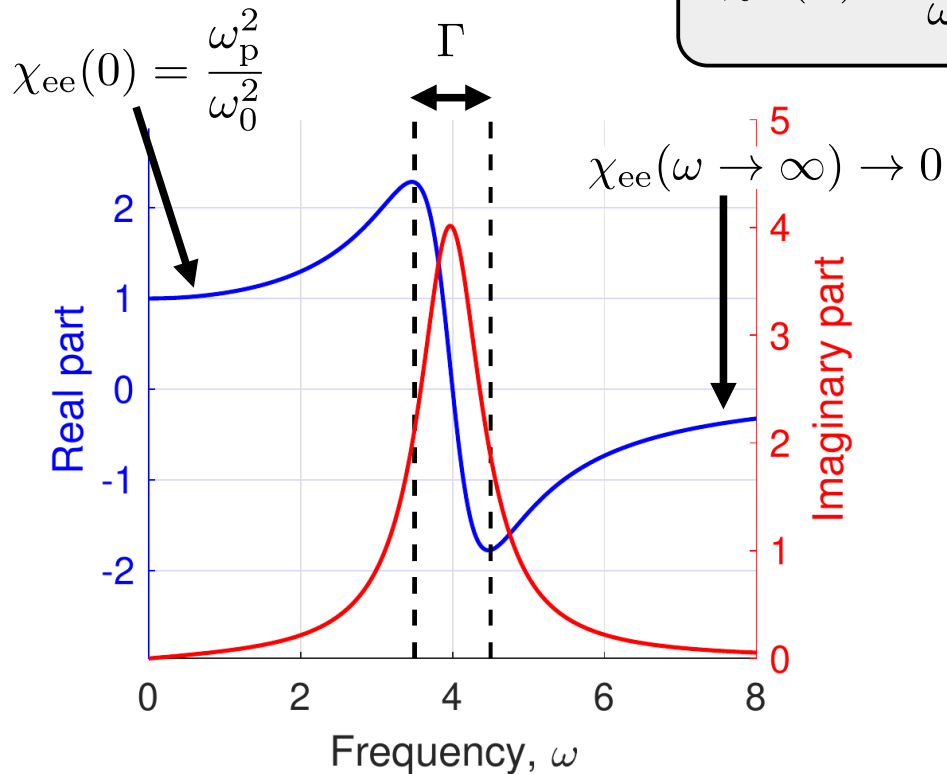
Lorentzian function

Polarized medium



Behavior of the Susceptibility Function

$$\chi_{ee}(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$



Γ is the FWHM of $|\chi_{ee}(\omega)|^2$

$$\Gamma = 1, \omega_0 = 4, \omega_p = 4$$

From Susceptibility to Permittivity

$$\chi_{ee}(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$

$$\epsilon_r(\omega) = 1 + \chi_{ee}(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$



Splitting real and imaginary parts

$$\epsilon_r(\omega) = \epsilon_r'(\omega) + j\epsilon_r''(\omega)$$



$$\epsilon_r'(\omega) = 1 + \omega_p^2 \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$



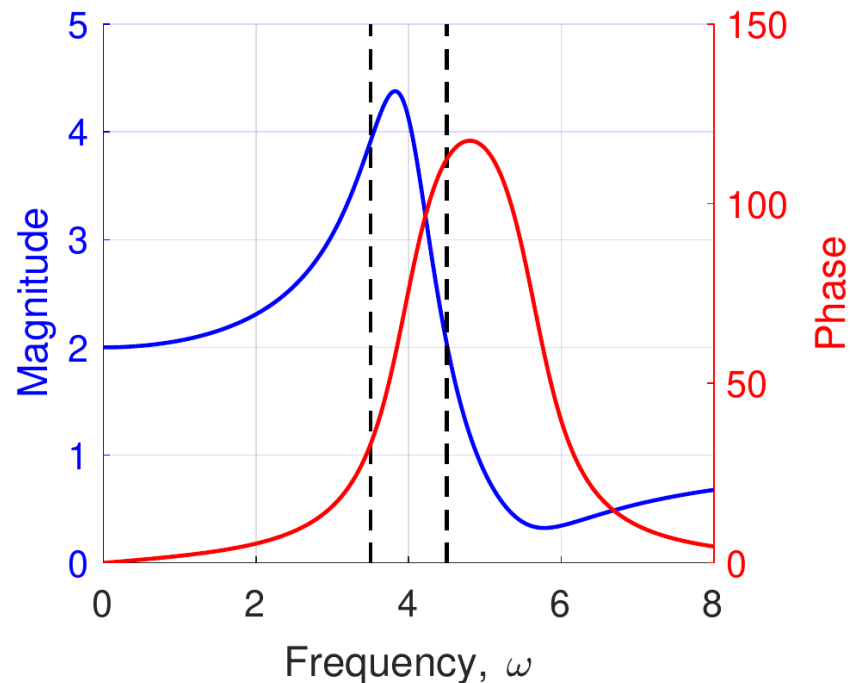
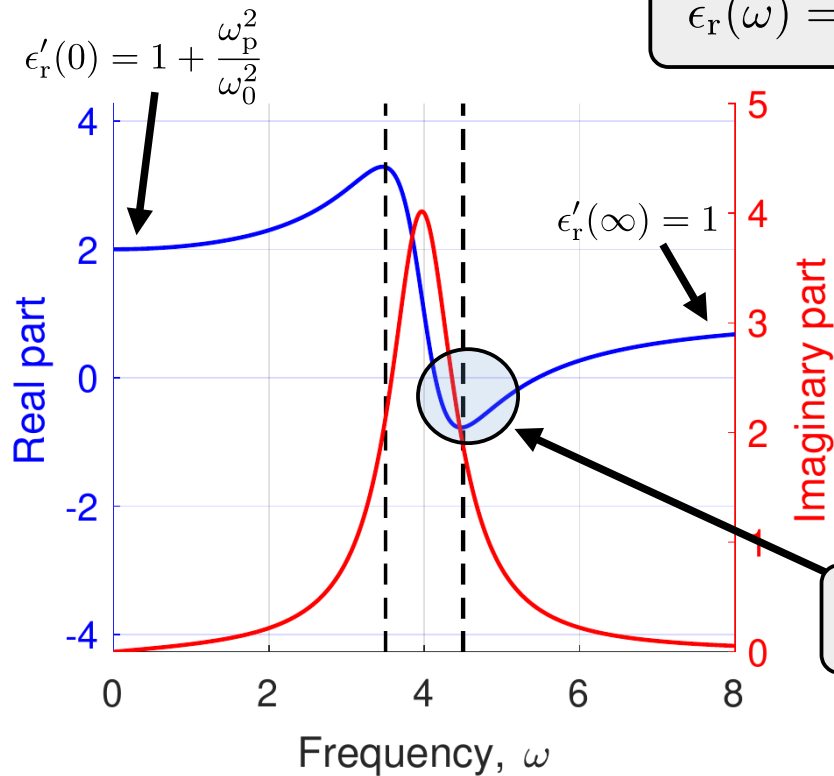
$$\epsilon_r''(\omega) = \omega_p^2 \frac{\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

Behavior of the Permittivity Function

$$\epsilon'_r(\omega) = 1 + \omega_p^2 \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}$$

$$\epsilon''_r(\omega) = \omega_p^2 \frac{\omega \Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}$$

$$\epsilon_r(\omega) = \epsilon'_r(\omega) + j\epsilon''_r(\omega)$$



$$\Gamma = 1, \omega_0 = 4, \omega_p = 4$$

Dispersive Refractive Index

$$n = n_0 + j\kappa = \pm\sqrt{\epsilon_r\mu_r}$$

↓ assuming $\mu_r = 1$

$$n^2 = (n_0 + j\kappa)^2 = \epsilon_r = \epsilon_r' + j\epsilon_r''$$

↓ associate real and imag. parts

$$\begin{aligned} \epsilon_r' &= n_0^2 - \kappa^2 \\ \epsilon_r'' &= 2n_0\kappa \end{aligned}$$

↓ solve for n_0 and κ

$$n_0 = \sqrt{\frac{1}{2} \left(\sqrt{\epsilon_r'^2 + \epsilon_r''^2} + \epsilon_r' \right)}$$

$$\kappa = \sqrt{\frac{1}{2} \left(\sqrt{\epsilon_r'^2 + \epsilon_r''^2} - \epsilon_r' \right)}$$

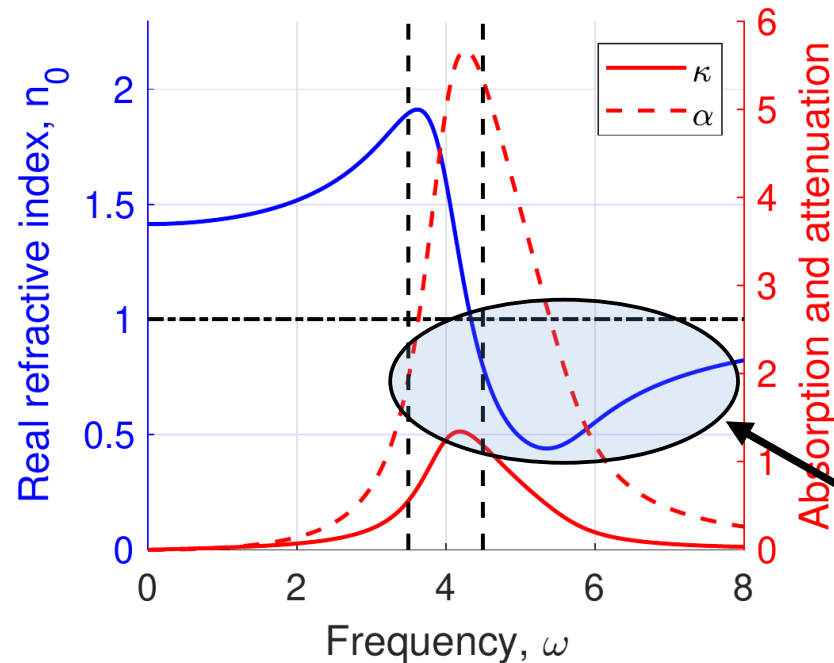
$$\mathbf{E} = \mathbf{E}_0 e^{jkz} = \mathbf{E}_0 e^{jk_0 n z} = \mathbf{E}_0 e^{jk_0 (n_0 + j\kappa) z}$$

$$\mathbf{E} = \mathbf{E}_0 e^{jk_0 n_0 z} e^{-k_0 \kappa z} = \mathbf{E}_0 e^{j\beta z} e^{-\alpha z}$$

where

$\beta = k_0 n_0$
(propagation constant)

$\alpha = k_0 \kappa$
(attenuation constant)



$$n_0(\omega) < 1$$

Absorption depends on real part of permittivity !

$$\Gamma = 1, \omega_0 = 4, \omega_p = 4$$

A Note on Absorption

$$\mathbf{E} = \mathbf{E}_0 e^{j\beta z} e^{-\alpha z}$$

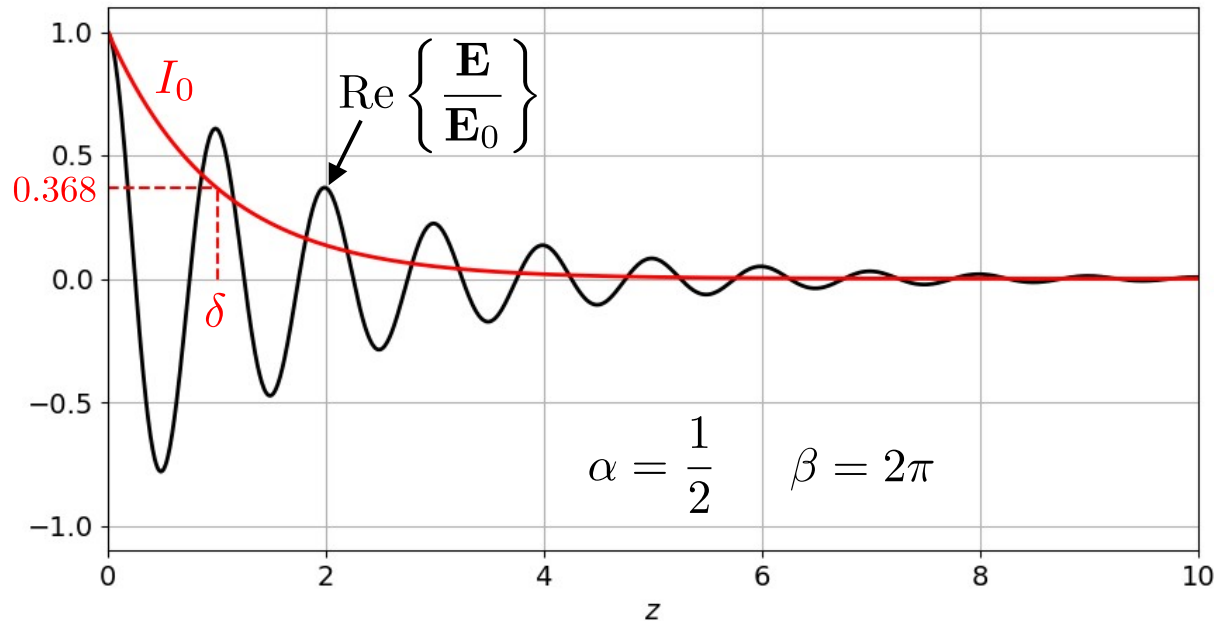
$$\longrightarrow I(z) = |\mathbf{E}|^2 = |\mathbf{E}_0|^2 e^{-2\alpha z}$$

$$\longrightarrow I_0(z) = \frac{I(z)}{|\mathbf{E}_0|^2}$$

Skin depth

$$\delta = \frac{1}{2\alpha}$$

Corresponds to a decrease of the intensity by $1 - e^{-1} \approx 63.2\%$



Decrease after
 n skin depths

$$I_0(\delta) \approx 36.8\%$$

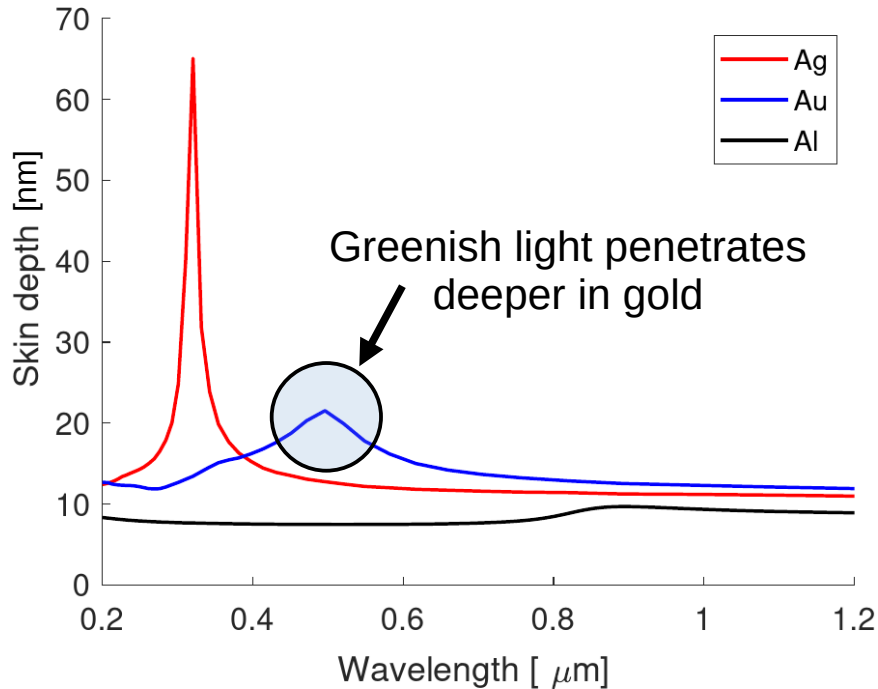
$$I_0(2\delta) \approx 13.5\%$$

$$I_0(3\delta) \approx 5.0\%$$

$$I_0(4\delta) \approx 1.8\%$$

Effects of Skin Depth

Skin depth of silver (Ag), gold (Au) and Aluminum (Al)



$$I_0(\delta) \approx 36.8\%$$

$$I_0(2\delta) \approx 13.5\%$$

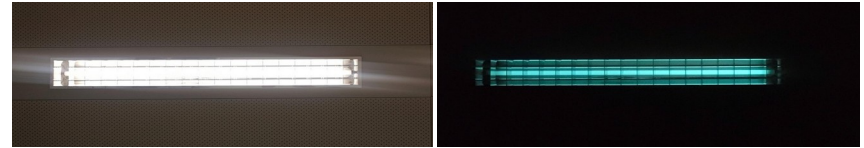
$$I_0(3\delta) \approx 5.0\%$$

$$I_0(4\delta) \approx 1.8\%$$

Transmission through 40 nm of gold

No gold

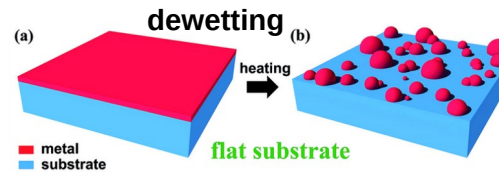
with gold



Reflection from bulk gold



Reflection from nano-layers of gold



Altomare, Marco, Nhat Truong Nguyen, and Patrik Schmuki. "Templated dewetting: designing entirely self-organized platforms for photocatalysis." *Chemical science* 7.12 (2016): 6865-6886.

Negative Index of Refraction

$$\nabla \times \mathbf{H} = +j\omega\epsilon_0\epsilon_r\mathbf{E}$$

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mu_r\mathbf{H}$$



Assuming plane wave of the type

$$\{\mathbf{E}, \mathbf{H}\} \propto e^{-j\mathbf{k}\cdot\mathbf{r}} = e^{-jn\mathbf{k}_0\cdot\mathbf{r}}$$



$$\mathbf{k} \times \mathbf{H} = -\omega\epsilon_0\epsilon_r\mathbf{E}$$

$$\mathbf{k} \times \mathbf{E} = +\omega\mu_0\mu_r\mathbf{H}$$

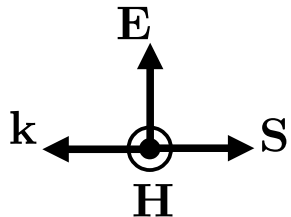
Poynting vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

$$\epsilon < 0, \mu < 0$$

$$\mathbf{k} \times \mathbf{H} = +\omega\epsilon\mathbf{E}$$

$$\mathbf{k} \times \mathbf{E} = -\omega\mu\mathbf{H}$$

Forms a left-hand triad



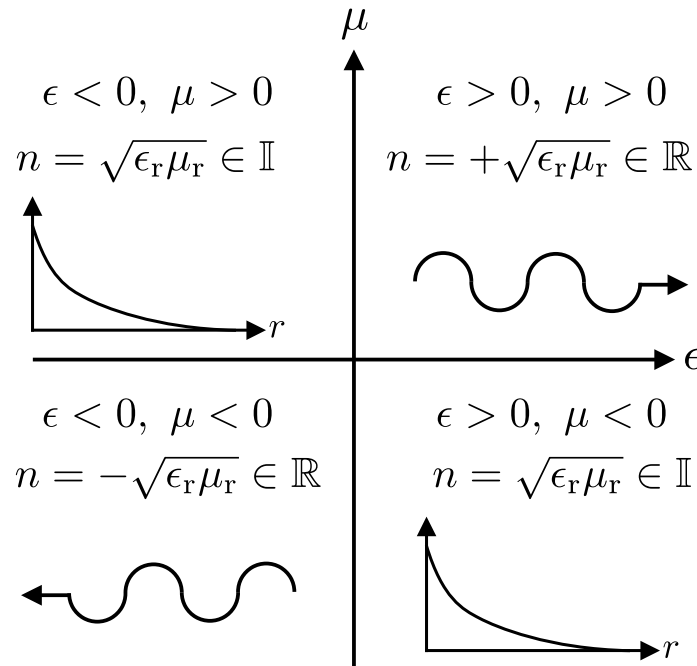
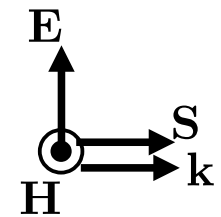
The refractive index must be negative

$$\epsilon > 0, \mu > 0$$

$$\mathbf{k} \times \mathbf{H} = -\omega\epsilon\mathbf{E}$$

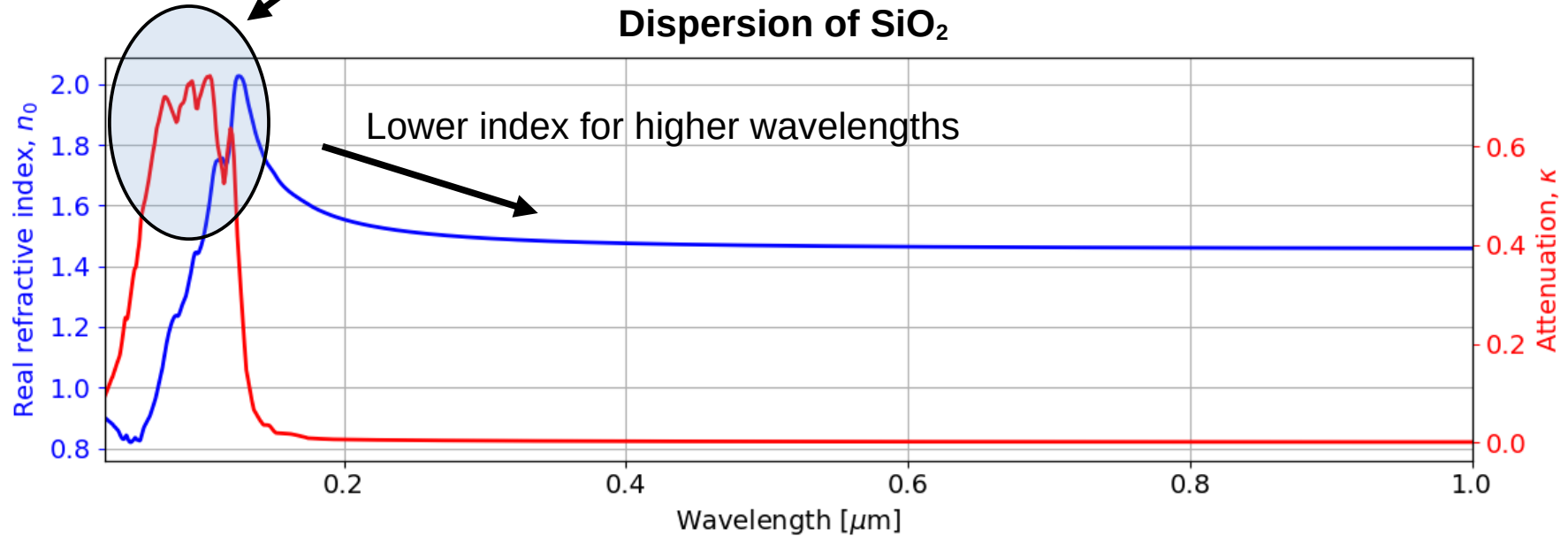
$$\mathbf{k} \times \mathbf{E} = +\omega\mu\mathbf{H}$$

Forms a right-hand triad

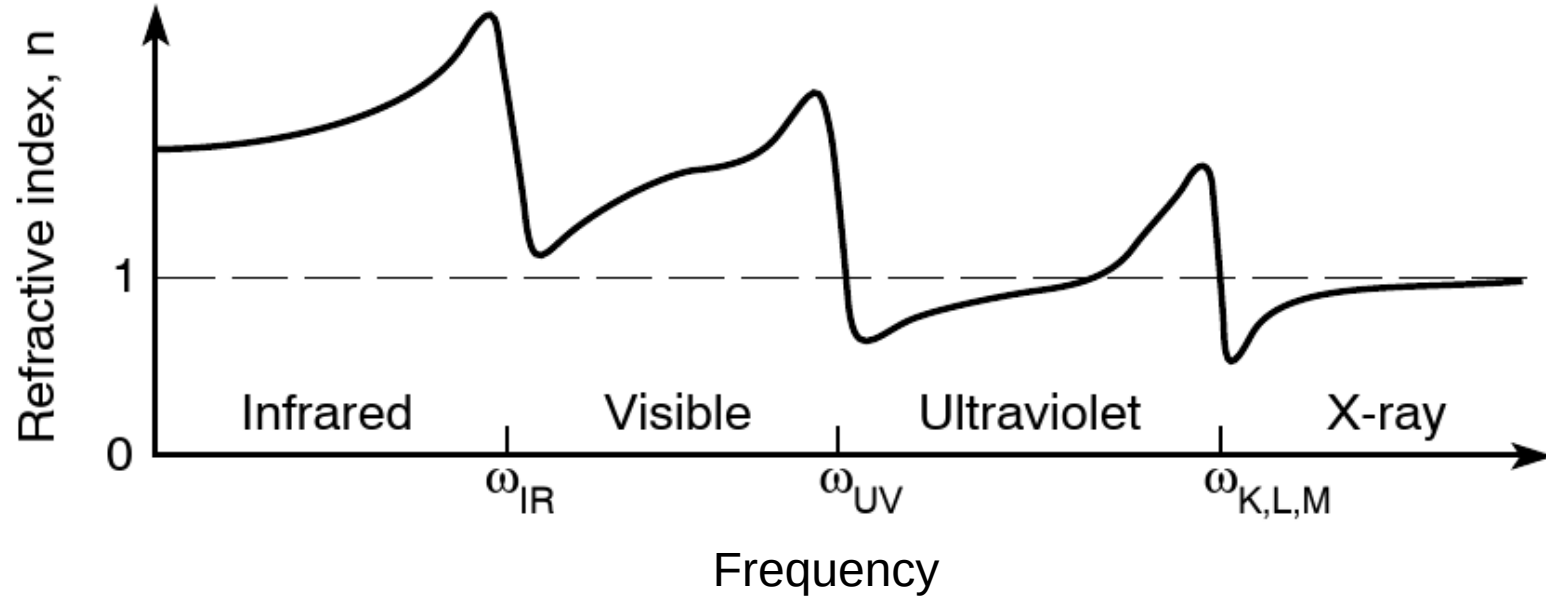


Materials Usually Have Multiple Resonances

$$\chi_{ee}(\omega) = \sum_{i=1}^N \frac{\omega_{p,i}^2}{\omega_{0,i}^2 - \omega^2 - j\omega\Gamma_i}$$



Materials Usually Have Multiple Resonances



Drude Model for Metals

In a metal, no restoring force
 $\omega_0 = 0$

$$\epsilon_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma} \quad \longrightarrow \quad \epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + j\omega\Gamma}$$

Dampening in terms of mean collision time $\Gamma = \frac{1}{\tau} \longrightarrow$

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + j\omega\tau^{-1}} = \underbrace{1 - \frac{\omega_p^2\tau^2}{1 + \omega^2\tau^2}}_{\epsilon_r'(\omega)} + j \underbrace{\frac{\omega_p^2\tau/\omega}{1 + \omega^2\tau^2}}_{\epsilon_r''(\omega)}$$

Since this is a conductor, we may express the permittivity in terms of the conductivity

$$\epsilon_r(\omega) = \epsilon_r'(\omega) + \frac{\sigma(\omega)}{j\omega\epsilon_0}$$

where $\sigma(\omega) = \omega\epsilon_0\epsilon_r'' = \frac{\sigma_0}{1 + \omega^2\tau^2}$

DC conductivity

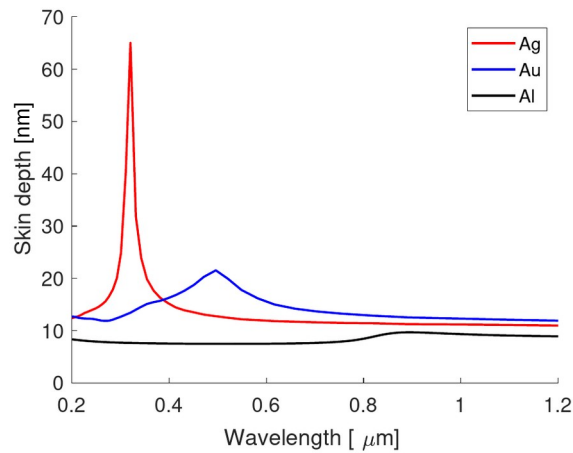
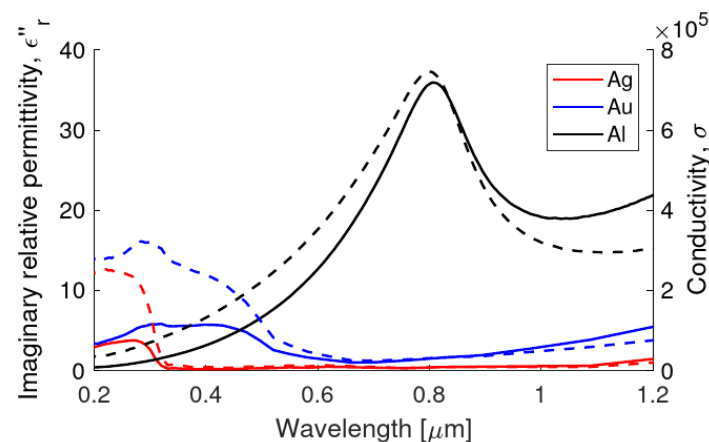
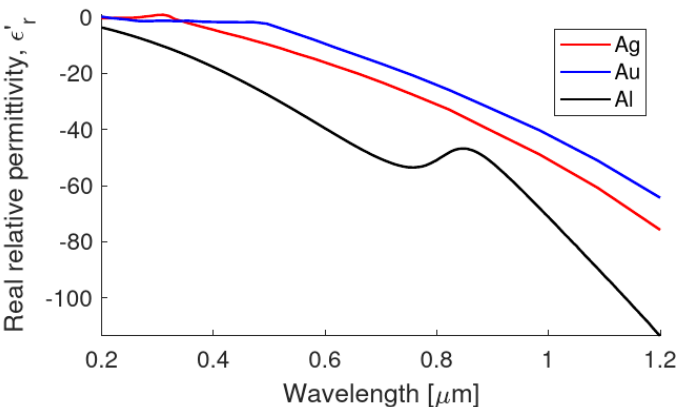
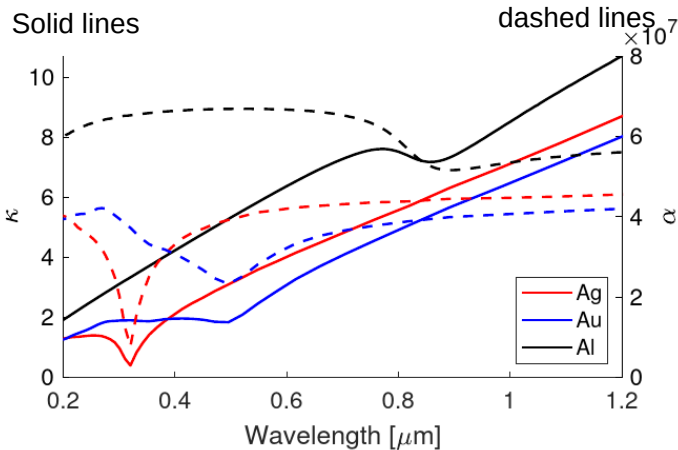
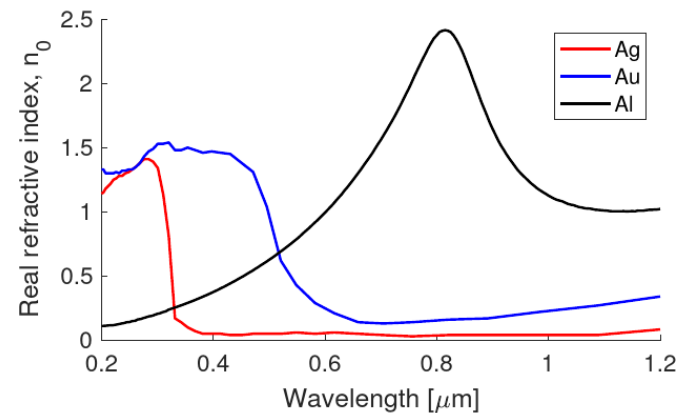
$$\sigma_0 = \epsilon_0\omega_p^2\tau$$

In a “perfect” metal, no collision
 $\tau \rightarrow \infty, \Gamma = 0$

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

A “good” metal as a very low ϵ_r'' and a negative ϵ_r'

Dispersion of Common Plasmonic Metals



Metal	Plasma frequency, ω_p	Damping, Γ
Silver (Ag)	137.56 nm	151.51 nm
Gold (Au)	137.36 nm	465.11 nm
Aluminum (Al)	84.06 nm	689.65 nm

What Have We Learned So Far....

- Material response is a convolution in the time domain leading to a product in frequency domain
- Particle polarizability is obtained from the sum of electric, restoring and frictional forces
- For an isolated particle, we use polarizabilities, for a medium, we use susceptibilities
- A susceptibility is a Lorentzian function with a resonance frequency and a dampening factor
- The dampening factor (loss) contributes to both the real and imaginary parts of the permittivity
- The real and imaginary parts of the permittivity both contribute to the real/imag parts of the refractive index
- A Lorentzian function exhibits a π phase shift across its resonance frequency
- This phase shift is also associated with an increase of the loss in the material
- The skin depth is inversely proportional to the imaginary part of the refractive index
- A propagation over a distance of one skin depth leads to $\sim 63\%$ reduction in total field intensity
- Resonances may be exploited to achieve negative permittivity and permeability and refractive indices lower than 1
- Negative permittivity and permeability leads to negative refractive index (left-handed material) with phase propagating in the opposite direction of the power flow
- A negative refractive index implies negative refraction leading to the "perfect lens" concept although loss and dispersion must be considered, which would make such a "perfect lens" not that "perfect"
- In the visible range, higher frequencies imply higher refractive indices leading to larger refraction angles
- An ideal metal has a permittivity with a low imaginary part and a negative real part

Kramers-Kronig Relations

Transfer Function in Time

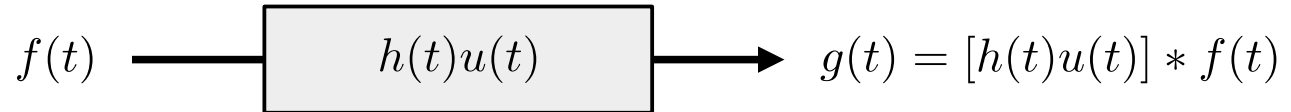
Transfer function of a time-dependent system



In the frequency domain, taking a Fourier transform, we have

$$\mathcal{F}\{g(t)\} = G(\omega) = H(\omega)F(\omega)$$

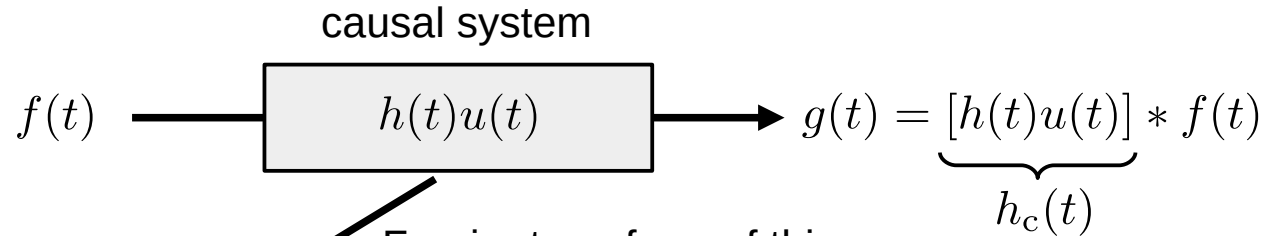
For a causal system



$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

Frequency Response of a Causal System

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



Fourier transform of this
causal transfer function

$$H_c(\omega) = \mathcal{F}\{h_c(t)\} = H(\omega) * \mathcal{F}\{u(t)\} = H(\omega) * \frac{1}{2\pi} \left[\pi\delta(\omega) + \frac{j}{\omega} \right]$$

applying the convolution

$$H_c(\omega) = \frac{1}{2}H(\omega) + \frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{\omega - \Omega} d\Omega$$

Derivation of the Kramers-Kronig Relations

$$H_c(\omega) = \frac{1}{2}H(\omega) + \frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{\omega - \Omega} d\Omega$$

Applied to the
susceptibility

$$\chi(t) = h_c(t) = h(t)u(t)$$

$$\hat{\chi}'(\omega) + j\hat{\chi}''(\omega) = \frac{1}{2}H(\omega) + \frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{\omega - \Omega} d\Omega$$

We are free to define $h(-t)$ as we desire without affecting $\chi(t)$!

Let's impose

$$h(-t) = h(t)$$



$$H(\omega) \in \mathbb{R}$$

Splitting real/imag. parts

$$\left\{ \begin{array}{l} \hat{\chi}'(\omega) = \frac{1}{2}H(\omega) \\ \hat{\chi}''(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{\omega - \Omega} d\Omega \end{array} \right\}$$

$$\hat{\chi}''(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{\chi}'(\Omega)}{\omega - \Omega} d\Omega$$

**Imaginary and real parts are
connected to each other !**

Let's impose

$$h(-t) = -h(t)$$



$$H(\omega) \in \mathbb{I}$$

Splitting real/imag. parts

$$\left\{ \begin{array}{l} \hat{\chi}''(\omega) = \frac{1}{2}H(\omega) \\ \hat{\chi}'(\omega) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{\omega - \Omega} d\Omega \end{array} \right\}$$

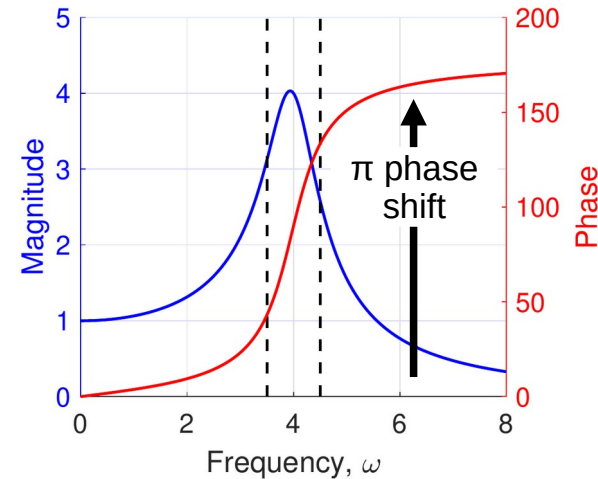
$$\hat{\chi}'(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{\chi}''(\Omega)}{\omega - \Omega} d\Omega$$

Consequences of the Kramers-Kronig Relations

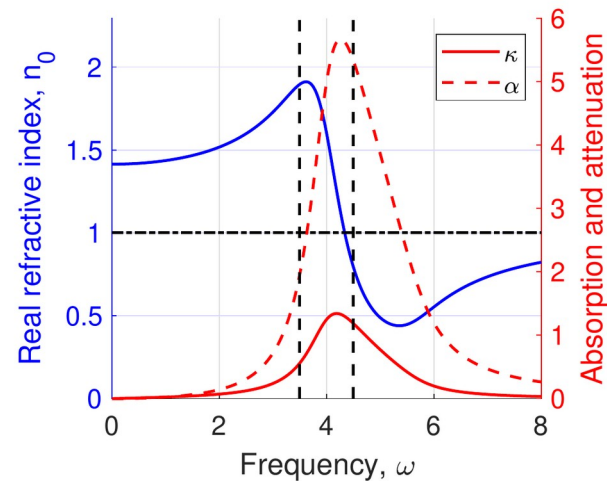
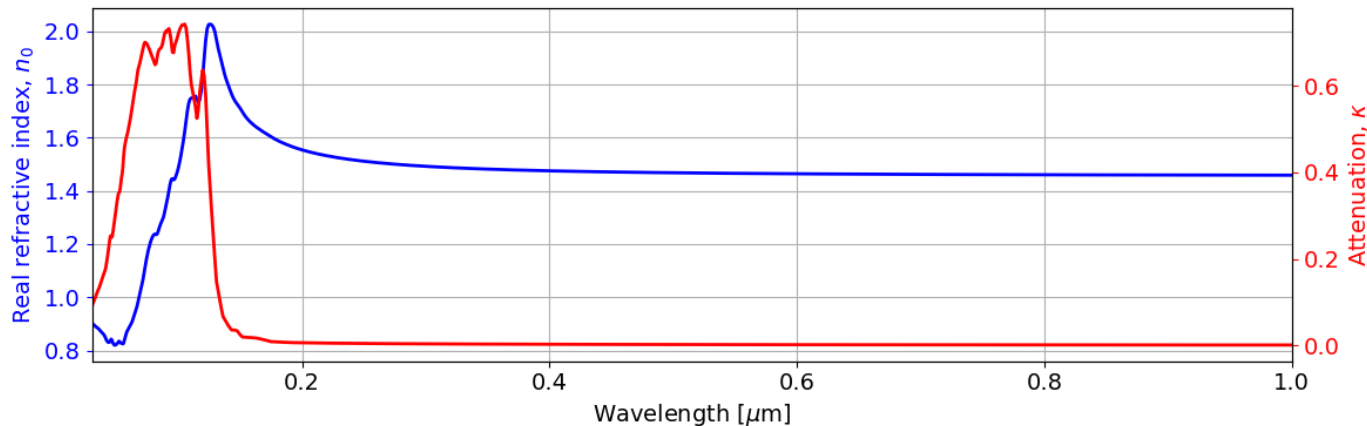
$$\hat{\chi}'(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{\chi}''(\Omega)}{\omega - \Omega} d\Omega$$

$$\hat{\chi}''(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{\chi}'(\Omega)}{\omega - \Omega} d\Omega$$

Real and imaginary parts are related to each other
 => phase shift implies loss



Dispersion of SiO₂



What Have We Learned So Far....

- Causality leads to a connection between the real and imaginary parts of the susceptibility function
- This implies a connection between the ability to create phase shift and the loss of the material
- When designing metamaterials, we typically avoid using materials that are too lossy (or we use them in a regime where their losses is small). This is because metamaterials are generally resonant structures whose loss is amplified near the resonances.

Group and Phase Velocities

Propagation Through a Dispersive Medium

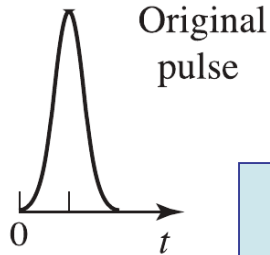
Dispersive permittivity

$$\epsilon_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$



$$n_0(\omega) = \sqrt{\frac{1}{2} \left(\sqrt{\epsilon_r'^2 + \epsilon_r''^2} + \epsilon_r' \right)}$$

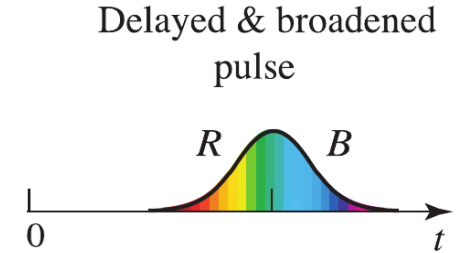
$$\kappa(\omega) = \sqrt{\frac{1}{2} \left(\sqrt{\epsilon_r'^2 + \epsilon_r''^2} - \epsilon_r' \right)}$$



Original pulse



Dispersive Medium



Delayed & broadened pulse

A pulse is made of many frequencies

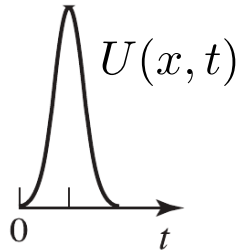
$$c = \frac{c_0}{n(\omega)}$$

Each frequency travels through the medium at different velocity

Derivation of Group and Phase Velocities

Consider a $U(x,t)$ wave expressed in terms of its fourier transform

$$U(x,t) = \int_{-\infty}^{\infty} \hat{U}(k) e^{j(kx - \omega t)} dk$$



→ $\hat{U}(k)$ has a narrow peak amplitude

For a monochromatic wave, we have $\omega(k) = kc$


In our case, we may expand ω in a Taylor series around k

$$\omega(k) \approx \omega_0 + (k - k_0)\omega'_0 \quad \text{where} \quad \omega'_0 = \left. \frac{\partial \omega(k)}{\partial k} \right|_{k=k_0}$$

Derivation of Group and Phase Velocities

$$U(x, t) = \int_{-\infty}^{\infty} \hat{U}(k) e^{j(kx - \omega t)} dk$$

$\omega(k) \approx \omega_0 + (k - k_0)\omega'_0$ where $\omega'_0 = \left. \frac{\partial \omega(k)}{\partial k} \right|_{k=k_0}$



$$U(x, t) \approx \int_{-\infty}^{\infty} \hat{U}(k) e^{j[kx - (\omega_0 + \omega'_0(k - k_0))t]} dk$$

Add the self-canceling terms $+k_0x - k_0x$



$$U(x, t) \approx e^{j(k_0x - \omega_0t)} \int_{-\infty}^{\infty} \hat{U}(k) e^{j(k - k_0)(x - \omega'_0t)} dk$$

Derivation of Group and Phase Velocities

$$U(x, t) \approx \underbrace{e^{j(k_0 x - \omega_0 t)}}_{\text{Monochromatic wave with phase velocity}} \int_{-\infty}^{\infty} \hat{U}(k) e^{j(k-k_0)(x-\omega'_0 t)} dk$$

Monochromatic wave
with phase velocity

$$v_p = \frac{\omega_0}{k_0} = c$$

Space and time variations
are given by

$$(x - \omega'_0 t)$$

corresponding to a
group velocity

$$v_g = \frac{\partial \omega(k)}{\partial k} = \left(\frac{\partial k(\omega)}{\partial \omega} \right)^{-1}$$

- Phase velocity
- Group velocity



Generalization to Arbitrary Direction of Propagation

Phase velocity

$$v_p = \frac{\omega}{k} = c$$

Group velocity

$$v_g = \frac{\partial \omega(k)}{\partial k} = \left(\frac{\partial k(\omega)}{\partial \omega} \right)^{-1}$$

The group velocity may be different along x,y and z

$$\mathbf{v}_g = v_{g,x} \hat{\mathbf{x}} + v_{g,y} \hat{\mathbf{y}} + v_{g,z} \hat{\mathbf{z}} = \left(\frac{\partial}{\partial k_x} \hat{\mathbf{x}} + \frac{\partial}{\partial k_y} \hat{\mathbf{y}} + \frac{\partial}{\partial k_z} \hat{\mathbf{z}} \right) \omega(\mathbf{k}) = \nabla_{\mathbf{k}} \omega(\mathbf{k})$$

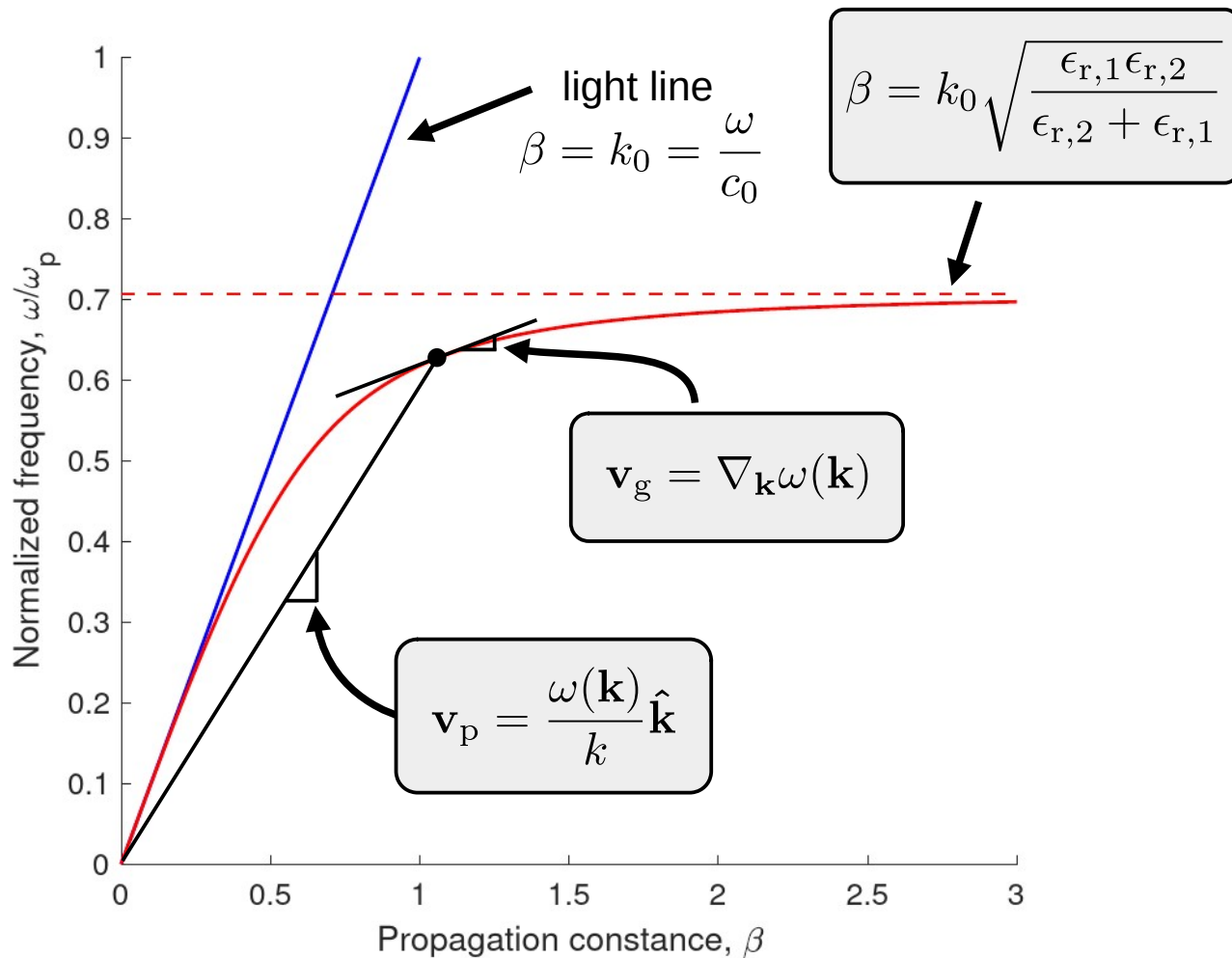
$$\mathbf{v}_p = \frac{\omega(\mathbf{k})}{k} \hat{\mathbf{k}}$$

The phase velocity is the slope of the ω - k function

$$\mathbf{v}_g = \nabla_{\mathbf{k}} \omega(\mathbf{k})$$

The group velocity is perpendicular to surfaces of constant ω

Dispersion of a Surface Plasmon



For an "ideal" metal

$$\epsilon_{r,1} = 1 - \frac{\omega_p^2}{\omega^2}$$

Example: Dispersive Refractive Index

Real part of the refractive index $\omega n(\omega) = c_0 k$ \longrightarrow $\omega dn + n d\omega = c_0 dk$ \longrightarrow $\frac{c_0}{n} = \frac{\omega}{n} \frac{dn}{dk} + \frac{d\omega}{dk}$

$$v_g = v_p \left(1 - \frac{k}{n} \frac{dn}{dk} \right) \longrightarrow \mathbf{v}_g = \mathbf{v}_p \left(1 - \frac{k}{n} \nabla_{\mathbf{k}} n \right)$$

In the absence of dispersion $\frac{dn}{dk} = 0$ \longrightarrow $v_g = v_p$

We may define group and phase refractive indices as $n_g = \frac{c_0}{v_g}$ and $n_p = \frac{c_0}{v_p}$

$$n_g = n_p - \omega \frac{dn_p}{d\omega} = n_p - \lambda_0 \frac{dn_p}{d\lambda_0}$$

Consequence of a Dispersive Refractive Index

$$n_g = n_p - \omega \frac{dn_p}{d\omega} = n_p - \lambda_0 \frac{dn_p}{d\lambda_0}$$

Phase velocity/index

$$v_p = \frac{\omega}{k} = c$$

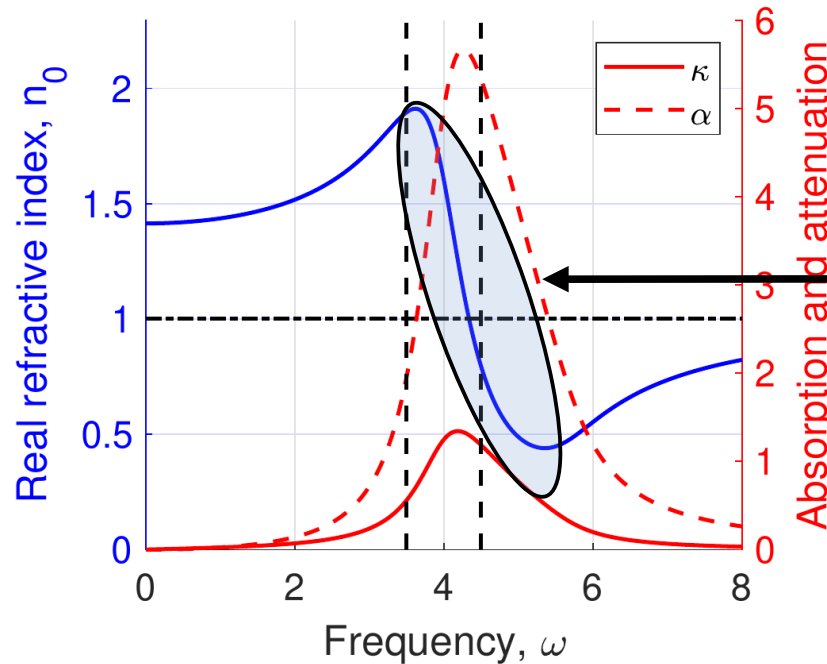
$$n_p = \frac{c_0}{v_p}$$

Group velocity/index

$$v_g = \frac{\partial \omega(k)}{\partial k}$$

$$n_g = \frac{c_0}{v_g}$$

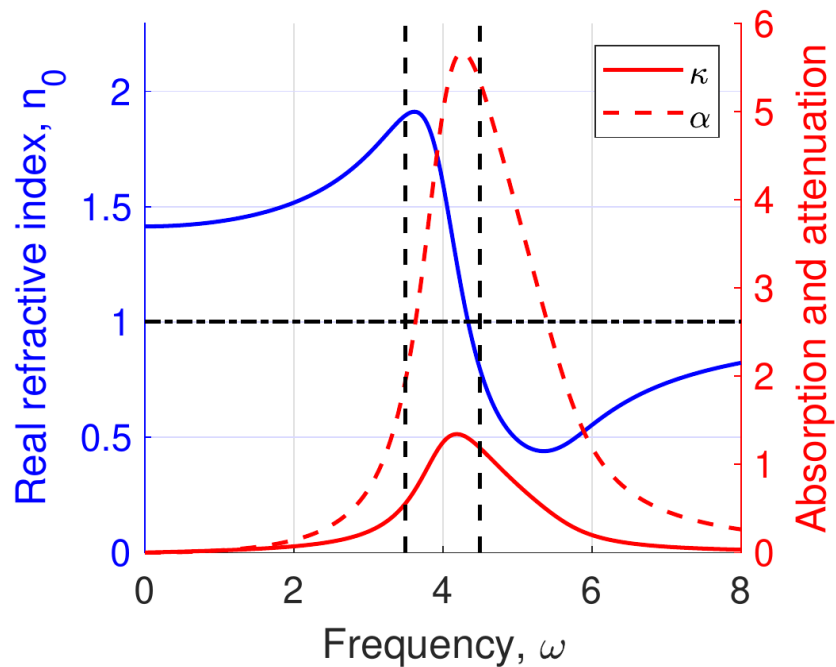
What if this becomes large compared to n_p ?



$$\omega \frac{dn_p}{d\omega} < 0$$

Light slows down

A Group Velocity Faster than the Speed of Light ?

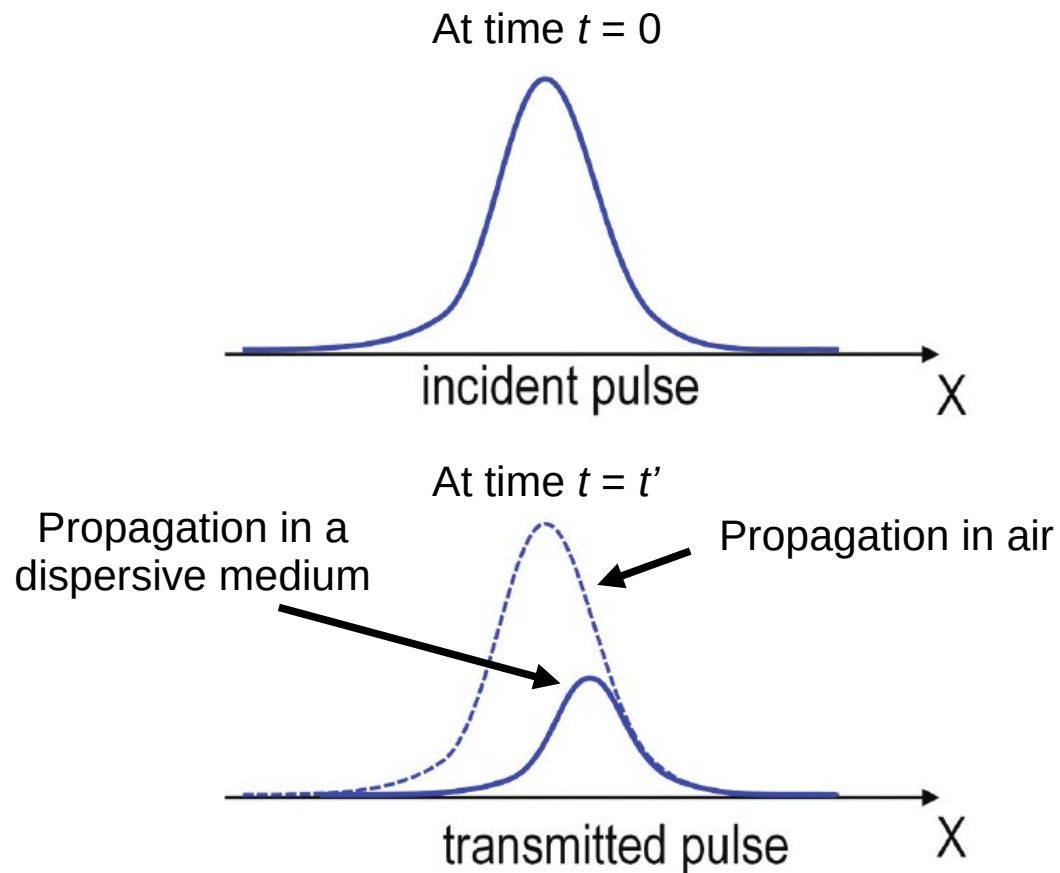


$$n_g = n_p - \omega \frac{dn_p}{d\omega}$$

What if $n_g < 1$?

$$v_g = \frac{c_0}{n_g}$$

$$v_g > c_0$$



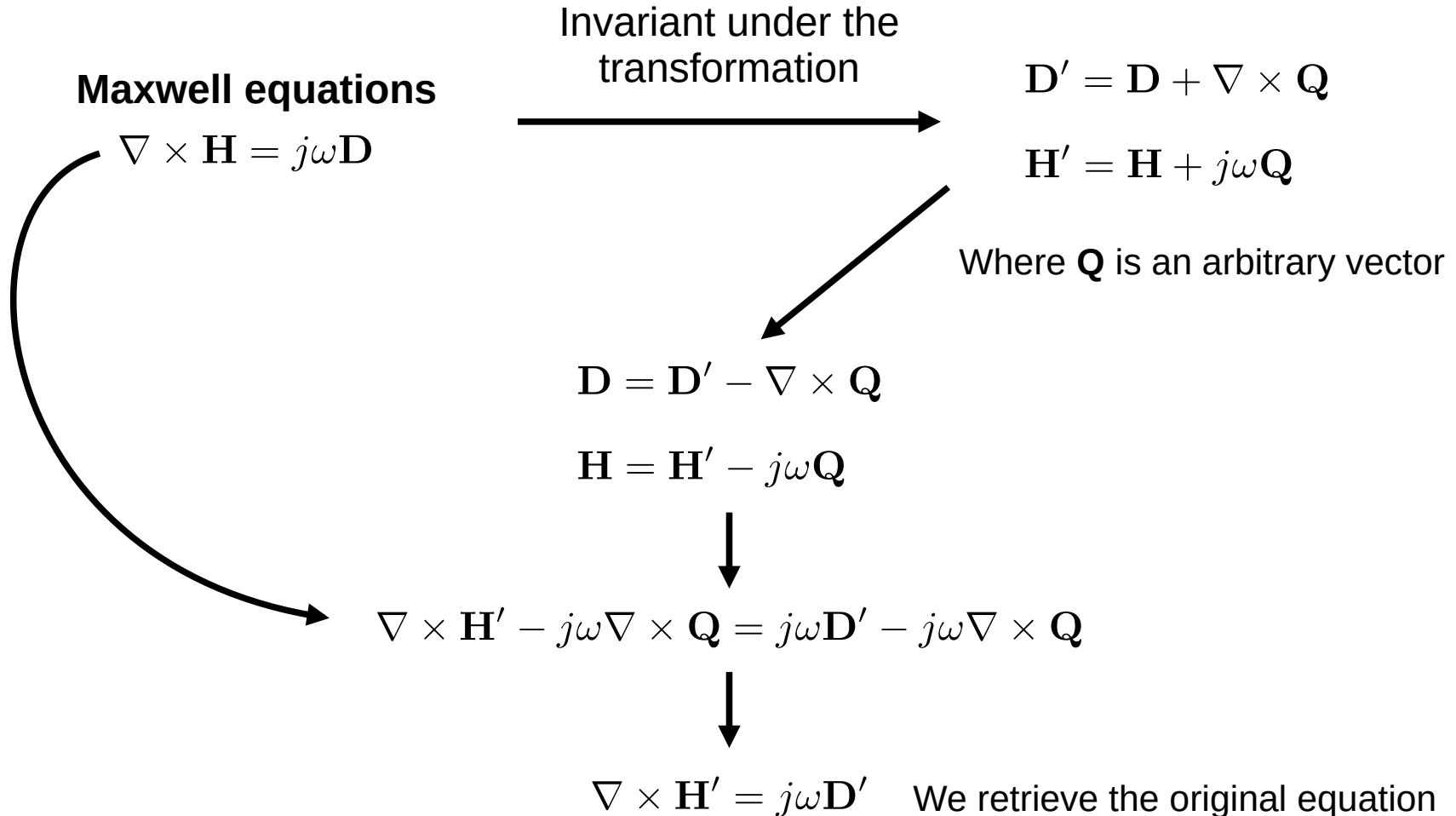
**The peak moves faster than light in air !
But the information does not !**

What Have We Learned So Far....

- Phase velocity = speed at which the phase of a pulse propagates in a dispersive medium
- The phase velocity is simply the speed of light in vacuum divided by the real part of the refractive index
- Group velocity = speed at which the energy of a pulse propagates in a dispersive medium
- The group velocity is the variation of ω with respect to k
- Dispersive media distort signals by changing the propagation velocity of the signal frequencies
- In a dispersion diagram, the phase and group velocity are easily identified as the ratio of ω/k and the slope of the dispersion curve, respectively.
- Small bandwidth signals propagating in dispersive media have different phase and group velocities
- We can also talk about phase and group refractive indices
- The group refractive index is proportional to the derivative of the phase (real part) refractive index w.r.t ω
- Large variation of the real part of the refractive index are associated with anomalous dispersion (slow light, fast light)
- The group refractive index may be lower than 1 leading to a group velocity larger than the speed of light in vacuum. This does not break causality because of the high loss associated with such effects

Spatial Dispersion

Non-Uniqueness of Constitutive Relations



What is a Tensor?

0th order tensor: scalar

$$a = [a]$$

1st order tensor: vector

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

2nd order tensor: matrix

$$a_{ij} = \begin{matrix} \xrightarrow{j} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ \downarrow i \end{matrix}$$

3rd order tensor

$$a_{ijk} = \begin{matrix} & & \begin{bmatrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{bmatrix} \\ & \begin{bmatrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{bmatrix} & \\ \begin{bmatrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{bmatrix} & & \nearrow k \\ \downarrow i & \xrightarrow{j} & \end{matrix}$$

What is Isotropy?

Isotropic = “equal in all orientations” \longrightarrow Rotation invariant

Rotation matrices

$$\overline{\overline{\mathbf{R}}}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$\overline{\overline{\mathbf{R}}}_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$\overline{\overline{\mathbf{R}}}_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

General rotation matrix (yaw, pitch, roll)

$$\overline{\overline{\mathbf{R}}} = \overline{\overline{\mathbf{R}}}_z(\alpha) \cdot \overline{\overline{\mathbf{R}}}_y(\beta) \cdot \overline{\overline{\mathbf{R}}}_x(\gamma) = \begin{bmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma \\ -\sin \beta & \cos \beta \sin \gamma & \cos \beta \cos \gamma \end{bmatrix}$$

Note that a rotation matrix is orthogonal

$$\overline{\overline{\mathbf{R}}}^{-1} = \overline{\overline{\mathbf{R}}}^T$$

$$\longrightarrow \det(\overline{\overline{\mathbf{R}}}) = \pm 1$$

What is an Isotropic Tensor?

0th order tensor

$$a_{\bar{\bar{R}}} = a_{\bar{\bar{R}}} \longrightarrow \text{Invariant under rotation} \longrightarrow \text{All scalar are isotropic}$$

1st order tensor

$$\mathbf{v} \cdot \bar{\bar{R}} = \mathbf{v}' \quad \text{Isotropic only if} \quad \mathbf{v} \cdot \bar{\bar{R}} = \mathbf{v} \longrightarrow \left[\bar{\bar{R}} - \bar{\bar{I}} \right] \cdot \mathbf{v} = \mathbf{0}$$

In tensor notation

$$R_{ij} v_j = v'_i$$

For an arbitrary rotation, only trivial solution is possible $\longrightarrow \mathbf{v} = \mathbf{0}$

In general, there is no isotropic vectors

What is an Isotropic Tensor?

2nd order tensor

Rotating a matrix $\bar{\bar{R}} \cdot \bar{\bar{M}} \cdot \bar{\bar{R}}^T = \bar{\bar{M}}'$

In tensor notation $R_{ik}R_{jl}M_{kl} = M'_{ij}$

Isotropic if $\bar{\bar{M}} = \bar{\bar{M}}' \longrightarrow \bar{\bar{R}} \cdot \bar{\bar{M}} \cdot \bar{\bar{R}}^T = \bar{\bar{M}} \longrightarrow$ true if $\boxed{\bar{\bar{M}} = \alpha \bar{\bar{I}}}$
 α is a constant

Demonstration $\bar{\bar{R}} \cdot \bar{\bar{M}} \cdot \bar{\bar{R}}^T = \bar{\bar{R}} \cdot (\alpha \bar{\bar{I}}) \cdot \bar{\bar{R}}^T = \alpha \bar{\bar{R}} \cdot \bar{\bar{R}}^T = \alpha \bar{\bar{I}}$

3rd order tensor

Rotating a tensor $R_{i\alpha}R_{j\beta}R_{k\gamma}A_{\alpha\beta\gamma} = A'_{ijk} \longrightarrow$ Isotropic if $\boxed{A_{ijk} = \alpha \varepsilon_{ijk}}$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1), \text{ or } (3, 1, 2), \\ -1 & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2), \text{ or } (2, 1, 3), \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

Used to define cross product

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \longrightarrow c_i = \varepsilon_{ijk} a_j b_k$$

Isotropic 4th Order Tensor

A 4th order tensor is isotropic if

$$A_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

α, β, γ : constants

Kronecker delta $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$

Symmetric part of a matrix $\bar{\bar{A}}_{\text{sym}} = \frac{1}{2} (\bar{\bar{A}} + \bar{\bar{A}}^T) \longrightarrow A_{ij,\text{sym}} = \frac{1}{2} (A_{ij} + A_{ji})$

Anti-symmetric part of a matrix $\bar{\bar{A}}_{\text{asym}} = \frac{1}{2} (\bar{\bar{A}} - \bar{\bar{A}}^T) \longrightarrow A_{ij,\text{asym}} = \frac{1}{2} (A_{ij} - A_{ji})$

alternative form

$$A_{ijkl} = a \delta_{ij} \delta_{kl} + b \underbrace{(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})}_{\text{symmetric}} + c \underbrace{(\delta_{ik} \delta_{jk} - \delta_{il} \delta_{jk})}_{\text{anti-symmetric}}$$

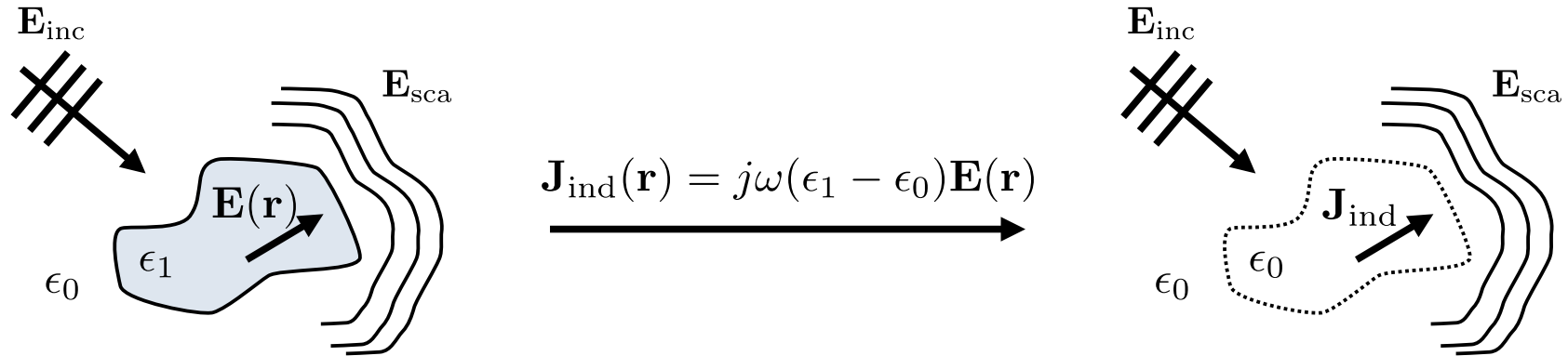
alternatively

$$A_{ijkl} = a \delta_{ij} \delta_{kl} + b (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + c \varepsilon_{ijm} \varepsilon_{mkl}$$

What Have We Learned So Far....

- The constitutive relations are not unique and may be transformed with an arbitrary vector \mathbf{Q}
- Tensors are used to describe effective material parameters
- Isotropy implies invariance under rotations
- Isotropic tensors may be used to represent the material responses of objects that are invariant under rotations (e.g., spheres)
- Vectors are not isotropic
- The identity matrix is isotropic
- The Levi-Civita symbol corresponds to a third-rank isotropic tensor. This symbol is also used to define vector products in tensor notations

Expansion of the Induced Current



In vacuum, the constitutive relations are

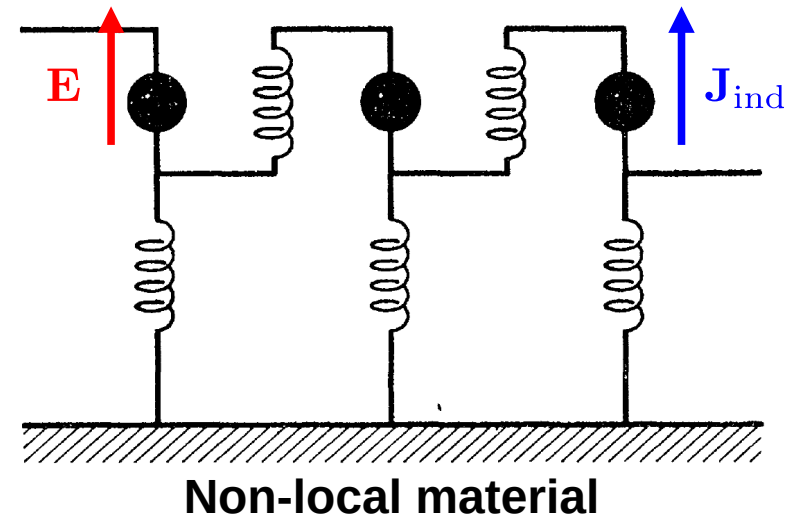
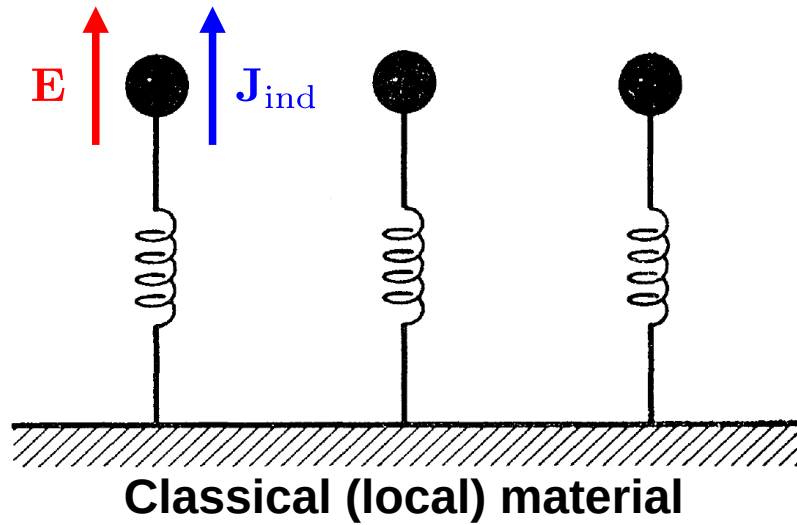
$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \frac{1}{j\omega} \mathbf{J}_{\text{ind}}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}$$

How to define the induced current in terms of effective material parameters?

$$\mathbf{J}_{\text{ind}} = A_{\text{eff}} \mathbf{E}_{\text{inc}}$$

Local vs Non-local Responses



$$\mathbf{J}_{\text{ind}} = j\omega\epsilon\mathbf{E}$$

Convolution in space

$$\mathbf{J}_{\text{ind}}(\mathbf{r}) = \int_{V'} \overline{\overline{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dV'$$

Electric response function

Taylor Expansion of the Induced Current Convolution

$$\mathbf{J}_{\text{ind}}(\mathbf{r}) = \int_{V'} \overline{\overline{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dV'$$

Taylor expansion of a vector function

$$f_i(x') = f_i(x) + \left. \frac{\partial f_i}{\partial x'_j} \right|_{x'=x} (x'_j - x_j) + \frac{1}{2!} \left. \frac{\partial^2 f_i}{\partial x'_j \partial x'_k} \right|_{x'=x} (x'_j - x_j) (x'_k - x_k) + \dots$$



$$\mathbf{E}(\mathbf{r}') = \mathbf{E}(\mathbf{r}) + (\nabla_i \mathbf{E})|_{\mathbf{r}=\mathbf{r}'} (r'_i - r_i) + \frac{1}{2} (\nabla_i \nabla_j \mathbf{E})|_{\mathbf{r}=\mathbf{r}'} (r'_i - r_i) (r'_j - r_j)$$



Not a function of \mathbf{r}' anymore!

Taylor Expansion of the Induced Current Convolution

$$\mathbf{J}_{\text{ind}}(\mathbf{r}) = \int_{V'} \overline{\overline{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dV' \leftarrow \mathbf{E}(\mathbf{r}') = \mathbf{E}(\mathbf{r}) + (\nabla_i \mathbf{E})|_{\mathbf{r}=\mathbf{r}'} (r'_i - r_i) + \frac{1}{2} (\nabla_i \nabla_j \mathbf{E})|_{\mathbf{r}=\mathbf{r}'} (r'_i - r_i) (r'_j - r_j)$$

$$\mathbf{J}_{\text{ind}}(\mathbf{r}) = \int_{V'} \overline{\overline{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') \cdot \left[\mathbf{E}(\mathbf{r}) + (\nabla_i \mathbf{E})|_{\mathbf{r}=\mathbf{r}'} (r'_i - r_i) + \frac{1}{2} (\nabla_i \nabla_j \mathbf{E})|_{\mathbf{r}=\mathbf{r}'} (r'_i - r_i) (r'_j - r_j) \right] dV'$$

Bringing un-primed terms out of the integral

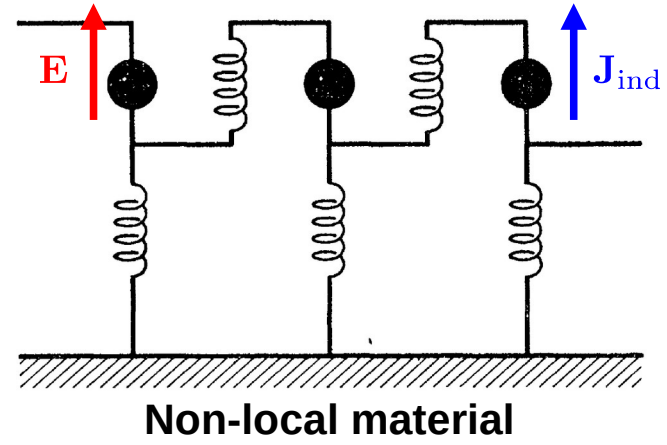
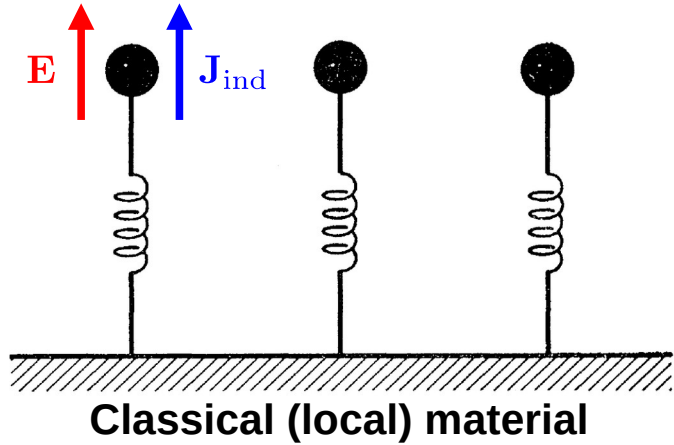
$$\mathbf{J}_{\text{ind}} = \mathbf{E} \int_{V'} \overline{\overline{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') dV' + (\nabla_i \mathbf{E}) \int_{V'} \overline{\overline{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') (r'_i - r_i) dV' + \frac{1}{2} (\nabla_i \nabla_j \mathbf{E}) \int_{V'} \overline{\overline{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') (r'_i - r_i) (r'_j - r_j) dV'$$

Corresponds to effective material parameter

Replacing integrals by effective material parameters

$$J_{\text{ind},i} = j\omega [a_{ij} E_j + a_{ijk} (\nabla_j E_k) + a_{ijkl} (\nabla_j \nabla_k E_l)]$$

Local vs Non-local Responses



$$\mathbf{J}_{\text{ind}} = j\omega\epsilon\mathbf{E}$$

Local response

$$\mathbf{J}_{\text{ind}}(\mathbf{r}) = \int_{V'} \bar{\bar{\mathbf{K}}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dV'$$

Taylor expansion

$$J_{\text{ind},i} = j\omega [a_{ij}E_j] + a_{ijk} (\nabla_j E_k) + a_{ijkl} (\nabla_j \nabla_k E_l)$$

Non-local responses

Non-local responses are associated with spatial derivatives

→ If $a_{ijk} \neq 0$, $a_{ijkl} \neq 0$ the structure is non-local

Isotropic Constitutive Relation

$$J_{\text{ind},i} = j\omega [a_{ij}E_j + a_{ijk}(\nabla_j E_k) + a_{ijkl}(\nabla_j \nabla_k E_l)]$$

The constitutive relation $\mathbf{D} = \epsilon_0 \mathbf{E} + \frac{1}{j\omega} \mathbf{J}_{\text{ind}}$ becomes

$$D_i = \epsilon_0 E_i + \frac{1}{j\omega} J_{\text{ind},i} = \underbrace{(\epsilon_0 \delta_{ij} + a_{ij})}_{\bar{\epsilon}} E_j + a_{ijk} \nabla_j E_k + a_{ijkl} \nabla_j \nabla_k E_l$$

$\bar{\epsilon}$ anisotropic permittivity, if isotropic $\longrightarrow \bar{\epsilon} = \epsilon \bar{\mathbf{I}}$

Let us consider that a_{ijk} is an isotropic tensor $\longrightarrow a_{ijk} = \alpha \epsilon_{ijk}$

$$a_{ijk} \nabla_k E_j = \alpha \epsilon_{ijk} \nabla_j E_k = \alpha \nabla \times \mathbf{E}$$

Let us also consider that a_{ijkl} is an isotropic tensor

$$a_{ijkl} \nabla_j \nabla_k E_l = [\beta_1 \delta_{ij} \delta_{kl} + \beta_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \gamma \epsilon_{ijm} \epsilon_{mkl}] \nabla_j \nabla_k E_l = \beta \nabla \nabla \cdot \mathbf{E} + \gamma \nabla \times \nabla \times \mathbf{E}$$

where $\beta_1 = \beta_2$

Isotropic Constitutive Relation

$$D_i = (\epsilon_0 \delta_{ij} + a_{ij}) E_j + a_{ijk} \nabla_j E_k + a_{ijkl} \nabla_j \nabla_k E_l$$



Assuming isotropic material parameters

$$\mathbf{D} = \epsilon \mathbf{E} + \alpha \nabla \times \mathbf{E} + \beta \nabla \nabla \cdot \mathbf{E} + \gamma \nabla \times \nabla \times \mathbf{E}$$

local

non-local

Let us transform this relation using the non-uniqueness of the constitutive relations

$$\mathbf{D}' = \mathbf{D} + \nabla \times \mathbf{Q}$$

Since \mathbf{Q} is an arbitrary vector, we may define

$$\mathbf{H}' = \mathbf{H} + j\omega \mathbf{Q}$$

$$\mathbf{Q}_1 = -\frac{\alpha}{2} \mathbf{E}$$

Isotropic Constitutive Relation – First Substitution

$$\mathbf{D} = \epsilon \mathbf{E} + \alpha \nabla \times \mathbf{E} + \beta \nabla \nabla \cdot \mathbf{E} + \gamma \nabla \times \nabla \times \mathbf{E}$$

$$\mathbf{Q}_1 = -\frac{\alpha}{2} \mathbf{E}$$

$$\begin{cases} \mathbf{D}' = \mathbf{D} + \nabla \times \mathbf{Q}_1 = \epsilon \mathbf{E} + \frac{\alpha}{2} \nabla \times \mathbf{E} + \beta \nabla \nabla \cdot \mathbf{E} + \gamma \nabla \times \nabla \times \mathbf{E} \\ \mathbf{H}' = \mathbf{H} + j\omega \mathbf{Q}_1 = \frac{1}{\mu_0} \mathbf{B} - j\omega \frac{\alpha}{2} \mathbf{E} \end{cases}$$

Using Maxwell equation

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\begin{cases} \mathbf{D}' = \epsilon \mathbf{E} - j\omega \frac{\alpha}{2} \mathbf{B} + \beta \nabla \nabla \cdot \mathbf{E} + \nabla \times (-j\omega \gamma \mathbf{B}) \\ \mathbf{H}' = \frac{1}{\mu_0} \mathbf{B} - j\omega \frac{\alpha}{2} \mathbf{E} \end{cases}$$

Isotropic Constitutive Relation – Second Substitution

We started from

$$\mathbf{D} = \epsilon \mathbf{E} + \alpha \nabla \times \mathbf{E} + \beta \nabla \nabla \cdot \mathbf{E} + \gamma \nabla \times \nabla \times \mathbf{E}$$

$$\begin{cases} \mathbf{D}' = \epsilon \mathbf{E} - j\omega \frac{\alpha}{2} \mathbf{B} + \beta \nabla \nabla \cdot \mathbf{E} + \nabla \times (-j\omega \gamma \mathbf{B}) \\ \mathbf{H}' = \frac{1}{\mu_0} \mathbf{B} - j\omega \frac{\alpha}{2} \mathbf{E} \end{cases}$$



Let us make another substitution

$$\mathbf{Q}_2 = j\omega \gamma \mathbf{B}$$

$$\begin{cases} \mathbf{D}'' = \mathbf{D}' + \nabla \times \mathbf{Q}_2 = \epsilon \mathbf{E} - j\omega \frac{\alpha}{2} \mathbf{B} + \beta \nabla \nabla \cdot \mathbf{E} \\ \mathbf{H}'' = \mathbf{H}' + j\omega \mathbf{Q}_2 = \frac{1}{\mu_0} \mathbf{B} - j\omega \frac{\alpha}{2} \mathbf{E} - \omega^2 \gamma \mathbf{B} \end{cases}$$

Bringing things together

$$\begin{cases} \mathbf{D} = \epsilon \mathbf{E} - j\nu \mathbf{B} + \beta \nabla \nabla \cdot \mathbf{E} \\ \mathbf{H} = \frac{1}{\mu} \mathbf{B} - j\nu \mathbf{E} \end{cases}$$

where

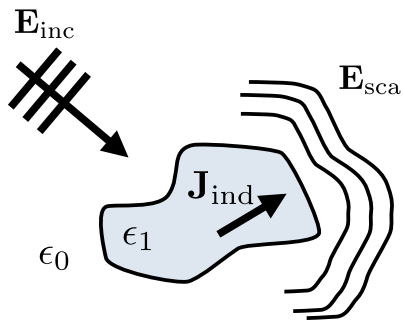
$$\begin{cases} \mu = \frac{\mu_0}{1 - \omega^2 \mu_0 \gamma} \\ \nu = \frac{\omega \alpha}{2} \end{cases}$$

Bi-(An)isotropic Constitutive Relations

We started from

$$\mathbf{D} = \epsilon \mathbf{E} + \alpha \nabla \times \mathbf{E} + \beta \nabla \nabla \cdot \mathbf{E} + \gamma \nabla \times \nabla \times \mathbf{E}$$

Remember that we started with a purely dielectric object



$$\begin{aligned} \mathbf{D} &= \epsilon_1 \mathbf{E} - j\nu \mathbf{B} + \beta \nabla \nabla \cdot \mathbf{E} \\ \mathbf{H} &= \frac{1}{\mu} \mathbf{B} - j\nu \mathbf{E} \end{aligned}$$

where

$$\begin{cases} \mu = \frac{\mu_0}{1 - \omega^2 \mu_0 \gamma} \\ \nu = \frac{\omega \alpha}{2} \end{cases}$$

Expressing these relations in terms of \mathbf{D} and \mathbf{B} and assuming that $\beta \approx 0$

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} + \xi \mathbf{H} \\ \mathbf{B} &= \zeta \mathbf{E} + \mu \mathbf{H} \end{aligned}$$

General bi-anisotropic relations

$$\begin{aligned} \mathbf{D} &= \bar{\bar{\epsilon}} \cdot \mathbf{E} + \bar{\bar{\xi}} \cdot \mathbf{H} \\ \mathbf{B} &= \bar{\bar{\zeta}} \cdot \mathbf{E} + \bar{\bar{\mu}} \cdot \mathbf{H} \end{aligned}$$

$$\epsilon = \epsilon' + \mu \nu^2 = \epsilon' + \frac{\omega^2 \alpha^2 \mu_0}{4(1 - \omega^2 \mu_0 \gamma)}$$

$$\xi = -j\nu\mu = -\frac{j\omega\alpha\mu_0}{2(1 - \omega^2\mu_0\gamma)}$$

$$\zeta = j\nu\mu = \frac{j\omega\alpha\mu_0}{2(1 - \omega^2\mu_0\gamma)}$$

In a static case: $\omega = 0 \longrightarrow$

$$\epsilon = \epsilon_1$$

$$\mu = \mu_0$$

$$\xi = \zeta = 0$$

A Word About Constitutive Relations

General form of the constitutive relations

Lorentz-covariant

$$\begin{bmatrix} c\mathbf{D} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathbf{P}}} & \overline{\overline{\mathbf{L}}} \\ \overline{\overline{\mathbf{M}}} & \overline{\overline{\mathbf{Q}}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E} \\ c\mathbf{B} \end{bmatrix}$$

Lorentz force

$$\mathbf{F} = q\mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$$c \begin{bmatrix} \overline{\overline{\mathbf{P}}}^{-1} & -\overline{\overline{\mathbf{P}}}^{-1} \cdot \overline{\overline{\mathbf{L}}} \\ \overline{\overline{\mathbf{M}}} \cdot \overline{\overline{\mathbf{P}}}^{-1} & \overline{\overline{\mathbf{Q}}} - \overline{\overline{\mathbf{M}}} \cdot \overline{\overline{\mathbf{P}}}^{-1} \cdot \overline{\overline{\mathbf{L}}} \end{bmatrix}$$

Mapping relations

$$\frac{1}{c} \begin{bmatrix} \overline{\overline{\mathbf{P}}} - \overline{\overline{\mathbf{L}}} \cdot \overline{\overline{\mathbf{Q}}}^{-1} \cdot \overline{\overline{\mathbf{M}}} & \overline{\overline{\mathbf{L}}} \cdot \overline{\overline{\mathbf{Q}}}^{-1} \\ -\overline{\overline{\mathbf{Q}}}^{-1} \cdot \overline{\overline{\mathbf{M}}} & \overline{\overline{\mathbf{Q}}}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\kappa}} & \overline{\overline{\chi}} \\ \overline{\overline{\gamma}} & \overline{\overline{\nu}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}$$

Sometimes useful to consider in anisotropic media because

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\mathbf{k} \perp \mathbf{D} \times \mathbf{B}$$

Better for modeling metamaterials

Bi-Anisotropic Medium vs Particle

Medium response

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}$$



$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

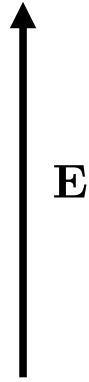
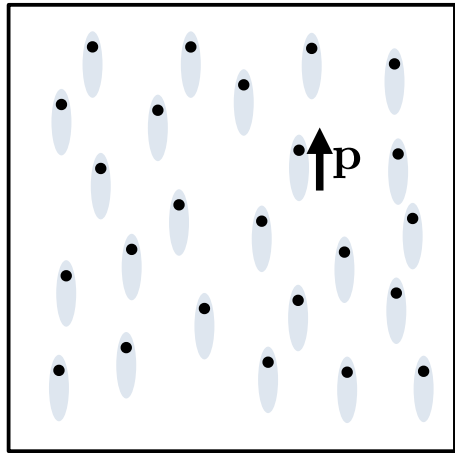


Medium polarization response

$$\begin{aligned} \mathbf{P} &= \epsilon_0 \bar{\bar{\chi}}_{ee} \cdot \mathbf{E} + \frac{1}{c_0} \bar{\bar{\chi}}_{em} \cdot \mathbf{H} \\ \mathbf{M} &= \bar{\bar{\chi}}_{mm} \cdot \mathbf{H} + \frac{1}{\eta_0} \bar{\bar{\chi}}_{me} \cdot \mathbf{E} \end{aligned}$$

Susceptibilities are unitless

Polarized medium



$$\mathbf{P} = N \langle \mathbf{p} \rangle$$

$$\bar{\bar{\epsilon}} = \bar{\bar{I}} + \bar{\bar{\chi}}_{ee}$$

$$\bar{\bar{\mu}} = \bar{\bar{I}} + \bar{\bar{\chi}}_{mm}$$

$$\bar{\bar{\xi}} = \frac{1}{c_0} \bar{\bar{\chi}}_{em}$$

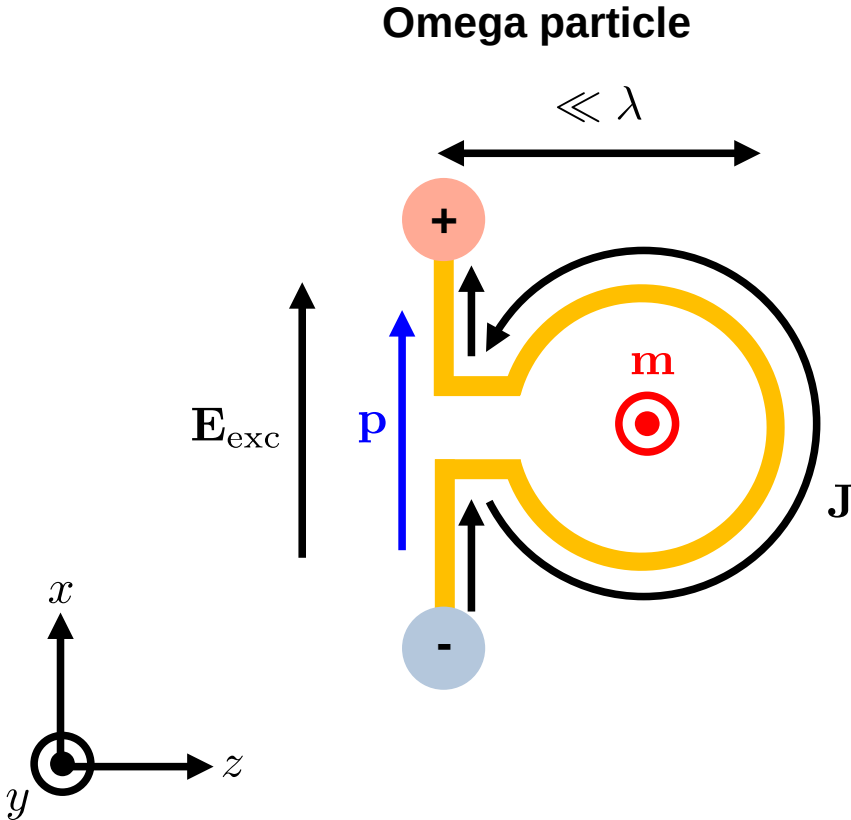
$$\bar{\bar{\zeta}} = \frac{1}{c_0} \bar{\bar{\chi}}_{me}$$

Particle polarization response

$$\begin{aligned} \mathbf{p} &= \epsilon_0 \bar{\bar{\alpha}}_{ee} \cdot \mathbf{E} + \frac{1}{c_0} \bar{\bar{\alpha}}_{em} \cdot \mathbf{H} \\ \mathbf{m} &= \bar{\bar{\alpha}}_{mm} \cdot \mathbf{H} + \frac{1}{\eta_0} \bar{\bar{\alpha}}_{me} \cdot \mathbf{E} \end{aligned}$$

Polarizabilities are unitless

Example of a Bi-Anisotropic Response



Particle polarization response

$$\mathbf{p} = \epsilon_0 \bar{\alpha}_{ee} \cdot \mathbf{E} + \frac{1}{c_0} \bar{\alpha}_{em} \cdot \mathbf{H}$$

$$\mathbf{m} = \bar{\alpha}_{mm} \cdot \mathbf{H} + \frac{1}{\eta_0} \bar{\alpha}_{me} \cdot \mathbf{E}$$

$$\mathbf{p} = \epsilon_0 \bar{\alpha}_{ee} \cdot \mathbf{E}_{\text{exc}}$$

$$\mathbf{m} = \frac{1}{\eta_0} \bar{\alpha}_{me} \cdot \mathbf{E}_{\text{exc}}$$

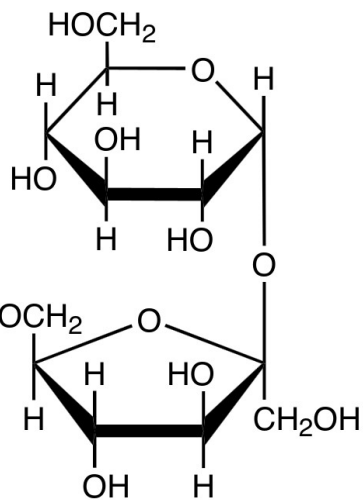
$$p_x = \epsilon_0 \alpha_{ee}^{xx} E_{\text{exc},x}$$

$$m_y = \frac{1}{\eta_0} \alpha_{me}^{yx} E_{\text{exc},x}$$

Chiral Response

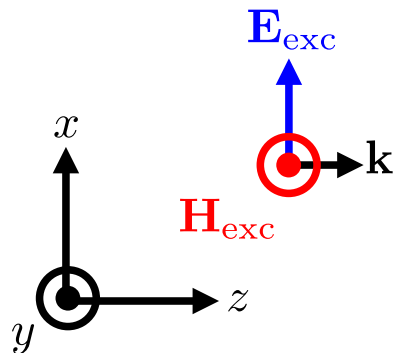
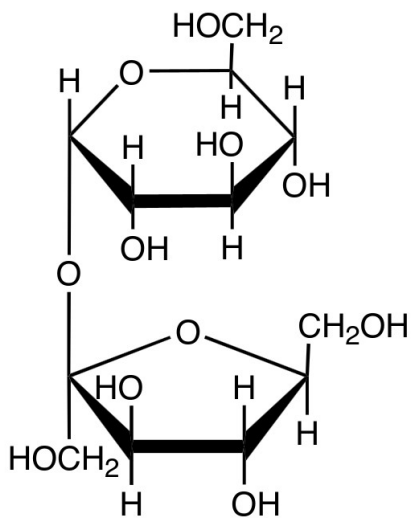
Chirality means lack of mirror symmetry

sucrose



Sucrose is a chiral molecule

Sucrose
mirror image



Solution of
water + sugar

The solution is bi-isotropic because sugar molecules are randomly oriented.

$$\mathbf{D} = \epsilon \mathbf{E} + \xi \mathbf{H}$$

$$\mathbf{B} = \zeta \mathbf{E} + \mu \mathbf{H}$$

$$\mathbf{D} = \epsilon E_{exc,x} \hat{x} + \underbrace{\xi H_{exc,y}}_{\text{cross polarization}} \hat{y}$$

We expect the generation
of cross polarization

Classification of Bi-Anisotropic Media

Split the coupling matrices

$$\overline{\overline{\chi}}_{em} = \overline{\overline{\nu}} - j\overline{\overline{\kappa}}$$

$$\overline{\overline{\chi}}_{me} = (\overline{\overline{\nu}} + j\overline{\overline{\kappa}})^T$$

ν and κ are decomposed as

$$\{\overline{\overline{\nu}}, \overline{\overline{\kappa}}\} = \overline{\overline{D}} + \overline{\overline{S}} + \overline{\overline{A}}$$

diagonal \nearrow symmetric \nearrow anti-sym. \nwarrow

$$\overline{\overline{S}} = \frac{1}{2} \left(\overline{\overline{A}} + \overline{\overline{A}}^T \right)$$

$$\overline{\overline{A}} = \frac{1}{2} \left(\overline{\overline{A}} - \overline{\overline{A}}^T \right)$$

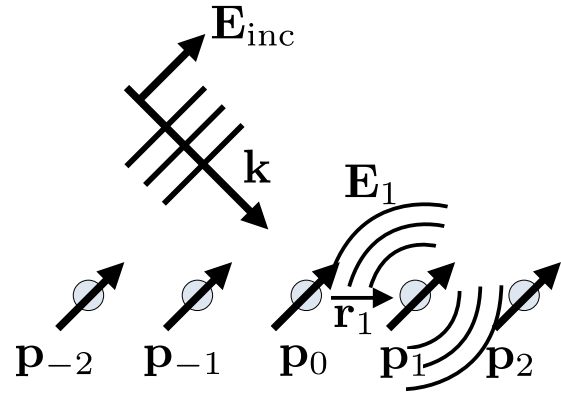
Type	Parameters	Medium
Reciprocal	$\overline{\overline{D}} = 0, \overline{\overline{S}} = 0, \overline{\overline{A}} \neq 0$	Omega
$\overline{\overline{\nu}} = 0$	$\overline{\overline{D}} = 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} = 0$	Pseudochiral
$\overline{\overline{\kappa}} \neq 0$	$\overline{\overline{D}} = 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} \neq 0$	Pseudochiral omega
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} = 0, \overline{\overline{A}} = 0$	Pasteur (or biisotropic)
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} = 0, \overline{\overline{A}} \neq 0$	Chiral omega
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} = 0$	Anisotropic chiral
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} \neq 0$	General reciprocal
Nonreciprocal	$\overline{\overline{D}} = 0, \overline{\overline{S}} = 0, \overline{\overline{A}} \neq 0$	Moving
$\overline{\overline{\nu}} \neq 0$	$\overline{\overline{D}} = 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} = 0$	Pseudo Tellegen
$\overline{\overline{\kappa}} = 0$	$\overline{\overline{D}} = 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} \neq 0$	Moving pseudo Tellegen
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} = 0, \overline{\overline{A}} = 0$	Tellegen
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} = 0, \overline{\overline{A}} \neq 0$	Moving Tellegen
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} = 0$	Anisotropic Tellegen
	$\overline{\overline{D}} \neq 0, \overline{\overline{S}} \neq 0, \overline{\overline{A}} \neq 0$	Nonreciprocal nonchiral

What Have We Learned So Far....

- Spatially dispersive medium are spatially nonlocal meaning that the response of a medium at a given spatial position depends on an excitation at another location
- Spatial dispersion (nonlocality) is modeled as a convolution in space of the material response in a similar way as temporal dispersion is modeled as a convolution in time.
- This spatial convolution may be simplified using a Taylor expansion leading to a dependence of the medium response on field derivatives
- In contrast to temporally dispersive effects, which must be causal and thus lead to loss as highlighted by the Kramers-Kronig relations, spatially dispersive effects do not suffer from this limitation
- Nonlocality comes from the presence of nonlocal material parameters. These parameters are associated with spatial derivatives of the electric field. As such it leads to material parameters that are \mathbf{k} dependent
- Note that the fact that the reflection/transmission coefficients are generally functions of the incidence angle (or \mathbf{k}) is never a sufficient condition to conclude that a material is nonlocal.
- Weak nonlocality leads to the appearance of artificial permittivity and permeability as well as electric-to-magnetic coupling terms (bi-anisotropy).
- Bi-isotropy is associated to chirality and thus rotation of polarization. Geometrically speaking, chirality appears when an object does not exhibit mirror symmetries (e.g., a spring, sugar molecules, DNA, etc).
- Bi-anisotropy is a general concept describing structures that exhibit electric and magnetic dipolar responses from both electric and magnetic field excitations
- You should be able to explain bi-anisotropy from the example of the Ω particle

Lattice Spatial Dispersion

2D Lattice of Electric Dipoles



This is a 2D lattice along x and y of period P

Field acting on the $n=0$ particle

$$\mathbf{E}_{\text{act}} = \mathbf{E}_{\text{inc}} + \mathbf{E}_1 + \mathbf{E}_2 + \dots = \mathbf{E}_{\text{inc}} + \sum_{n \neq 0} \mathbf{E}_n$$

Field produced by each dipole at the position \mathbf{r}_0

Using the Green function formalism, we have

$$\mathbf{E}_1 = -j\omega\mu \int \overline{\overline{\mathbf{G}}}(\mathbf{r}_1) \cdot \mathbf{J}(\mathbf{r}_1) dV' \quad \begin{cases} \mathbf{P} = \mathbf{p}_1 \delta(\mathbf{r}_1) \\ \mathbf{J} = j\omega\mathbf{P} = j\omega\mathbf{p}_1 \delta(\mathbf{r}_1) \end{cases}$$

$$\mathbf{E}_1 = \omega^2 \mu \overline{\overline{\mathbf{G}}}(\mathbf{r}_1) \cdot \mathbf{p}_1$$

The acting field is thus

$$\mathbf{E}_{\text{act}} = \mathbf{E}_{\text{inc}} + \omega^2 \mu \sum_{n \neq 0} \overline{\overline{\mathbf{G}}}(\mathbf{r}_n) \cdot \mathbf{p}_n$$

Dipole phase dependent on incident wave propagation direction
 $\mathbf{p}_n = \mathbf{p} e^{-j\mathbf{k}_{\parallel} \cdot \mathbf{r}_n}$

Lattice Green function

$$\overline{\overline{\mathbf{G}}}_L(\mathbf{k}_{\parallel}) = \omega^2 \mu \sum_{n \neq 0} \overline{\overline{\mathbf{G}}}(\mathbf{r}_n) e^{-j\mathbf{k}_{\parallel} \cdot \mathbf{r}_n}$$

$$\mathbf{E}_{\text{act}} = \mathbf{E}_{\text{inc}} + \overline{\overline{\mathbf{G}}}_L(\mathbf{k}_{\parallel}) \cdot \mathbf{p}$$

Effective Nonlocal Polarizability

Let's consider that the dipole moment is

$$\mathbf{p} = \bar{\alpha} \cdot \mathbf{E}_{\text{act}}$$

$$\mathbf{E}_{\text{act}} = \mathbf{E}_{\text{inc}} + \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \cdot \mathbf{p}$$

$$\mathbf{p} = \bar{\alpha} \cdot \left[\mathbf{E}_{\text{inc}} + \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \cdot \mathbf{p} \right]$$

$$\left[\bar{\alpha}^{-1} - \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \right] \cdot \mathbf{p} = \mathbf{E}_{\text{inc}}$$

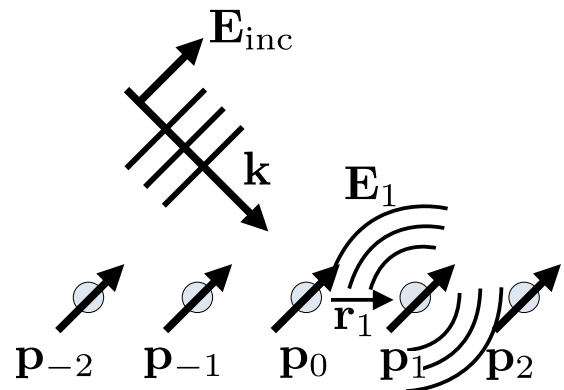
$$\mathbf{p} = \underbrace{\left[\bar{\alpha}^{-1} - \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \right]^{-1}}_{\bar{\alpha}_{\text{eff}}(\mathbf{k}_{\parallel})} \cdot \mathbf{E}_{\text{inc}}$$

$$\bar{\alpha}_{\text{eff}}(\mathbf{k}_{\parallel})$$

Lattice nonlocality

Effective polarizability

$$\bar{\alpha}_{\text{eff}}(\mathbf{k}_{\parallel}) = \left[\bar{\alpha}^{-1} - \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \right]^{-1}$$



Lattice Green function

$$\bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) = \omega^2 \mu \sum_{n \neq 0} \bar{\bar{G}}(\mathbf{r}_n) e^{-j\mathbf{k}_{\parallel} \cdot \mathbf{r}_n}$$

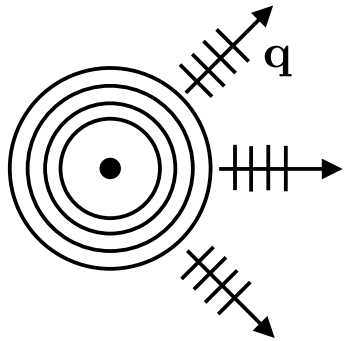
Lattice Green Function in Reciprocal Space

Lattice Green function

$$\bar{\bar{G}}_L(\mathbf{k}_{\parallel}) = \omega^2 \mu \sum_{n \neq 0} \bar{\bar{G}}(\mathbf{r}_n) e^{-j\mathbf{k}_{\parallel} \cdot \mathbf{r}_n}$$

Dyadic Green function

$$\bar{\bar{G}}(\mathbf{r}) = \left(\bar{\bar{I}} + \frac{\nabla \nabla}{k^2} \right) \frac{e^{-jkr}}{4\pi r}$$



Weyl identity (plane wave expansion of the Green function)

$$\frac{e^{-jkr}}{4\pi r} = \frac{1}{8\pi^2 j} \iint_{-\infty}^{\infty} e^{-j\mathbf{q}_{\parallel} \cdot \mathbf{r}} \frac{e^{-jq_z |z|}}{q_z} d^2 \mathbf{q}_{\parallel} \quad \text{where } q_z = \sqrt{k^2 - |\mathbf{q}_{\parallel}|^2}$$

The dyadic Green function in reciprocal space (at $z = 0$) is

$$\bar{\bar{G}}(\mathbf{r}) = \frac{1}{8\pi^2 j} \iint_{-\infty}^{\infty} e^{-j\mathbf{q}_{\parallel} \cdot \mathbf{r}} \frac{1}{q_z} \left(\bar{\bar{I}} - \frac{\mathbf{q}\mathbf{q}}{k^2} \right) d^2 \mathbf{q}_{\parallel}$$

We use \mathbf{q} instead of \mathbf{k} as the wavevector to avoid confusion with the wavevector of the incident field.

Lattice Green Function in Reciprocal Space

Lattice Green function

$$\overline{\overline{G}}_L(\mathbf{k}_{\parallel}) = \omega^2 \mu \sum_{n \neq 0} \overline{\overline{G}}(\mathbf{r}_n) e^{-j\mathbf{k}_{\parallel} \cdot \mathbf{r}_n}$$

Dyadic Green function in reciprocal space

$$\overline{\overline{G}}(\mathbf{r}) = \frac{1}{8\pi^2 j} \iint_{-\infty}^{\infty} e^{-j\mathbf{q}_{\parallel} \cdot \mathbf{r}} \frac{1}{q_z} \left(\overline{\overline{I}} - \frac{\mathbf{q}\mathbf{q}}{k^2} \right) d^2\mathbf{q}_{\parallel}$$



$$\overline{\overline{G}}_L(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{8\pi^2 j} \iint_{-\infty}^{\infty} \frac{1}{q_z} \left(\overline{\overline{I}} - \frac{\mathbf{q}\mathbf{q}}{k^2} \right) \underbrace{\sum_{n \neq 0} e^{-j(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel}) \cdot \mathbf{r}_n}}_{d^2\mathbf{q}_{\parallel}}$$

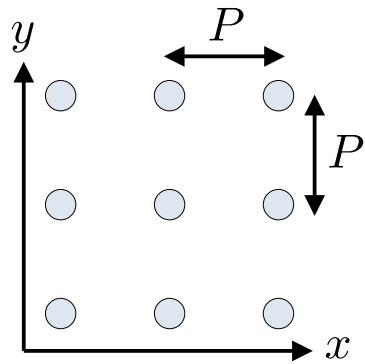
We next use Poisson summation to simplify this expression

where $q_z = \sqrt{k^2 - |\mathbf{q}_{\parallel}|^2}$

Poisson Summation

How to compute ?

$$\sum_{n \neq 0} e^{-j(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel}) \cdot \mathbf{r}_n}$$



Surface area

$$A = P^2$$

1D Poisson summation

$$\sum_{n=-\infty}^{\infty} f(na) = \frac{1}{a} \sum_{m=-\infty}^{\infty} F\left(\frac{2\pi m}{a}\right) \leftarrow \text{Fourier transform of } f$$

2D version for a lattice

$$\sum_n e^{-j\mathbf{Q} \cdot \mathbf{r}_n} = \frac{4\pi^2}{A} \sum_{\Lambda} \delta(\mathbf{Q} - \Lambda)$$

Lattice vector

$$\Lambda = \left(\frac{2\pi m}{P}, \frac{2\pi n}{P} \right)$$

Removing the n=0 element

$$\sum_{n \neq 0} e^{-j\mathbf{Q} \cdot \mathbf{r}_n} = \frac{4\pi^2}{A} \sum_{\Lambda} \delta(\mathbf{Q} - \Lambda) - 1$$

This applies that we can express

$$\sum_{n \neq 0} e^{-j(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel}) \cdot \mathbf{r}_n} = \frac{4\pi^2}{A} \sum_{\Lambda} \delta(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel} - \Lambda) - 1$$

Lattice Green Function in Reciprocal Space

$$\bar{\bar{G}}_L(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{8\pi^2 j} \iint_{-\infty}^{\infty} \frac{1}{q_z} \left(\bar{\bar{I}} - \frac{\mathbf{q}\mathbf{q}}{k^2} \right) \sum_{n \neq 0} e^{-j(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel}) \cdot \mathbf{r}_n} d^2 \mathbf{q}_{\parallel} \quad \leftarrow \sum_{n \neq 0} e^{-j(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel}) \cdot \mathbf{r}_n} = \frac{4\pi^2}{A} \sum_{\Lambda} \delta(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel} - \Lambda) - 1$$

$$\bar{\bar{G}}_L(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{8\pi^2 j} \iint_{-\infty}^{\infty} \frac{1}{q_z} \left(\bar{\bar{I}} - \frac{\mathbf{q}\mathbf{q}}{k^2} \right) \left[\frac{4\pi^2}{A} \sum_{\Lambda} \delta(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel} - \Lambda) - 1 \right] d^2 \mathbf{q}_{\parallel}$$

$$\bar{\bar{G}}_L(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{2jA} \iint_{-\infty}^{\infty} \frac{1}{q_z} \left(\bar{\bar{I}} - \frac{\mathbf{q}\mathbf{q}}{k^2} \right) \sum_{\Lambda} \delta(\mathbf{q}_{\parallel} + \mathbf{k}_{\parallel} - \Lambda) d^2 \mathbf{q}_{\parallel} - \frac{A\omega^2 \mu}{32\pi^4 j} \iint_{-\infty}^{\infty} \frac{1}{q_z} \left(\bar{\bar{I}} - \frac{\mathbf{q}\mathbf{q}}{k^2} \right) d^2 \mathbf{q}_{\parallel}$$

Integrating the Dirac delta over \mathbf{q} results in nonzero values only when $\mathbf{q}_{\parallel} = \Lambda - \mathbf{k}_{\parallel}$

$$\bar{\bar{G}}_L(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{2jA} \sum_{\Lambda} \frac{1}{q_z} \left(\bar{\bar{I}} - \frac{\mathbf{Q}\mathbf{Q}}{k^2} \right) - \bar{\bar{G}}^{\text{self}}$$

where $\mathbf{Q} = (\mathbf{q}_{\parallel}, q_z)$ with $q_z = \sqrt{k^2 - |\mathbf{q}_{\parallel}|^2}$ and $\mathbf{q}_{\parallel} = \Lambda - \mathbf{k}_{\parallel}$ and $\Lambda = \left(\frac{2\pi m}{P}, \frac{2\pi n}{P} \right)$

Simplifications for Isotropic Case

Effective polarizability

$$\bar{\alpha}_{\text{eff}}(\mathbf{k}_{\parallel}) = \left[\bar{\alpha}^{-1} - \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \right]^{-1}$$

For an isotropic case

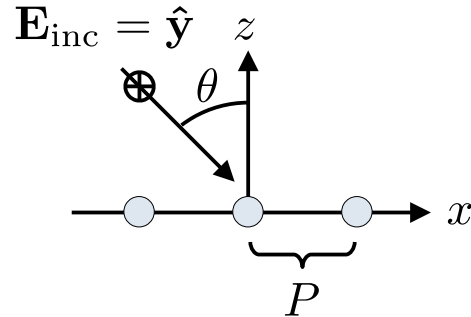
$$\alpha_{\text{eff}}(\mathbf{k}_{\parallel}) = \frac{1}{\alpha^{-1} - S_{\text{L}}(\mathbf{k}_{\parallel})}$$

Polarization vector

↓

$$\text{where } S_{\text{L}}(\mathbf{k}_{\parallel}) = \hat{\mathbf{p}} \cdot \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \cdot \hat{\mathbf{p}}$$

Let's consider a TE-polarized incident wave in the xz-plane



The polarization vector is $\hat{\mathbf{p}} = \hat{\mathbf{y}}$

The lattice sum simplifies to

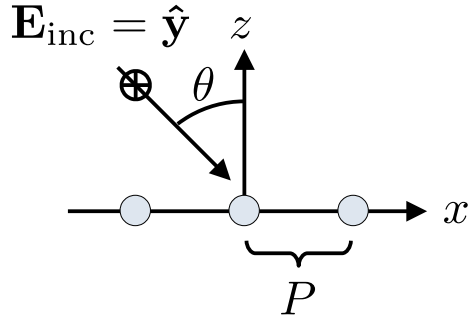
$$S_{\text{L}}(\mathbf{k}_{\parallel}) = \hat{\mathbf{y}} \cdot \bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) \cdot \hat{\mathbf{y}} \quad \text{where } \mathbf{k}_{\parallel} = (k_x, 0)$$

Lattice Green function in reciprocal space

$$\bar{\bar{G}}_{\text{L}}(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{2jA} \sum_{\Lambda} \frac{1}{q_z} \left(\bar{\bar{I}} - \frac{\mathbf{Q}\mathbf{Q}}{k^2} \right) - \bar{\bar{G}}^{\text{self}}$$

Simplifications for Isotropic Case

Lattice Green function in reciprocal space



$$S_L(\mathbf{k}_{\parallel}) = \hat{\mathbf{y}} \cdot \overline{\overline{G}}_L(\mathbf{k}_{\parallel}) \cdot \hat{\mathbf{y}} \quad \leftarrow \quad \overline{\overline{G}}_L(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{2jA} \sum_{\Lambda} \frac{1}{q_z} \left(\overline{\overline{1}} - \frac{\mathbf{Q}\mathbf{Q}}{k^2} \right) - \overline{\overline{G}}^{\text{self}}$$

$$S_L(\mathbf{k}_{\parallel}) = \frac{\omega^2 \mu}{2jA} \sum_{\Lambda} \frac{1}{q_z} \left(1 - \frac{q_y^2}{k^2} \right) - G_{yy}^{\text{self}}$$

Since $k_y = 0$

$$q_y = \frac{2\pi n}{P}$$

$$S_L(\mathbf{k}_{\parallel}) = \frac{\pi}{j\lambda\epsilon_0 P^2} \sum_{m,n} \frac{\left[1 - \left(\frac{n\lambda}{P} \right)^2 \right]}{\sqrt{1 - \left(\frac{m\lambda}{P} - \sin\theta \right)^2 - \left(\frac{n\lambda}{P} \right)^2}} - G_{yy}^{\text{self}}$$

Real part depends on the particle

$$\text{Im} \{ G_{yy}^{\text{self}} \} = -\frac{k^3}{6\pi\epsilon}$$

From the lattice sum, we can now calculate the effective polarizability

$$\alpha_{\text{eff}}(\mathbf{k}_{\parallel}) = \frac{1}{\alpha^{-1} - S_L(\mathbf{k}_{\parallel})}$$

For more details, see section 5.2.2 in Tretyakov, "Analytical modeling in applied electromagnetics", 2003

where $\mathbf{Q} = (\mathbf{q}_{\parallel}, q_z)$ with $q_z = \sqrt{k^2 - |\mathbf{q}_{\parallel}|^2}$ and $\mathbf{q}_{\parallel} = \Lambda - \mathbf{k}_{\parallel}$ and $\Lambda = \left(\frac{2\pi m}{P}, \frac{2\pi n}{P} \right)$

What Have We Learned So Far....

- Nonlocal responses may arise in isolated objects but an additional form of nonlocality appears in arrays of particles due to the coupling between them
- Such coupling effects may be investigated using a Green function (lattice sum) approach allowing us to obtain the field acting on a given particle of the array as the sum of the incident field plus an infinite sum of the contribution from all other particles
- The lattice sum allows us to re-express the induced dipole moments in the array directly in terms of the incident field instead of the local (acting) field
- This leads to the definition of an effective polarizability expressed in terms of the polarizability of the isolated particle and the lattice sum
- This effective polarizability depends on the angle of incidence (\mathbf{k} dependence), which corresponds to a lattice nonlocal effect
- It follows that even if the particle itself exhibits only local responses (e.g., no bi-anisotropy, only local electric polarizability), placing this particle in an array will generally lead to nonlocal responses due to the coupling between them
- The lattice sum may be expressed in reciprocal space (Fourier space) in terms of a sum of propagating and evanescent waves instead of a sum along the x and y direction (direct space)