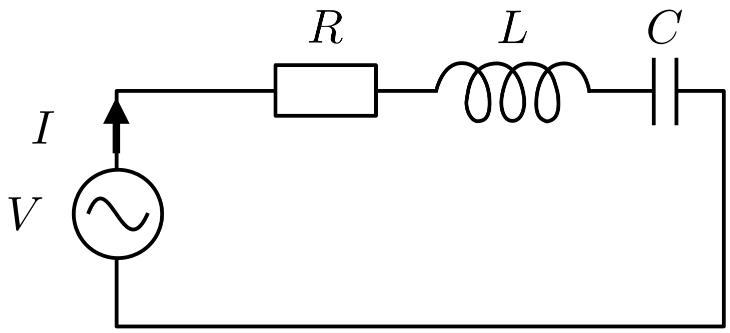
Lecture 5

Antenna Theory

RLC Resonant Circuits

The RLC Circuit



Equivalent impedance of the system

$$Z = R + j\omega L + \frac{1}{j\omega C}$$

Complex power delivered by the source

$$P = \frac{1}{2} V I^* = \frac{1}{2} Z |I|^2 = \frac{1}{2} |I|^2 \left[R + j \left(\omega L - \frac{1}{\omega C} \right) \right]$$

Ohm law

$$V = ZI$$

Capacitor voltage

$$V_{\mathrm{C}} = \frac{I}{j\omega C}$$

Averaged electric energy

$$W_{\mathrm{e}} = \frac{1}{4} C |V_{\mathrm{C}}|^2 = \frac{1}{4\omega^2 C} |I|^2$$

Averaged magnetic energy

$$W_{\mathrm{m}} = \frac{1}{4} L |I|^2$$

$$P = P_{\mathrm{loss}} + j 2\omega \left(W_{\mathrm{m}} - W_{\mathrm{e}} \right)$$

Reactive power corresponds to energy being stored

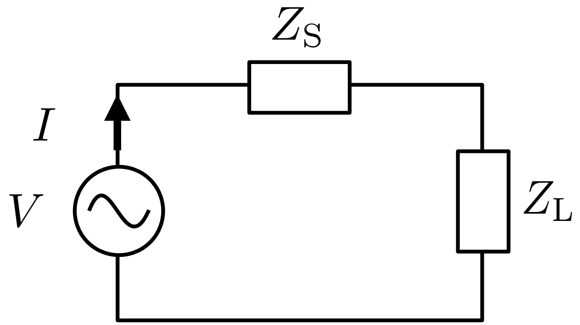
Real power (loss): $P_{\mathrm{loss}} = \frac{1}{2} |I|^2 R$

At resonance

$$W_{\mathrm{e}} = W_{\mathrm{m}}$$
$$Z \in R$$

$$\omega = \frac{1}{\sqrt{LC}}$$

Conjugate Matching



How to maximize the power delivered to the load *R*L ?

$$P_L = \frac{1}{2} R_L |I|^2 = \frac{1}{2} R_L \left(\frac{|V|}{|Z_S + Z_L|} \right)^2 = \frac{1}{2} \frac{|V|^2 R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}$$

P_L is maximum when the denumerator is minimum

Since the reactance can be positive or negative, we directly have $X_S = -X_L$

$$P_L = \frac{1}{2} \frac{|V|^2}{R_S^2 R_L + 2R_S + R_L}$$

The minima is found when the derivative of the denumerator by R_L is zero

$$\frac{d}{dR_L} \left(R_S^2 R_L + 2R_S + R_L \right) = 1 - \frac{R_S^2}{R_L^2} = 0 \quad \longrightarrow \quad R_S = \pm R_L$$

Since the resistance is only positive, we have $R_S = R_L$

In general, we have $Z_S = Z_L^*$

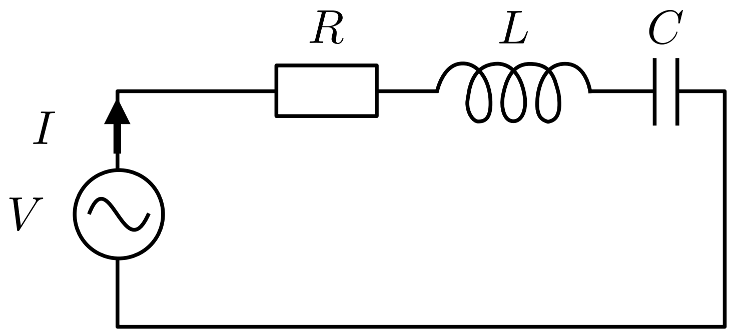
The Q Factor

How to Define the Q Factor

The most general definition of the Q factor is

$$Q = 2\pi \times \frac{\text{energy stored}}{\text{energy dissipated per cycle}} = \omega \times \frac{\text{energy stored}}{\text{power loss}}$$

Power loss could be a combination of Ohmic (heat) loss and scattering loss



$$P = P_{\text{loss}} + j2\omega\left(W_{\text{m}} - W_{\text{e}}\right)$$

The Q factor of an RLC circuit

$$Q = \omega \frac{W_{\text{e}} + W_{\text{m}}}{P_{\text{loss}}}$$

Q Factor in Terms of Lifetime

Consider a system containing the energy *U*. The corresponding power loss is $-\partial U/\partial t$

$$Q = \omega \times \frac{\text{energy stored}}{\text{power loss}} = \omega \frac{U}{-\frac{\partial U}{\partial t}} \longrightarrow \frac{\partial}{\partial t}U + \frac{\omega}{Q}U = 0$$

The solution to this differential equation is

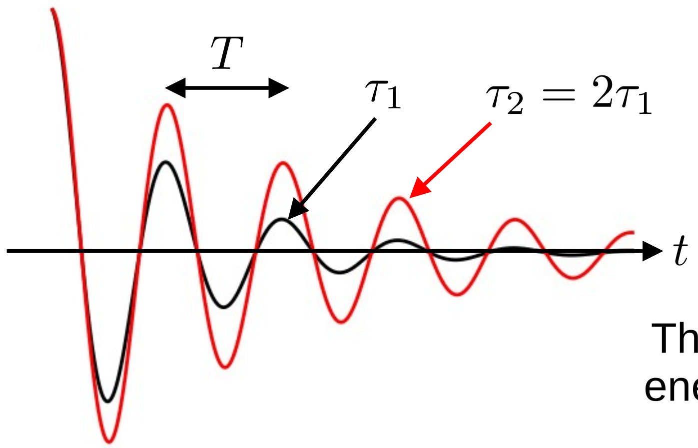
$$U = U_0 e^{-\omega t/Q} = U_0 e^{-t/\tau}$$

where $\tau = \frac{Q}{\omega} = \frac{T}{2\pi}Q$

$$Q = \frac{2\pi}{T}\tau = \omega\tau$$

τ is the lifetime and
T is the period

Energy decay of a damped oscillator



The higher the lifetime, the slower the energy decays, the higher the Q factor

Q Factor in Terms of Bandwidth

A function with a decaying amplitude cannot be composed of a single frequency. It must have a bandwidth

Consider a damped time-harmonic field $\Psi(t) = Ae^{-\frac{t}{2\tau}}e^{j\omega_0 t}$

we assume the field starts to decay at *t = 0*

Its Fourier transform is $\tilde{\Psi}(\omega) = \int_0^\infty \Psi(t)e^{-j\omega t}\, dt = A\frac{j2\tau}{j - 2\tau(\omega - \omega_0)}$

The intensity is $|\tilde{\Psi}(\omega)|^2 = |A|^2\frac{4\tau^2}{1 + 4\tau^2(\omega - \omega_0)^2}$ → at the resonance → $|\tilde{\Psi}(\omega_0)|^2 = |A|^2 4\tau^2 = |A|^2 4\frac{Q^2}{\omega_0^2}$

The frequencies at the half-maximum intensity are $\omega = \omega_0 \pm \frac{1}{2\tau} = \omega_0 \pm \frac{\omega_0}{2Q}$ that's twice the bandwidth

$$Q = \frac{2\pi}{T}\tau = \omega\tau$$

It follows that the bandwidth is $\Delta\omega = \frac{1}{\tau} = \frac{\omega_0}{Q}$ →

$$Q = \frac{\omega_0}{\Delta\omega}$$

The Q Factor

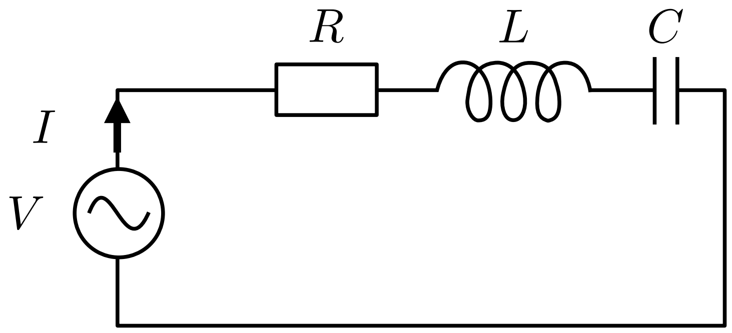
Three useful definitions of the Q factor

$$Q = \omega \times \frac{\text{energy stored}}{\text{power loss}}$$

$$Q = \frac{2\pi}{T}\tau = \omega\tau$$

$$Q = \frac{\omega}{\Delta\omega}$$

$$\Delta\omega = \frac{\text{power loss}}{\text{energy stored}}$$



The Q factor of an RLC circuit

$$Q = \omega\frac{W_\mathrm{e} + W_\mathrm{m}}{P_\mathrm{loss}} \rightarrow \Delta\omega = 2\frac{P_\mathrm{loss}}{W}$$

$$P_\mathrm{in} = P_\mathrm{loss} + j2\omega\left(W_\mathrm{m} - W_\mathrm{e}\right)$$

at resonance $W = W_\mathrm{e} = W_\mathrm{m}$

The Q Factor of a Series RLC Circuit

The Q factor is

$$Q = \omega \frac{W_\mathrm{e} + W_\mathrm{m}}{P_\mathrm{loss}}$$

For a series RLC circuit

$$\begin{cases} W_\mathrm{e} = \dfrac{1}{4\omega^2 C}|I|^2 \\ W_\mathrm{m} = \dfrac{1}{4}L|I|^2 \\ P_\mathrm{loss} = \dfrac{1}{2}|I|^2 R \end{cases}$$

$$Q = \frac{1 + LC\omega^2}{2\omega RC}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{\omega RC} = \frac{\omega L}{R}$$

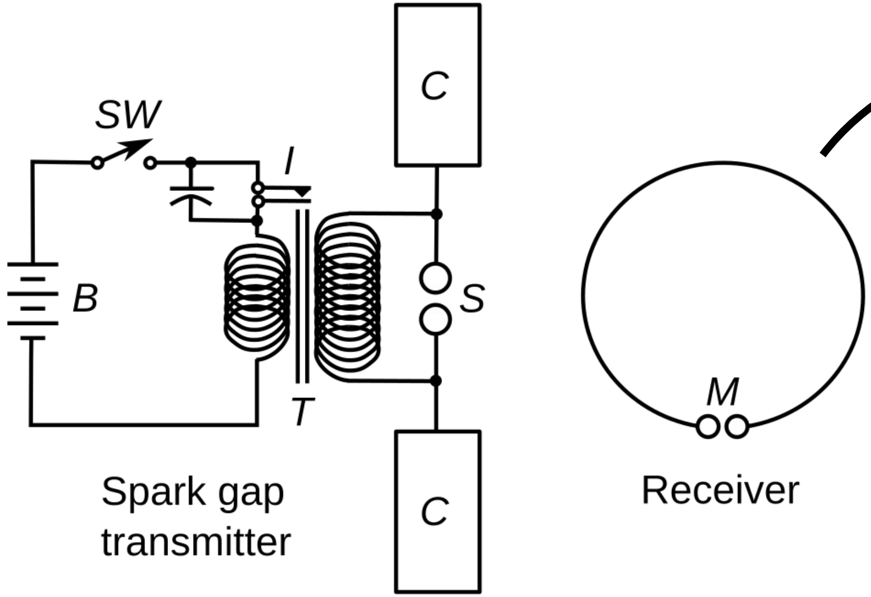
What Have We Learned So Far….

- The resonance frequency of an LC circuit is proportional to 1/sqrt(LC)
- The resonance wavelength is therefore proportional to sqrt(LC)
- The total power in an RLC circuit is a combination of loss (P_{loss}) and stored energy in the form of imaginary (reactive) electric and magnetic power.
- At resonance, the time-average amount of stored electric and magnetic energy is equal
- The Q factor is proportional to stored energy by dissipated power (loss + scattering)
- The Q factor is proportional to the lifetime of the system
- The Q factor is inversely proportional to the bandwidth
- The Q factor represents a very powerful tool to understand and predict the behavior of an electromagnetic system

Antenna Theory

What is an Antenna ?

1886 Hertz: first transmission of EM waves



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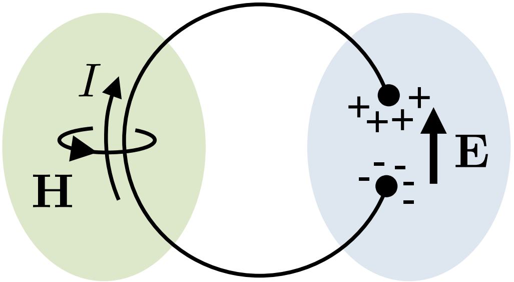
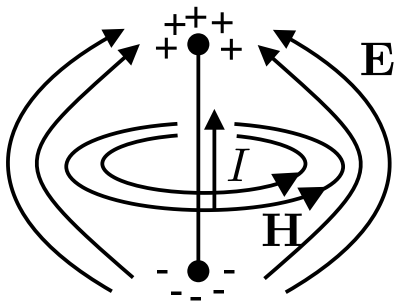


Stores magnetic energy => **inductor**

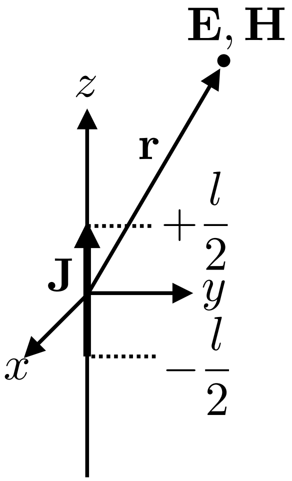
Stores electric energy => **capacitor**

can be straighten out



An antenna is an RLC circuit!

The Ideal Dipole Antenna



Consider an antenna of length *l* « λ with <u>uniform</u> current *I* → $\mathbf{J}(\mathbf{r}') = \hat{\mathbf{z}} I \delta(x')\delta(y')$

The potential vector is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_{V_0} \mathbf{J}(\mathbf{r}')\frac{e^{-jkr}}{r}\, dV' = \hat{\mathbf{z}}\frac{I\mu}{4\pi}\int_{-l/2}^{l/2}\frac{e^{-jkr}}{r}\, dz' \underset{l \ll r}{\approx} \hat{\mathbf{z}}\frac{lI\mu e^{-jkr}}{4\pi r}$$

The magnetic field is

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} = \frac{1}{\mu}\nabla\times(A_z\hat{\mathbf{z}}) = \frac{1}{\mu}(\nabla A_z)\times\hat{\mathbf{z}} = \frac{lI}{4\pi}\nabla\left(\frac{e^{-jkr}}{r}\right)\times\hat{\mathbf{z}}$$

In spherical coordinates, the gradient becomes

$$\mathbf{H} = \frac{lI}{4\pi}\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r}\right)\hat{\mathbf{r}}\times\hat{\mathbf{z}} = \frac{lI}{4\pi}\left[jk + \frac{1}{r}\right]\frac{e^{-jkr}}{r}\sin\theta\hat{\boldsymbol{\phi}}$$

The corresponding electric field is

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla\times\mathbf{H} = \frac{lI}{4\pi}\left[j\omega\mu\left(1+\frac{1}{jkr}-\frac{1}{k^2r^2}\right)\sin\theta\hat{\boldsymbol{\theta}} + 2\eta\left(\frac{1}{r}-\frac{j}{kr^2}\right)\cos\theta\hat{\mathbf{r}}\right]\frac{e^{-jkr}}{r}$$

Note that

$$\hat{\mathbf{r}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right) = -\sin\theta\hat{\boldsymbol{\phi}}$$

The Fields and Power of an Ideal Dipole Antenna

$$\begin{cases} \mathbf{H} = \dfrac{lI}{4\pi}\left[jk + \dfrac{1}{r}\right]\dfrac{e^{-jkr}}{r}\sin\theta\hat{\boldsymbol{\phi}} \\ \mathbf{E} = \dfrac{lI}{4\pi}\left[j\omega\mu\left(1 + \dfrac{1}{jkr} - \dfrac{1}{k^2r^2}\right)\sin\theta\hat{\boldsymbol{\theta}} + 2\eta\left(\dfrac{1}{r} - \dfrac{j}{kr^2}\right)\cos\theta\hat{\mathbf{r}}\right]\dfrac{e^{-jkr}}{r} \end{cases}$$

These corresponds to the fields of an electric dipole

In the far field, we have

$$\mathbf{E}_{\text{ff}} = \frac{lI}{4\pi}j\omega\mu\frac{e^{-jkr}}{r}\sin\theta\hat{\boldsymbol{\theta}}$$

$$\mathbf{H}_{\text{ff}} = \frac{lI}{4\pi}jk\frac{e^{-jkr}}{r}\sin\theta\hat{\boldsymbol{\phi}}$$

Fields are in phase

In the near field, we have

$$\mathbf{E}_{\text{nf}} = -j\eta\frac{lI}{4\pi}\frac{e^{-jkr}}{kr^3}\left(\sin\theta\hat{\boldsymbol{\theta}} + 2\cos\theta\hat{\mathbf{r}}\right)$$

$$\mathbf{H}_{\text{nf}} = \frac{lI}{4\pi}\frac{e^{-jkr}}{r^2}\sin\theta\hat{\boldsymbol{\phi}}$$

Fields are out of phase

Complex Poynting vector

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{E}\times\mathbf{H}^*\right)$$

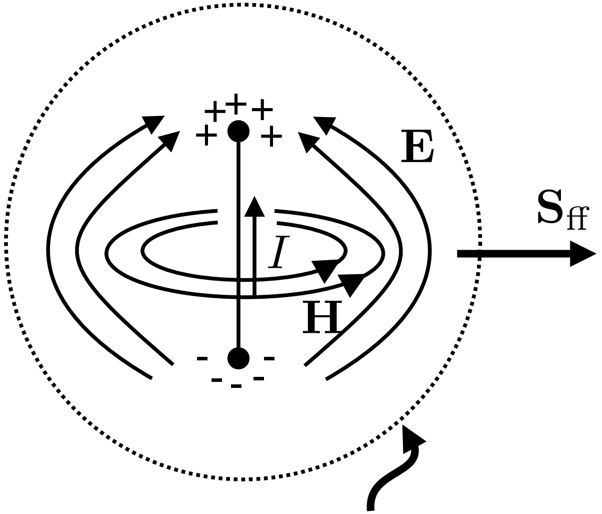
$$\mathbf{S}_{\text{ff}} = \frac{\omega\mu k}{2}\left(\frac{Il}{4\pi}\right)^2\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

Real power

$$\mathbf{S}_{\text{nf}} = -\frac{j\eta}{2k}\left(\frac{Il}{4\pi}\right)^2\frac{1}{r^5}\left(\sin^2\theta\hat{\mathbf{r}} - \sin 2\theta\hat{\boldsymbol{\theta}}\right)$$

Reactive (imaginary) power

Far-Field Condition



Boundary between reactive/near-field region and far-field region

Real power

$$\mathbf{S}_{\text{ff}} = \frac{\omega\mu k}{2}\left(\frac{Il}{4\pi}\right)^2 \frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

Reactive (imaginary) power

$$\mathbf{S}_{\text{nf}} = -\frac{j\eta}{2k}\left(\frac{Il}{4\pi}\right)^2 \frac{1}{r^5}\left(\sin^2\theta\hat{\mathbf{r}} - \sin 2\theta\hat{\boldsymbol{\theta}}\right)$$

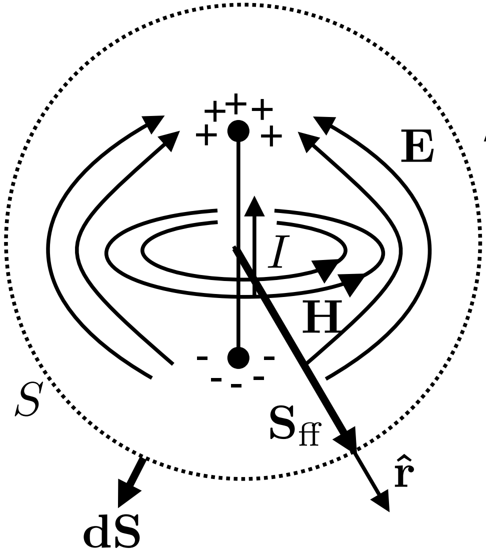
Ratio of power going through a sphere of radius *r*

$$\left|\frac{\mathbf{S}_{\text{ff}}\cdot\hat{\mathbf{r}}}{\mathbf{S}_{\text{nf}}\cdot\hat{\mathbf{r}}}\right| = (kr)^3$$

Let's assume we want the far-field power to be 1000 times greater than the near-field power → $kr = 10$

$$r \approx 1.6\lambda$$

Radiation Resistance



A dipole antenna radiates in the far field and stores energy in the near field

$$\mathbf{S}_{\text{ff}} = \frac{\omega\mu k}{2}\left(\frac{Il}{4\pi}\right)^2 \frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \qquad \mathbf{S}_{\text{nf}} = -\frac{j\eta}{2k}\left(\frac{Il}{4\pi}\right)^2 \frac{1}{r^5}\left(\sin^2\theta\hat{\mathbf{r}} - \sin 2\theta\hat{\boldsymbol{\theta}}\right)$$

Integrating the Poynting vector over a sphere surrounding the antenna gives the total power

$$P_{\text{rad}} = \oiint_S \mathbf{S}_{\text{ff}} \cdot \mathbf{dS} = \frac{\pi\eta}{3} I^2 \left(\frac{l}{\lambda}\right)^2$$

Imagine a fictitious resistance such that

$$P_{\text{rad}} = \frac{1}{2} R_{\text{rad}} I^2$$

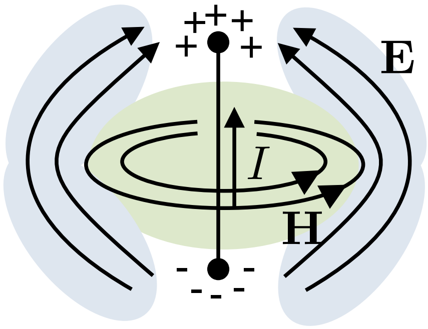
$$R_{\text{rad}} = \frac{2\pi\eta}{3}\left(\frac{l}{\lambda}\right)^2$$

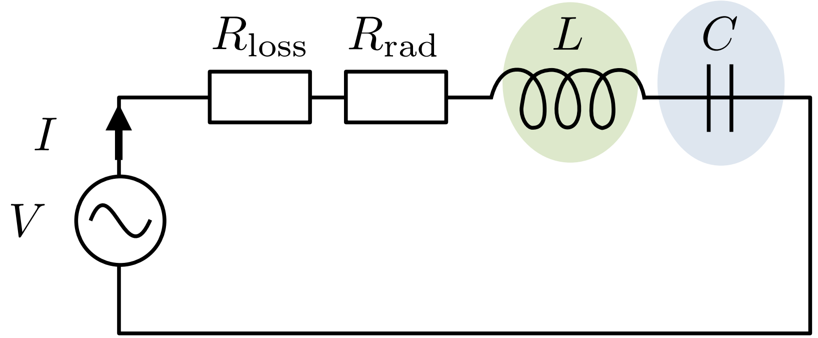
$R_{\text{rad}} \approx 0.0087\ \Omega$ for $l = \frac{\lambda}{300}$

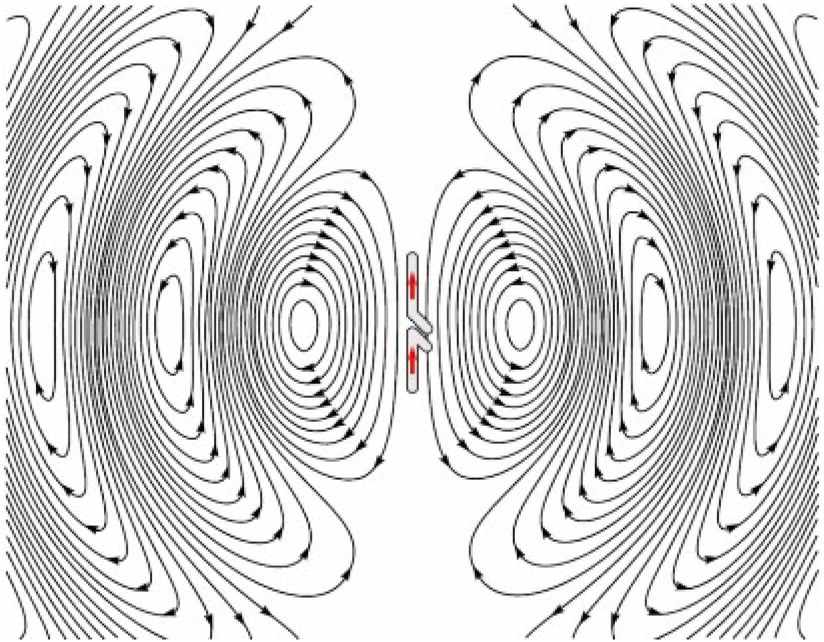
Example: AM radio f ≈ 1 MHz and λ ≈ 300 m

Remember that we have assumed that $l \ll \lambda$

Simple Model of an Ideal Dipole Antenna







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efficiency

$$R_{\text{rad}} = \frac{2\pi\eta}{3}\left(\frac{l}{\lambda}\right)^2 \rightarrow \eta = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}}$$

$$R_{\text{rad}} \approx 0.0087\ \Omega$$

$$R_{\text{loss}} \approx 0.03\ \Omega$$

(for a 1 m long wire)

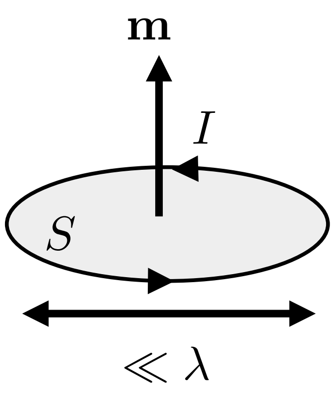
$\longrightarrow$ $\eta \approx 22.6\%$

about half of that for a monopole antenna!

Ideal Loop Antenna

An ideal loop antenna behaves as a magnetic dipole

$$\mathbf{m} = IS\hat{\mathbf{z}}$$



Far field of a magnetic dipole

$$\mathbf{E}_{\mathrm{FF}} = \frac{\omega^2\mu}{c}(\mathbf{m}\times\hat{\mathbf{r}})\frac{e^{-jkr}}{4\pi r}$$

The far fields are

$$\begin{cases} E_\phi = -\dfrac{\omega^2 IS\mu}{c}\sin\theta\dfrac{e^{-jkr}}{4\pi r} \\ H_\theta = -\dfrac{E_\phi}{\eta} \end{cases}$$

Poynting vector

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \hat{\boldsymbol{r}}\frac{\pi^2\eta I^2 S^2}{2\lambda^4 r^2}\sin^2\theta$$

Radiated power

$$P_{\mathrm{rad}} = \int_0^{2\pi}\int_0^{\pi}\mathbf{S}\cdot\hat{\boldsymbol{r}}r^2\sin\theta d\theta d\phi = \frac{4\eta\pi^3 I^2 S^2}{3\lambda^4}$$

Radiation resistance

$$R_{\mathrm{rad}} = \frac{8\pi^3\eta}{3}\left(\frac{S}{\lambda^2}\right)^2$$

For a loop with *N* turns

$$R_{\mathrm{rad}} = \frac{8\pi^3\eta}{3}\left(\frac{NS}{\lambda^2}\right)^2$$

Comparison Between E and M Dipole Antennas

E Dipole radiation resistance

$$R_{\text{rad}} = \frac{2\pi\eta}{3}\left(\frac{l}{\lambda}\right)^2$$

Capacitive

Loop diameter D to achieve the same R_{rad} as an E dipole

$$D = \frac{1}{\pi}\sqrt{\frac{2l\lambda}{N}}$$

M Dipole radiation resistance

$$R_{\text{rad}} = \frac{8\pi^3\eta}{3}\left(\frac{NS}{\lambda^2}\right)^2$$

Inductive

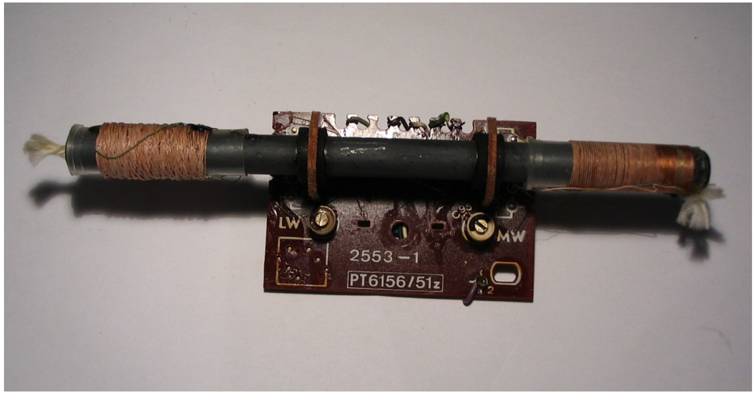
For a single turn (N=1), a M dipole is typically much bigger than an E dipole. Increasing the number of turns is a simple way to increase the radiation resistance but it also increases the loss.

Another way is to use a ferrite core to increase R_{rad} without having too many turns thus limiting losses

$$R_{\text{rad}} = \frac{8\pi^3\eta}{3}\left(\mu_{\text{eff}}\frac{NS}{\lambda^2}\right)^2$$

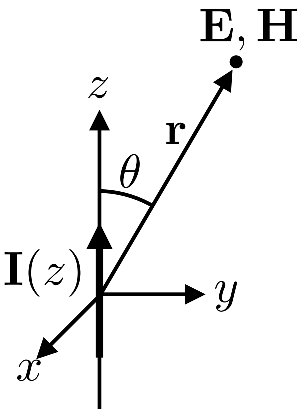
where μ_{eff} is the effective relative permeability of the ferrite core

Ferrite loop antenna



https://en.wikipedia.org/wiki/Loop_antenna

Far Fields from Line Currents



Consider a current *I(z)* that is not uniform. We know that the fields are given by the vector potential component A_z

$$A_z = \frac{\mu}{4\pi}\int I(z')\frac{e^{-jkR}}{R}\,dz'$$

we assume that $r \gg z'$ which implies that

$$R = |\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{r}\cdot\mathbf{r}' = r - z'\cos\theta$$

For the amplitude $R \approx r$

$$A_z = \frac{\mu}{4\pi}\int I(z')\frac{e^{-jk(r-z'\cos\theta)}}{r}\,dz' = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int I(z')e^{jkz'\cos\theta}\,dz'$$

essentially a Fourier integral

Following the same procedure as before to find the fields and keeping only the far-field components

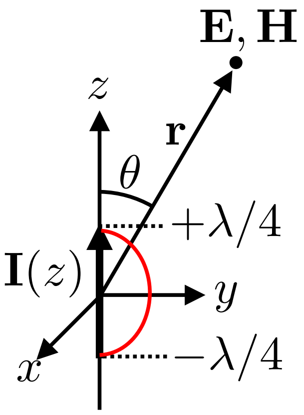
$$\mathbf{H} = \frac{jk}{\mu}\sin(\theta)A_z\hat{\boldsymbol{\phi}}$$

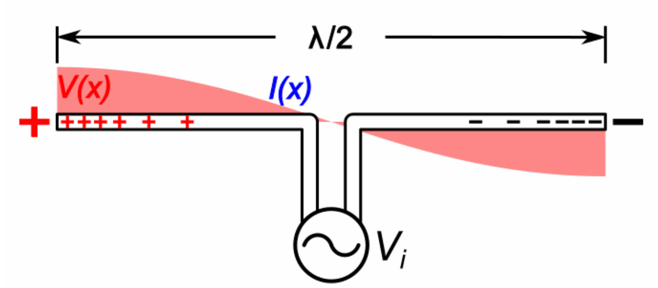
$$\mathbf{E} = j\omega\sin(\theta)A_z\hat{\boldsymbol{\theta}}$$

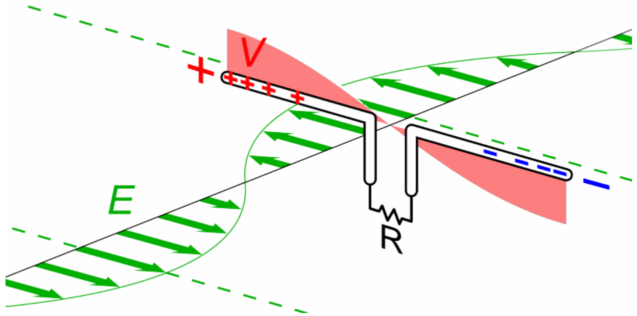
where

$$A_z = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int I(z')e^{jkz'\cos\theta}\,dz'$$

The Half-Wave Dipole Antenna







The current on the antenna is $I(z) = I\cos(kz)$

The corresponding vector potential along *z* is

$$A_z = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int I(z')e^{jkz'\cos\theta}\,dz' = \frac{I\mu e^{-jkr}}{2\pi r k\sin^2\theta}\cos\left(\frac{\pi}{2}\cos\theta\right)$$

The fields are given by

$$E_\theta = j\omega\sin(\theta)A_z = j\frac{I\eta e^{-jkr}}{2\pi r\sin\theta}\cos\left(\frac{\pi}{2}\cos\theta\right)$$

$$H_\phi = E_\theta/\eta \quad \text{(because we are in the far field)}$$

The Poynting vector is

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \hat{\boldsymbol{r}}\frac{I^2\eta}{8\pi^2 r^2\sin^2\theta}\cos^2\left(\frac{\pi}{2}\cos\theta\right)$$

The radiated power is

$$P_{\text{rad}} = \oiint_S \mathbf{S}\cdot d\mathbf{S} \approx \frac{\eta I^2}{4\pi}1.22$$

The radiation resistance is $R_{\text{rad}} \approx 73.2\ \Omega$

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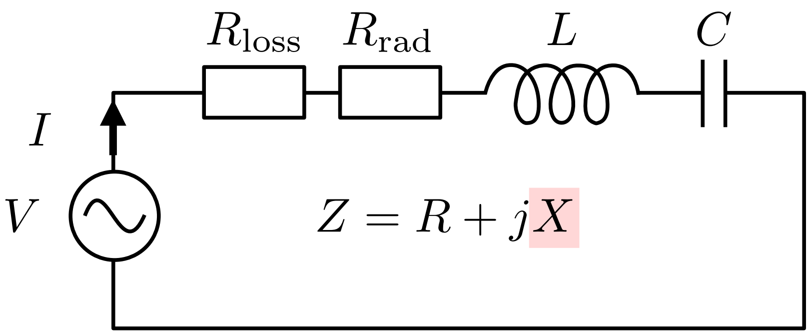
The Impedance of a Half-Wave Dipole Antenna



We have found that $R_{\text{rad}} \approx 73.2\ \Omega$, what about its reactance?

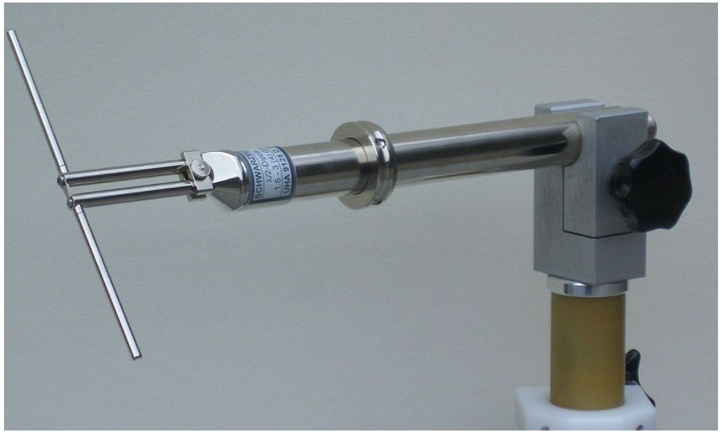
for length of λ/2

$$Z = 73.2 + j42.5\ \Omega$$

The actual resonance is at

$$l = 0.485\lambda$$

shortening the antenna makes it more capacitive!



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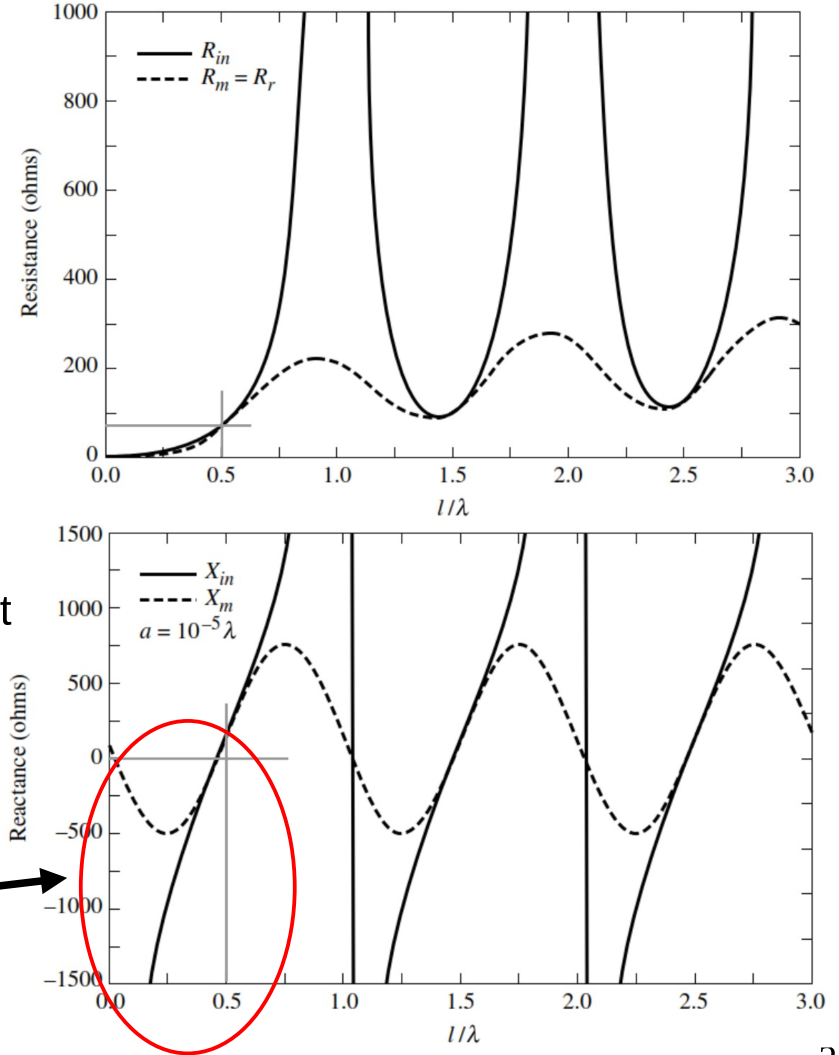


Balanis, "Antenna Theory: Analysis and Design", 2005.

How to Make the Antenna Shorter ?

Quarter-wave monopole antennas

A quarter-wave monopole antenna above a ground plane has an effective length of a half-wave dipole (image theory) but half its radiation resistance

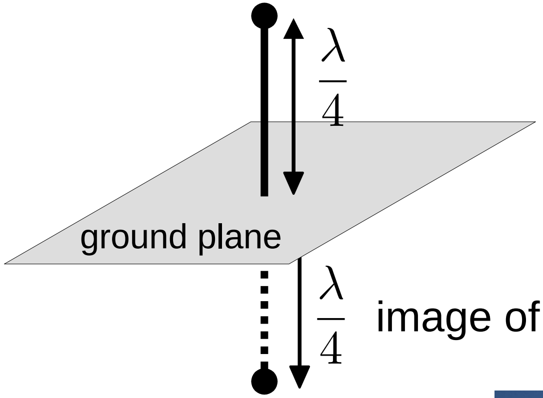
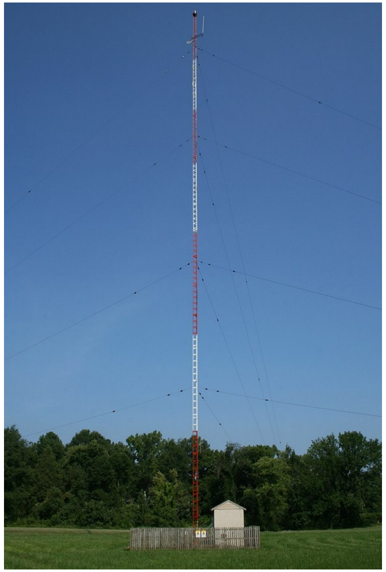


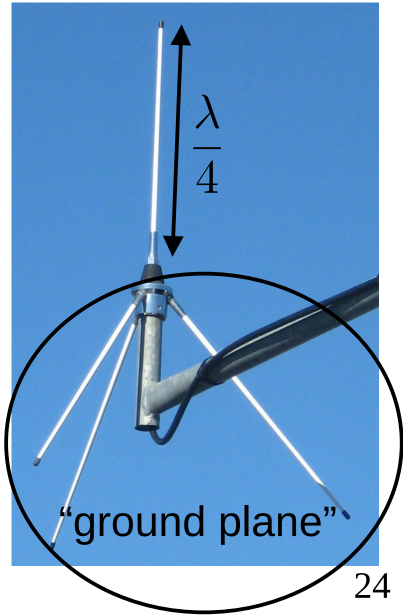
image of the monopole

Radiated power is integrated over one hemisphere, so it is divided by two compared to a dipole antenna

$$P_{\text{rad}} = \oiint_S \mathbf{S} \cdot \mathbf{dS}$$

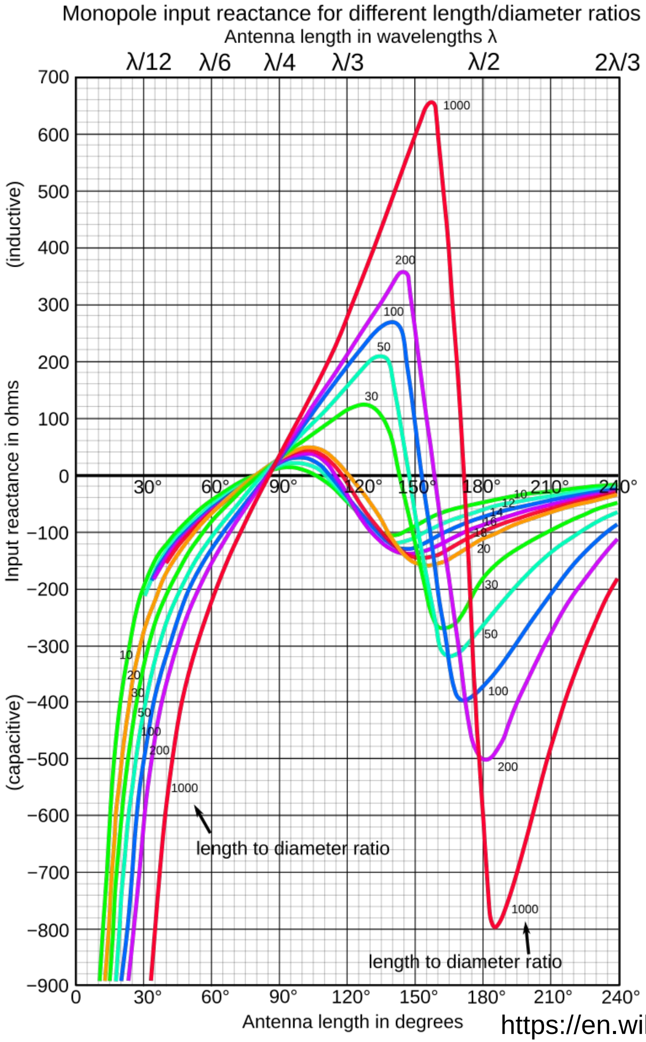
whip antennas



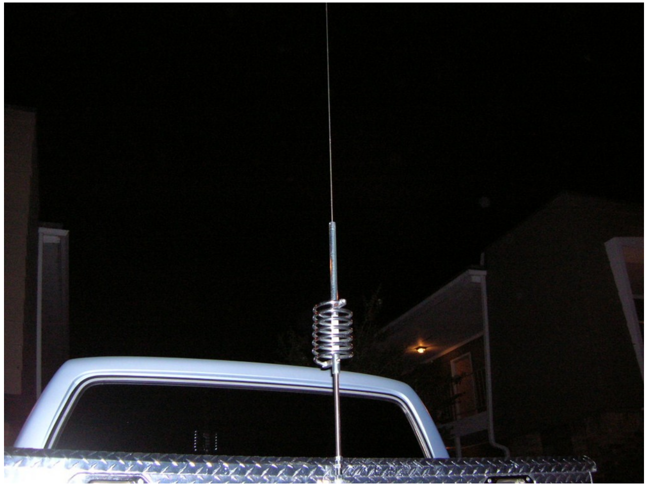


https://en.wikipedia.org/wiki/Whip_antenna

How to Make it Even Shorter ?



By making a dipole/monopole antenna shorter, it becomes more capacitive. A solution is to add a “loading coil” (inductance) to cancel the excess capacitance.



This additional inductance stores magnetic energy: Q ↑ and BW ↓

$$Q = \omega \frac{W_{\mathrm{e}} + W_{\mathrm{m}}}{P_{\mathrm{loss}}}$$

$$Q = \frac{\omega}{\Delta\omega}$$

https://en.wikipedia.org/wiki/Monopole_antenna#/media/File:Monopole_measured_input_reactance_graph.svg

The Chu-Wheeler-Harrington Limit

Q factor of an RLC circuit

$$Q = \frac{1}{\omega RC}$$

Radiation resistance of a dipole antenna

$$R_{\text{rad}} = \frac{2\pi\eta}{3}\left(\frac{l}{\lambda}\right)^2$$

$$Q_{\text{rad}} = \frac{3\epsilon}{4\pi^2 C}\frac{\lambda^3}{l^2}$$

$$Q_{\text{rad}} > \frac{3}{4\pi^2}\left(\frac{\lambda}{l}\right)^3$$

The capacitance of a small wire of length *l* and radius *r*

$$C \approx \frac{\epsilon l \pi}{2\left[\ln\left(\frac{l}{2r}\right) - 1\right]}$$

$l \gg r$

$$C > \epsilon l$$

good rule of thumb for very small wires

If the antenna length is halved, the radiation resistance is divided by 4 and the bandwidth by 8

Stutzman, “Antenna Theory and Design”, 2013

Example

antenna

$\sim 14\ \mathrm{cm}$

about half a wavelength

antenna

Typical cell phone band in the US

$$\omega_0 \approx 850\ \mathrm{MHz} \qquad \Delta\omega \approx 80\ \mathrm{MHz} \longrightarrow Q = \frac{\omega_0}{\Delta\omega} \approx 11$$

$$\lambda \approx 35\ \mathrm{cm}$$

$$Q_{\mathrm{rad}} > \frac{3}{4\pi^2}\left(\frac{\lambda}{l}\right)^3 \longrightarrow l \approx \lambda\left(\frac{3}{4\pi^2 Q}\right)^{\frac{1}{3}} \approx 7\ \mathrm{cm}$$

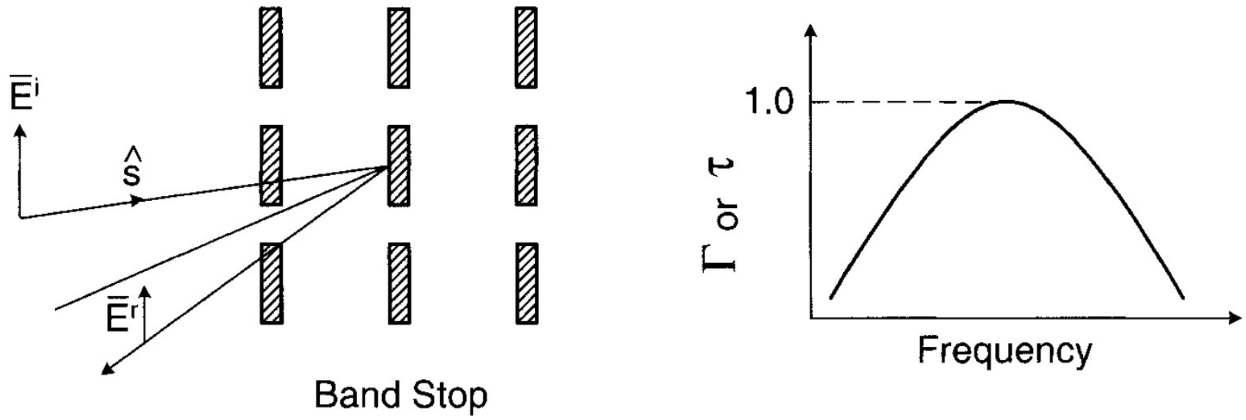
These gaps create capacitive coupling that increase the Q factor

$$Q = \omega\frac{W_{\mathrm{e}} + W_{\mathrm{m}}}{P_{\mathrm{loss}}}$$

$$C = \epsilon\frac{S}{d}$$

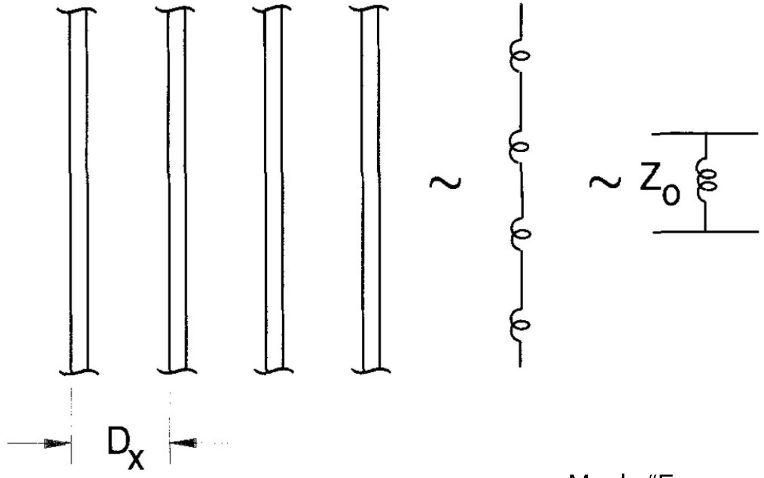
The gaps are wide to keep *C* low

A “Metasurface” is an Antenna Array

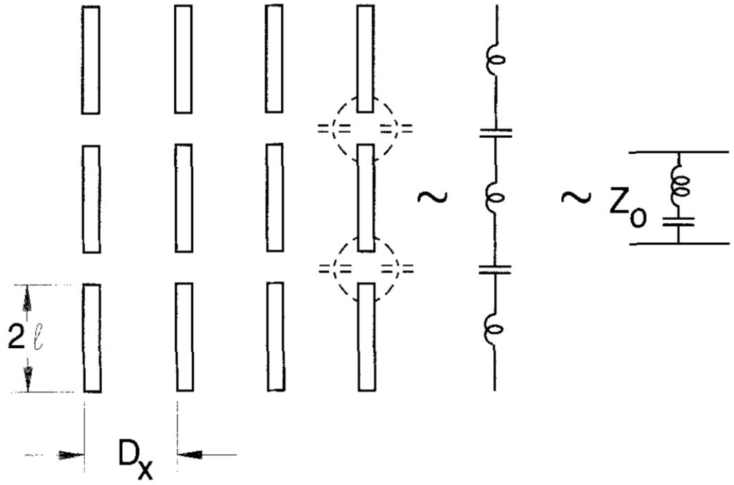


An array of half-wave dipoles resonates leading to sharp frequency features in reflection

An array of long rods does not resonate because it is only inductive

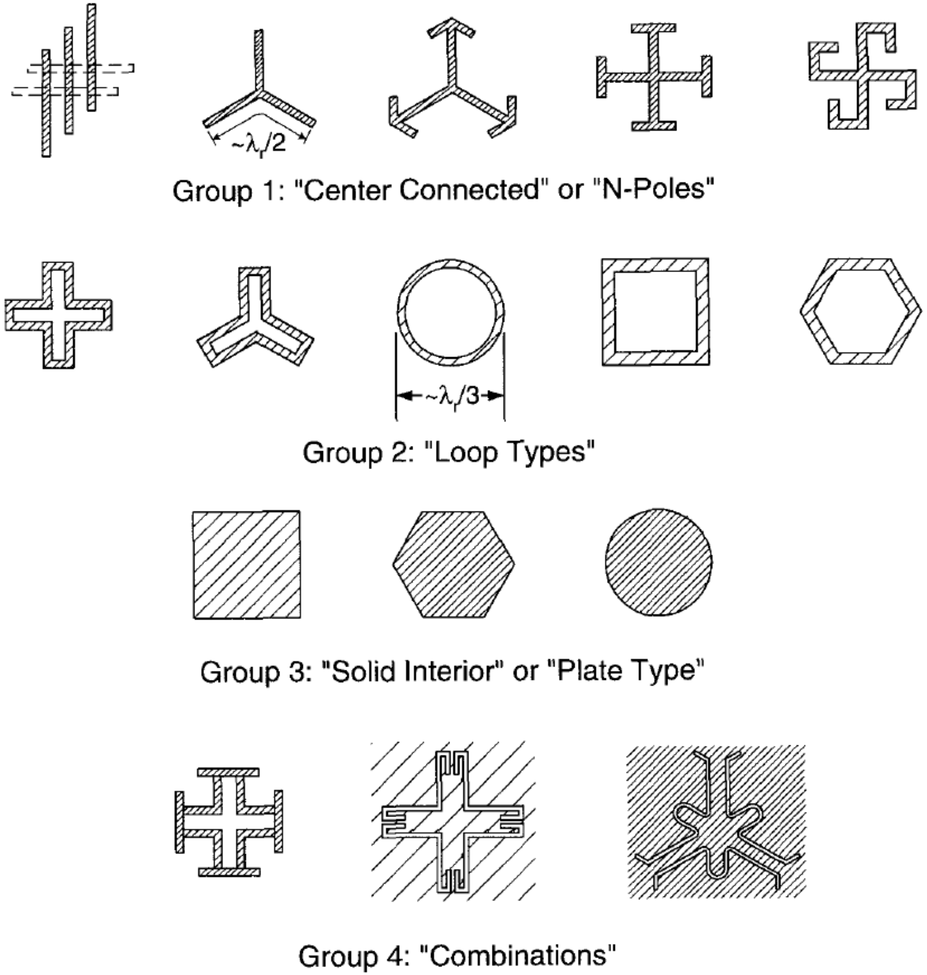


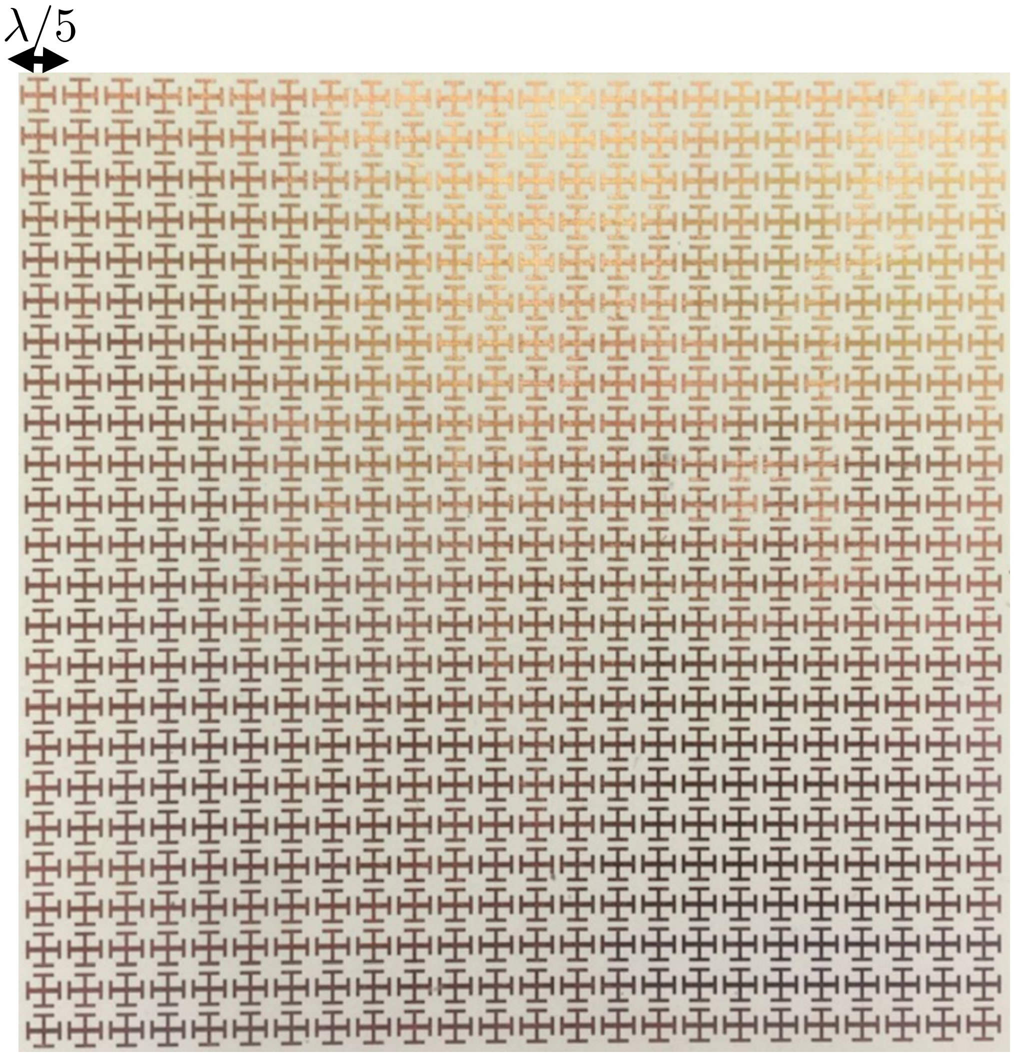
An array of dipoles is modeled as LC circuits



Munk, “Frequency selective surfaces: theory and design”, 2000

Metasurface Scattering Particles





Munk, “Frequency selective surfaces: theory and design”, 2000

Antenna Directivity

The radiated power by an antenna is

$$P_{\text{rad}} = \oiint_S \mathbf{S}\cdot\mathbf{dS} = \frac{1}{2}\text{Re}\left(\oiint_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{dS}\right) = \frac{1}{2\eta}\oiint_S (|E_\theta|^2 + |E_\phi|^2)r^2 \underbrace{\sin\theta d\theta d\phi}$$

$d\Omega$ differential solid angle

In the far field

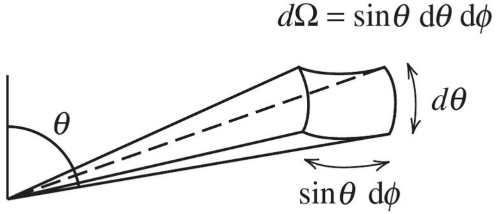
$$|\mathbf{H}| = \frac{|\mathbf{E}|}{\eta}$$

$$P_{\text{rad}} = \oiint_S U(\theta,\phi)d\Omega$$

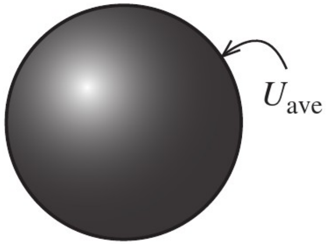
where

radiation intensity

$$U(\theta,\phi) = \frac{1}{2\eta}(|E_\theta|^2 + |E_\phi|^2)r^2$$



Consider an isotropic source with an angle-invariant radiation intensity $U(\theta,\phi) = U_{\text{av}}$



(*a*) Radiation intensity distributed isotropically.

$$P_{\text{rad}} = \oiint_S U(\theta,\phi)d\Omega = U_{\text{av}}\oiint_S d\Omega = 4\pi U_{\text{av}}$$

$$U_{\text{av}} = \frac{P_{\text{rad}}}{4\pi}$$

Stutzman, “Antenna Theory and Design”, 2013

Antenna Directivity

The directivity is the ratio of max to average radiation

Average radiation

$$U_{\text{av}} = \frac{P_{\text{rad}}}{4\pi}$$

Directivity

$$D = \frac{U_{\max}}{U_{\text{av}}} = 4\pi\frac{U_{\max}}{P_{\text{rad}}} = \frac{4\pi}{\Omega_A}$$

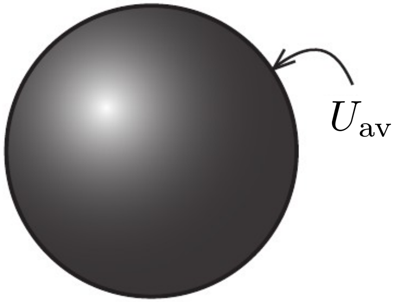
Antenna solid angle

$$\Omega_A = \frac{P_{\text{rad}}}{U_{\max}}$$

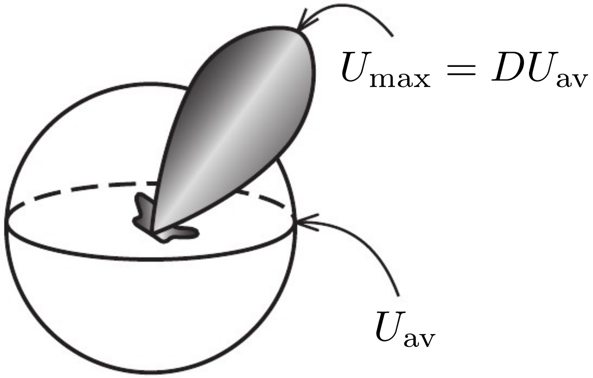
$$P_{\text{rad}} = \oiint_S U(\theta,\phi)\,d\Omega$$

$$P_{\text{rad}} = U_{\max}\oiint_S U_0(\theta,\phi)\,d\Omega$$

$$P_{\text{rad}} = U_{\max}\Omega_A$$



(a) Radiation intensity distributed isotropically.



(b) Radiation intensity from an actual antenna.

Stutzman, “Antenna Theory and Design”, 2013

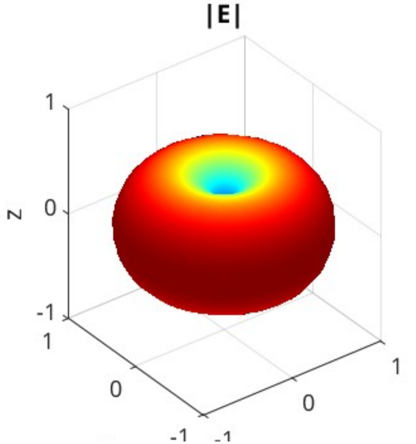
Directivity of an Ideal Dipole Antenna

The electric far field of an ideal dipole antenna is

$$\mathbf{E} = j\frac{I\eta e^{-jkr}}{2\pi r \sin\theta}\cos\left(\frac{\pi}{2}\cos\theta\right)\hat{\boldsymbol{\theta}}$$

its radiation intensity is

$$U(\theta,\phi) = \frac{\omega\mu k}{2}\left(\frac{Il}{4\pi}\right)^2 \sin^2\theta$$

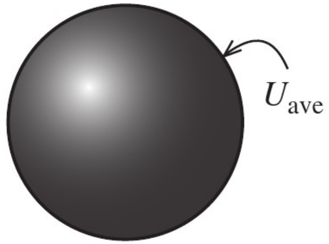


average radiation

$$U_{\text{av}} = \frac{P_{\text{rad}}}{4\pi} = \frac{\eta}{12} I^2 \left(\frac{l}{\lambda}\right)^2$$

max radiation when *θ = 90°*

$$U_{\text{max}} = \frac{\omega\mu k}{2}\left(\frac{Il}{4\pi}\right)^2 = \frac{\eta}{8} I^2 \left(\frac{l}{\lambda}\right)^2$$



(*a*) Radiation intensity distributed isotropically.

Dipole directivity

$$D = \frac{U_{\text{max}}}{U_{\text{av}}} = \frac{3}{2}$$

Dipole solid angle

$$\Omega_{\text{A}} = \frac{4\pi}{D} = \frac{8\pi}{3}$$

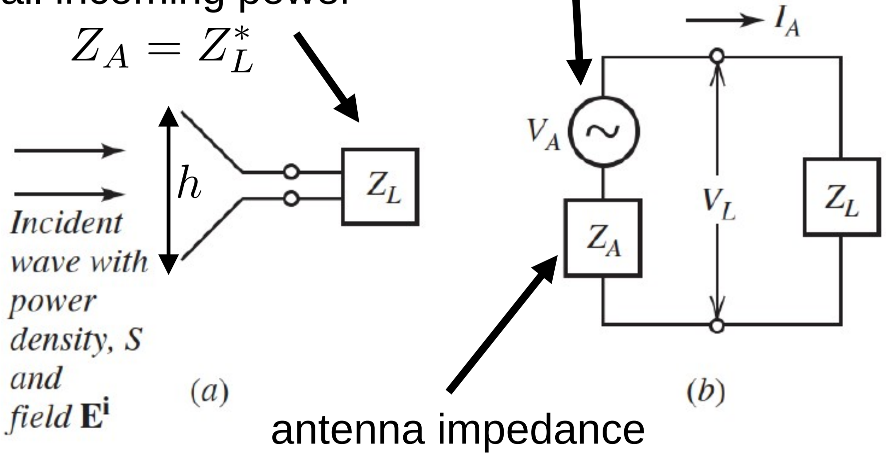
Stutzman, “Antenna Theory and Design”, 2013

Effective Length and Aperture

matched load to absorb all incoming power

$Z_A = Z_L^*$

voltage due to excitation



antenna impedance

Figure 4-1 Equivalent circuit for a receiving antenna. (*a*) Receive antenna connected to a receiver with load impedance Z_L. (*b*) Equivalent circuit.

h is the effective length of the antenna not its exact length!

How much power is delivered to the load Z_L

$$P = \frac{1}{2}|I_A|^2 R_L = \frac{1}{2}\frac{|V_A|^2}{(R_A+R_L)^2+(X_A+X_L)}R_L$$

$$Z_A = Z_L^*$$

$$P = \frac{|V_A|^2}{8R_A}$$

In an ideal case $V_A = E_\mathrm{i} h$ $\longrightarrow$ $P = h^2\frac{|E_\mathrm{i}|^2}{8R_A}$

The received power may also be expressed in terms of an effective aperture and the Poynting vector of the incident wave

$$P = A_\mathrm{e} S_\mathrm{i} \quad \text{where } S_\mathrm{i} = \frac{1}{2}|\mathbf{E}\times\mathbf{H}^*| = \frac{|E_\mathrm{i}|^2}{2\eta}$$

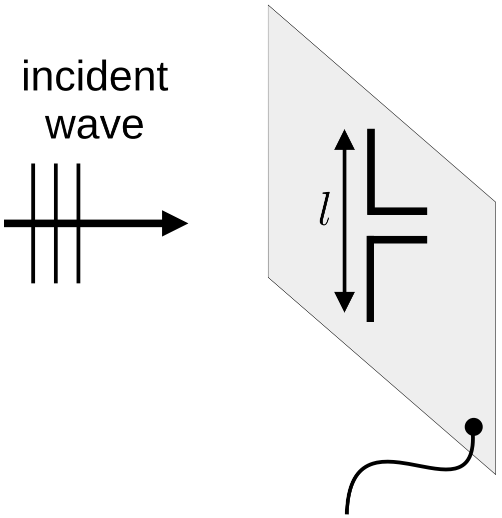
Equating both power

$$A_\mathrm{e} = \frac{\eta h^2}{4R_\mathrm{rad}}$$

Stutzman, “Antenna Theory and Design”, 2013

Effective Aperture

The effective aperture allows us to compute the power received from an incident wave



Effective aperture, A_e

$$P = A_e S_i$$

where

$$A_e = \frac{\eta h^2}{4R_{rad}}$$

h is the antenna effective length

For an electric dipole antenna

$$R_{rad} = \frac{2\pi\eta}{3}\left(\frac{l}{\lambda}\right)^2$$

and we can approximate $h \approx l$

$$A_e = \frac{3}{8\pi}\lambda^2$$

electric dipole solid angle

we know that

$$\Omega_A = \frac{8\pi}{3}$$

In general, for any antenna we have

$$A_e = \frac{\lambda^2}{\Omega_A}$$

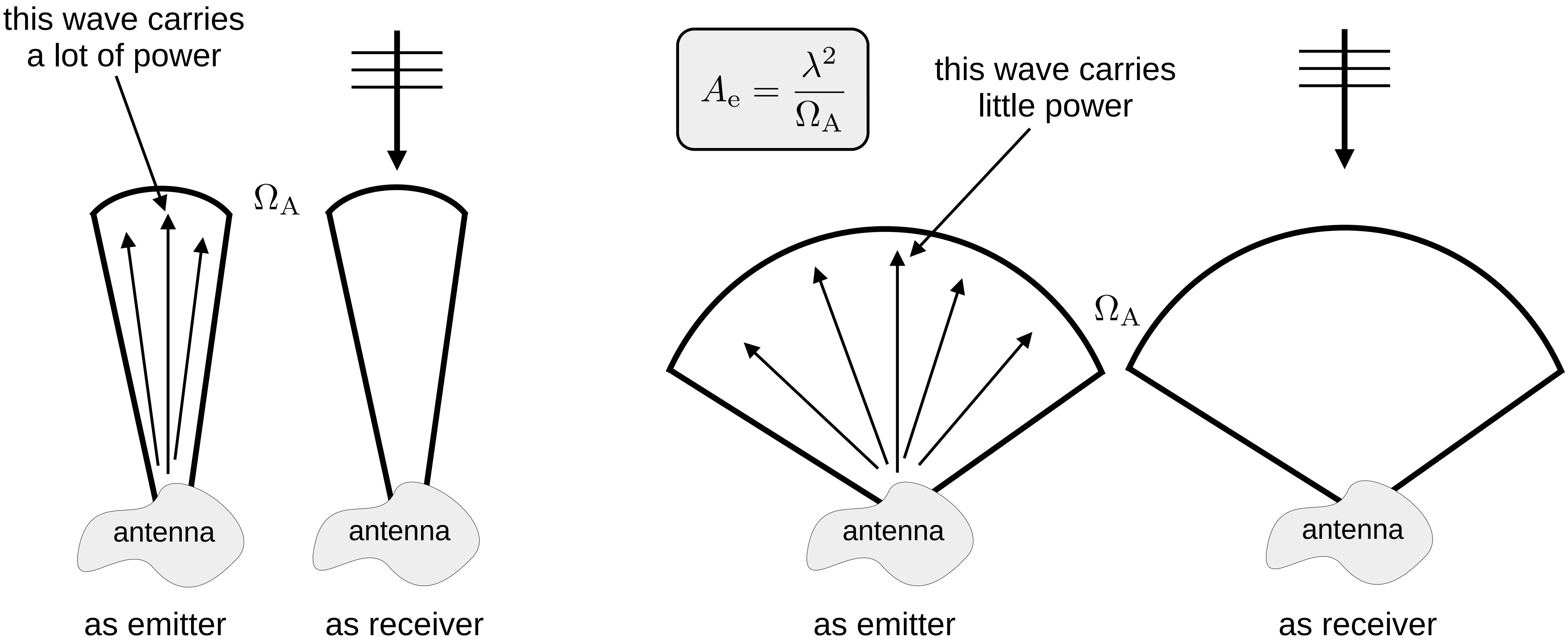
for an isotropic source

$$A_e = \frac{\lambda^2}{4\pi}$$

For a dipole antenna picking up a 1 MHz signal (λ = 300 m). The effective aperture is a circle of radius 58.5 m.

Stutzman, “Antenna Theory and Design”, 2013

Understanding the Effective Aperture



$$A_e = \frac{\lambda^2}{\Omega_A}$$

Small solid angle: emitted power doesn't spread much in space. Can capture a lot of the incoming power

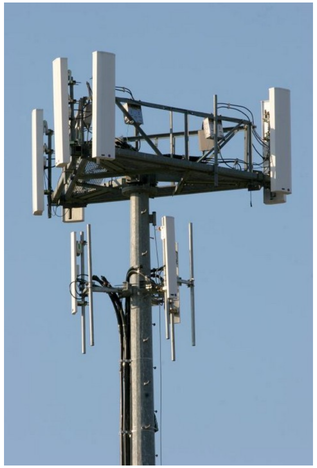
Large solid angle: emitted power spread a lot in space. Cannot capture a lot of the incoming power

What Have We Learned So Far….

- An antenna is an RLC circuit. It stores electric (L) and magnetic (C) energy in its near-field. The material making the antenna is lossy leading to R_{loss} and P_{loss}. The antenna radiates EM waves to the far-field leading to R_{rad} and P_{rad}.
- In the near-field, **E** and **H** are 90° out-of-phase leading to imaginary power (stored energy). In the far-field, **E** and **H** are in phase leading to real power (radiation).
- The region that defines the transition between near and far field is ~1.6 λ. This corresponds to where the radiation power is 1000 larger than the reactive power.
- The radiation resistance is a measure of how well an antenna radiates. For a small dipole antenna, it is proportional to $(l/\lambda)^2$. For a small loop antenna to $(S/\lambda^2)^2$. For a half-wave dipole to ~73 Ω.
- Loop antennas have typically a much small R_{rad} than dipole antennas but it may be increased by making many turns and using a ferrite core.
- Shortening a dipole antenna makes it more capacitive, which may be compensated by adding an inductance. This leads to more stored energy, a higher Q factor and a smaller bandwidth.
- The Wheeler limit tells us that the Q factor is bound to be more than $(\lambda/l)^3$ for a small antenna
- Metamaterials are arrays of small disconnected patches (antennas) so that they can all resonate. Arrays of metal (connected) strips do not resonate, they are modeled as simple inductance.
- The solid angle Ω_a of an antenna tells how the radiated power spreads in space. Laser beams have very small Ω_a, dipole antenna have very large $\Omega_a = 8\pi/3$
- The (maximum) effective aperture A_e tells us how much power can be captured by a system knowing the incident power density. It is generally $A_e = \lambda^2/\Omega_a$. For a dipole, we have $A_e = 3\lambda^2/(8\pi)$

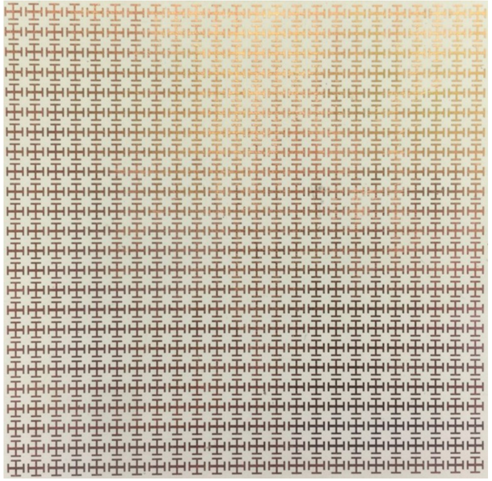
Antenna Arrays

Examples of Antenna Arrays

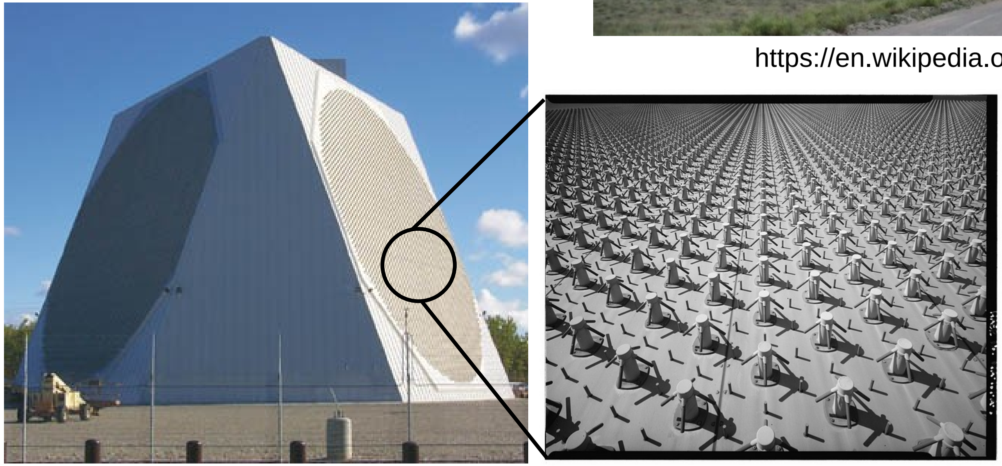


https://www.emfrf.com

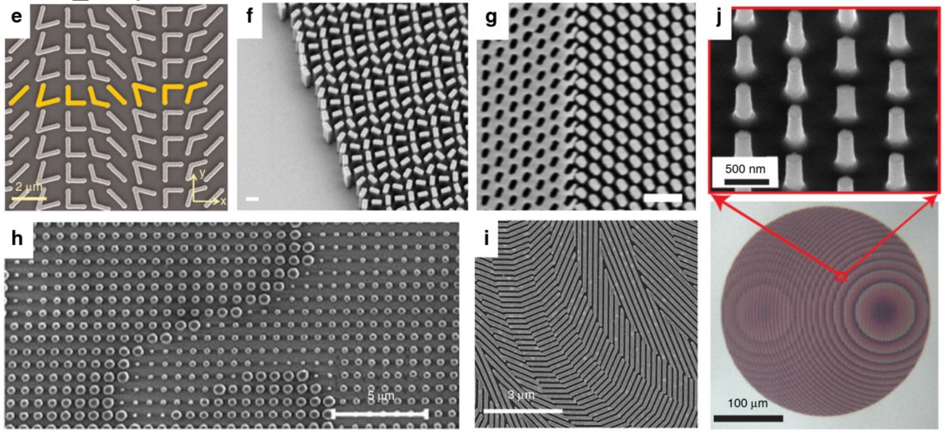
https://en.wikipedia.org/wiki/Antenna_array



https://doi.org/10.1063/1.4972195

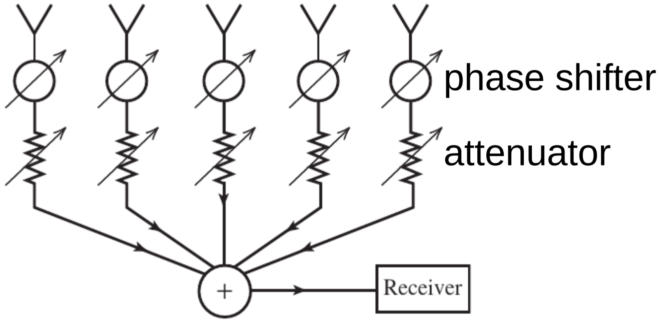


https://en.wikipedia.org/wiki/PAVE_PAWS



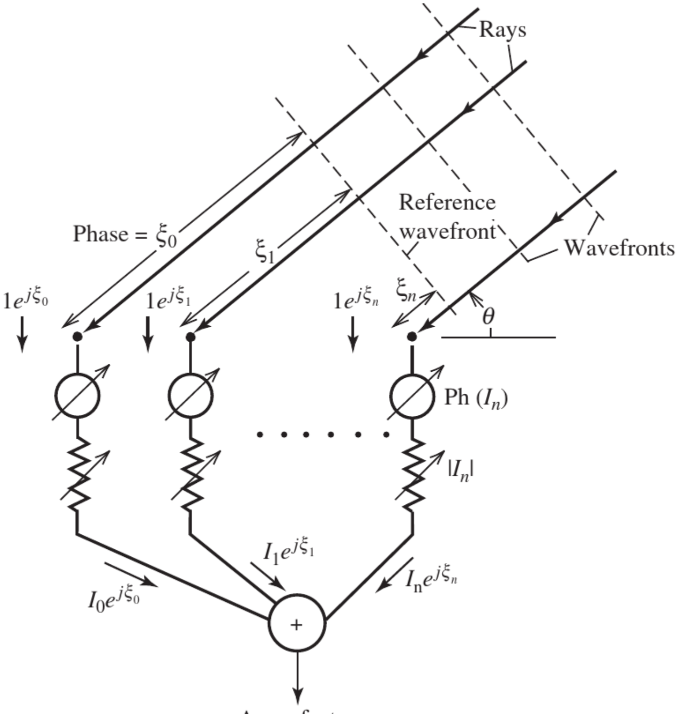
https://doi.org/10.1038/s41377-018-0058-1

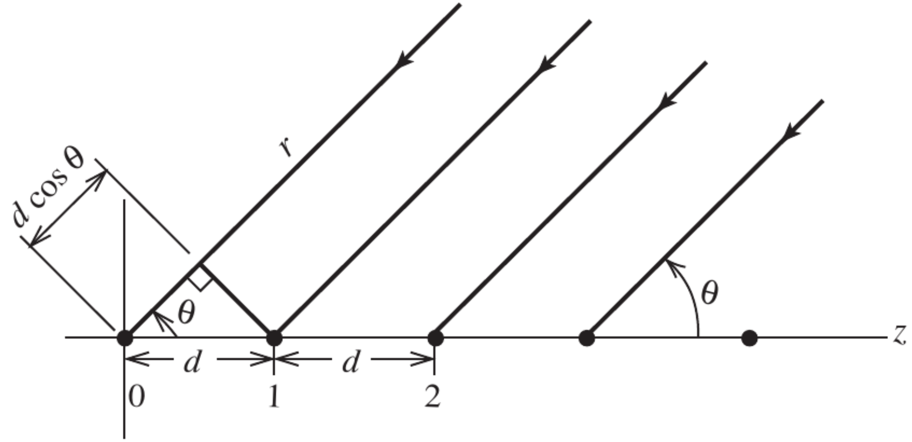
1D Antenna Arrays as Receivers



The <u>array factor</u> allows us to compute the radiation pattern of an antenna array assuming isotropic sources. It is expressed as the sum of the path lengths as

$$\mathrm{AF} = I_0 + I_1 e^{j\beta d\cos\theta} + I_2 e^{j\beta 2d\cos\theta} + \cdots = \sum_{n=0}^{N-1} I_n e^{j\beta n d\cos\theta}$$





For an incoming plane wave $I_0 = I_1 = \cdots = I_n$

Stutzman, “Antenna Theory and Design”, 2013

1D Antenna Arrays as Emitters

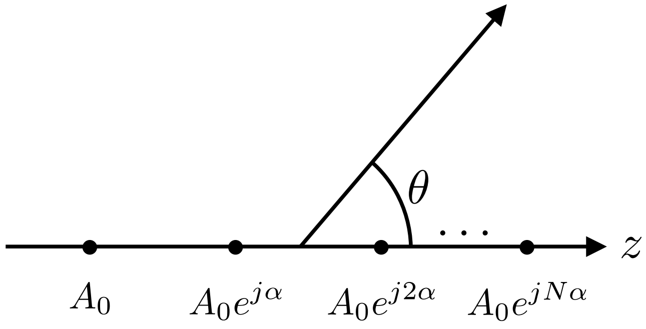
Consider an array used as an emitter with each element having the same amplitude and a linear phase progression

$$\mathrm{AF} = \sum_{n=0}^{N-1} I_n e^{j\beta n d \cos\theta} \quad I_n = A_0 e^{jn\alpha} \longrightarrow \mathrm{AF} = A_0 \underbrace{\sum_{n=0}^{N-1} e^{jn\psi}}_{\text{geometric series}}$$

where $\psi = \beta d \cos\theta + \alpha$

$$\mathrm{AF} = A_0 \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = A_0 e^{j(N-1)\psi/2} \frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

the maximum is
$$\mathrm{AF}(\psi = 0) = A_0 N$$



Normalized array factor

$$f(\psi) = \frac{\sin(N\psi/2)}{N\sin(\psi/2)}$$

The normalized array factor is found by removing the constant phase and dividing by the maximum value

Stutzman, "Antenna Theory and Design", 2013

Plotting the Normalized Array Factor

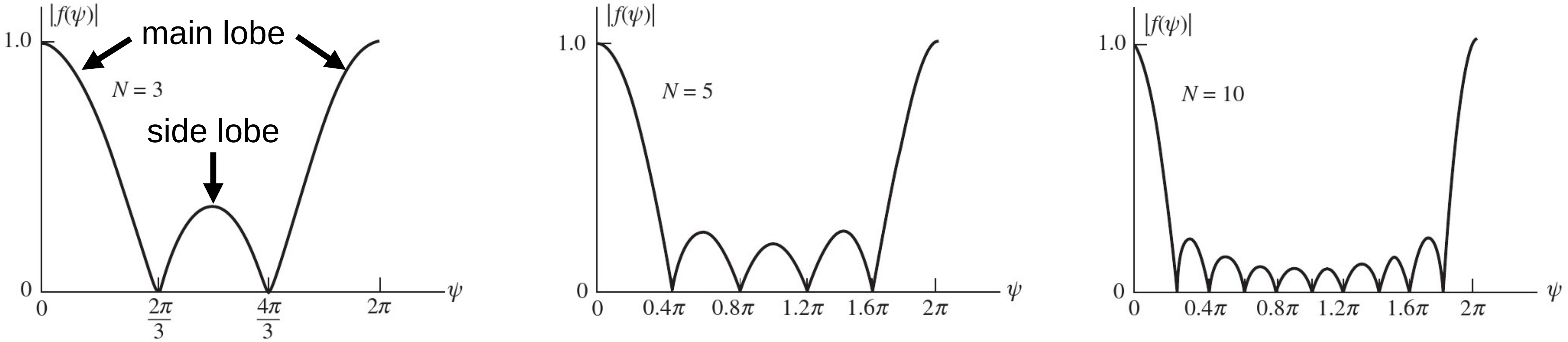
Normalized array factor

The array factor is 2π periodic

$$f(\psi) = \frac{\sin(N\psi/2)}{N\sin(\psi/2)}$$

where $\psi = \beta d \cos\theta + \alpha$

3-element array **5-element array** **10-element array**

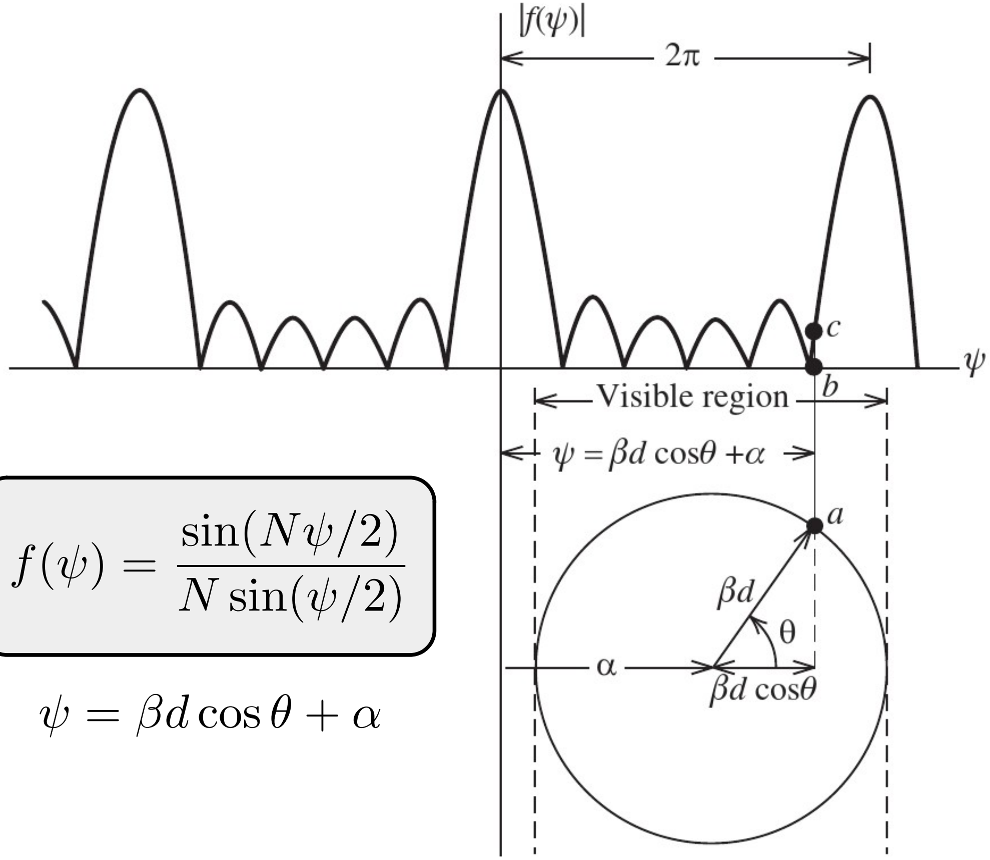


As *N* increases: the main lobe narrows, there are more side lobes, the amplitude of the side lobes decreases

Total number of <u>full</u> lobes: *N-1.* The width of minor lobes is *2π/N*

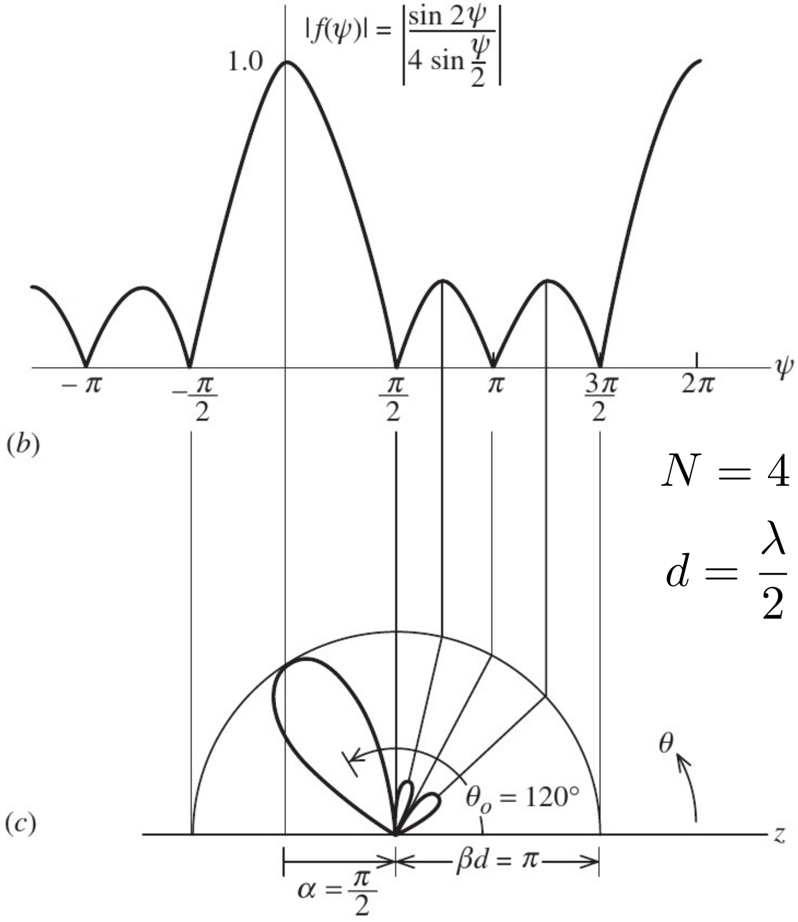
Stutzman, "Antenna Theory and Design", 2013

Finding the Radiation Pattern



$$f(\psi) = \frac{\sin(N\psi/2)}{N\sin(\psi/2)}$$

$$\psi = \beta d \cos\theta + \alpha$$



$$N = 4$$

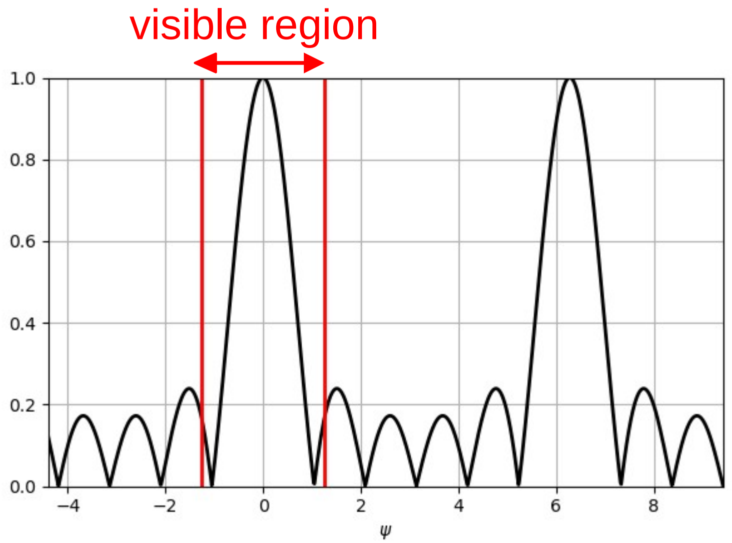
$$d = \frac{\lambda}{2}$$

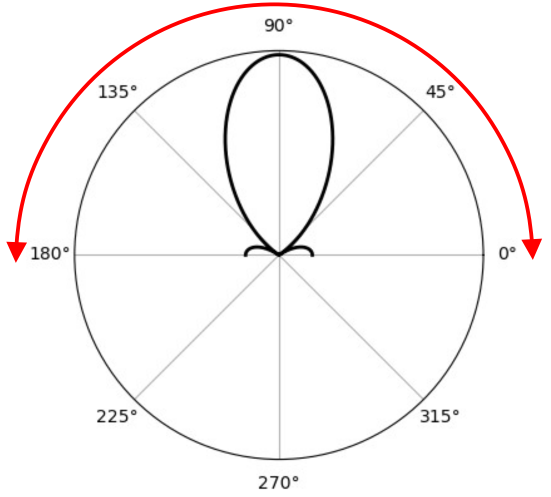
The visible region is defined as $0 < \theta < \pi \longrightarrow -\beta d < \beta d \cos\theta < \beta d \longrightarrow \alpha - \beta d < \psi < \alpha + \beta d$

Stutzman, "Antenna Theory and Design", 2013

Plotting the Array Factor and Radiation Pattern in Python

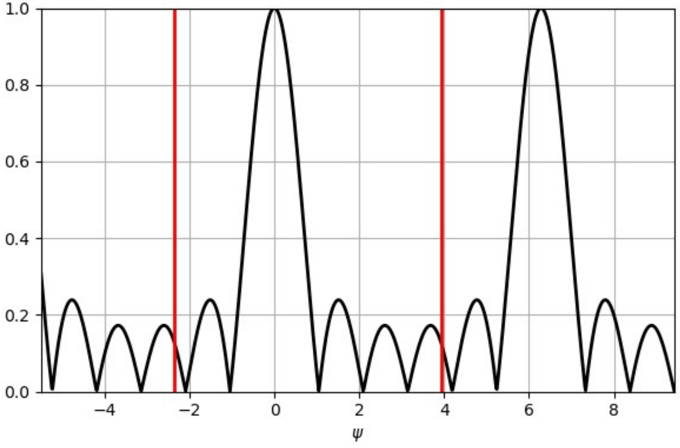
```
from numpy import *
from matplotlib.pyplot import *

N = 6    # number of elements
d = 0.2  # distance between elements in wavelength
a = 0    # linear phase shift

##  Plot the normalized array factor
k = 2*pi # we assume lambda = 1 [m]
Pmax = a+k*d
Pmin = a-k*d

RPmax = 2*pi if Pmax < 2*pi else Pmax
RPmin = 0 if Pmin > 2*pi else Pmin

psi = linspace(RPmin-pi,RPmax+pi,1000)
f = sin(N*psi/2)/(N*sin(psi/2))

fig = figure(figsize=(6,8))
fig.add_subplot(211)
plot(psi,abs(f),'k',lw=2)
plot([Pmax,Pmax],[0,1],'r',lw=2)
plot([Pmin,Pmin],[0,1],'r',lw=2)
grid()
xlim([psi[0],psi[-1]])
xlabel('$\psi$')
ylim([0,1])

##  Plot the radiation pattern
theta = linspace(0,pi,1000)
psi = k*d*cos(theta) + a
f = sin(N*psi/2)/(N*sin(psi/2))

ax = fig.add_subplot(212,polar=True)
plot(theta, abs(f),'k', lw=2)
ax.set_rmax(1.02)
ax.set_rticks([])
tight_layout()
show()
```

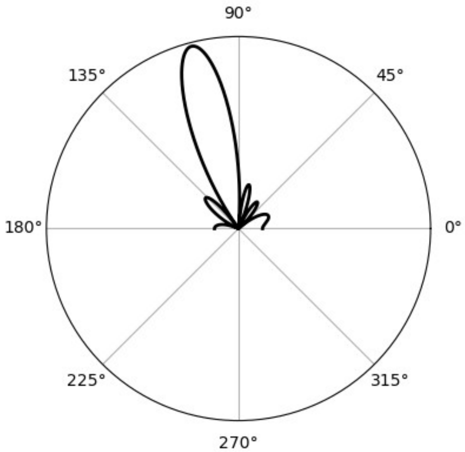




Why Does a Linear Phase Shift Tilts the Beam ?

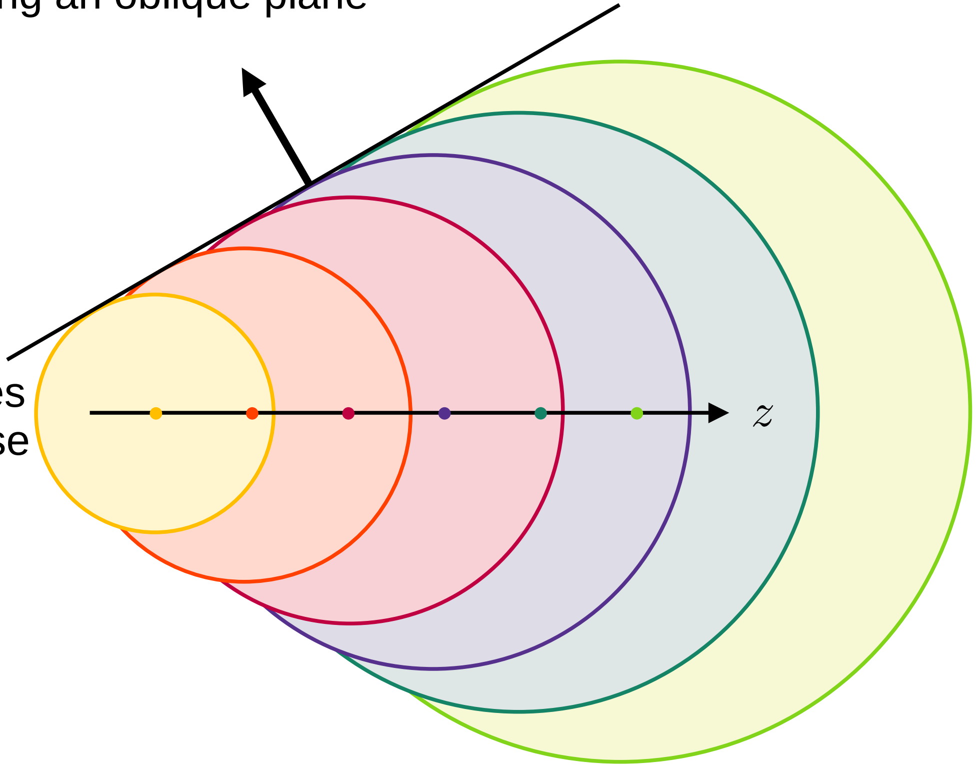
$$d = \frac{\lambda}{2} \qquad \alpha = \frac{\pi}{4} \qquad N = 6$$



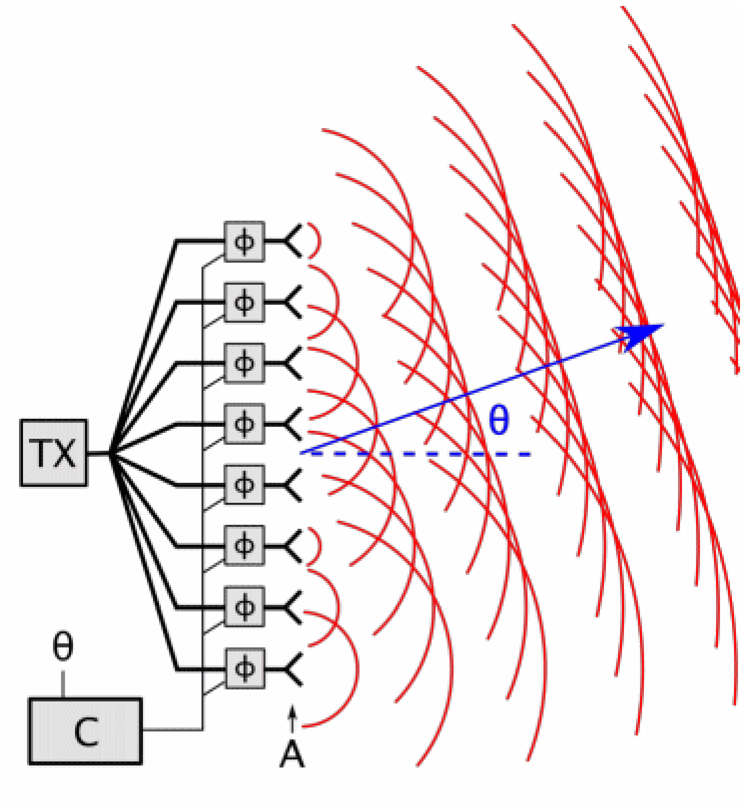


the radiated waves constructively interfere forming an oblique plane

each source radiates with a different phase



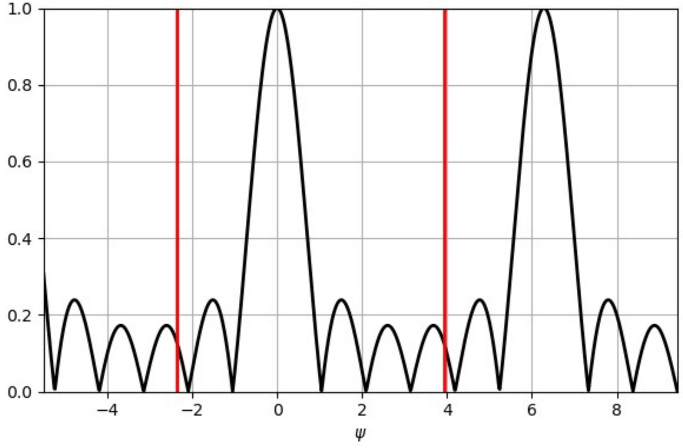
Why Does a Linear Phase Shift Tilts the Beam ?

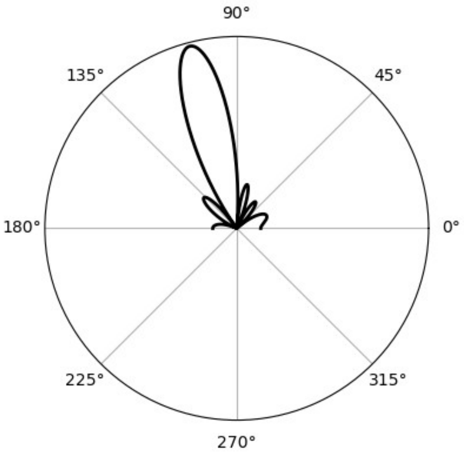


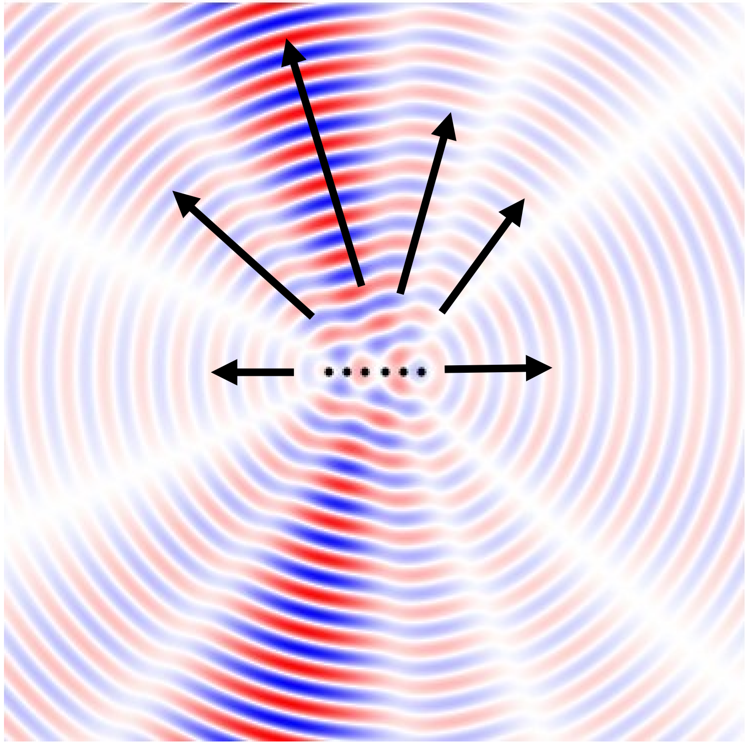
https://commons.wikimedia.org/wiki/File:Phased_array_animation_with_arrow_10frames_371x400px_100ms.gif#/media/File:Phased_array_animation_with_arrow_10frames_371x400px_100ms.gif

Field Visualization

$$d = \frac{\lambda}{2} \qquad \alpha = \frac{\pi}{4} \qquad N = 6$$







```
from numpy import *
from matplotlib.pyplot import *

d = 1/2  # source spacing in wavelength
a = pi/4 # linear phase shift
n = 6    # number of sources
L = 10   # window size in wavelength

N = (n-1)/2
l = linspace(-L,L,1001)
X,Y = meshgrid(l+d*N,l)

xs = arange(-N,N+1)
f = 0
for i in range(n):
    r = sqrt((X-i*d)**2+Y**2)
    f += exp(-1j*2*pi*r)*exp(1j*a*i)

    plot(xs[i]*d,0,'ok',ms=2)

ex = [-L,L,-L,L]
imshow(real(f),cmap='bwr',extent=ex)
axis('off')
show()
```

This script does not include field amplitude decay in terms of *r*

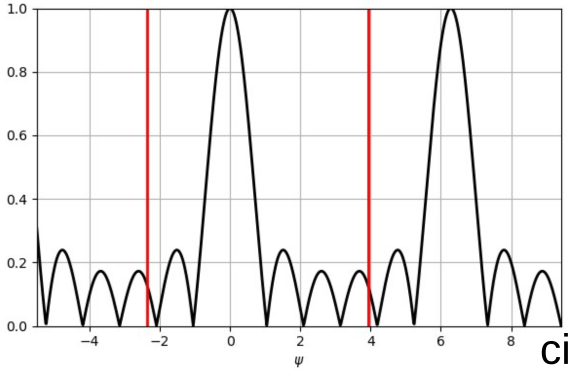
Increasing the Number of Sources

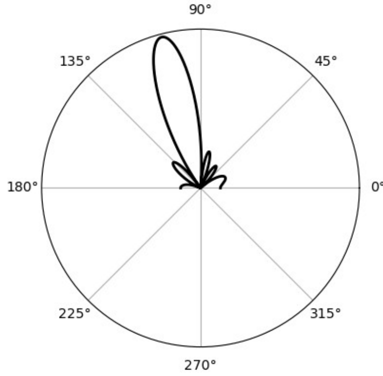
$$N = 6$$

$$d = \frac{\lambda}{2} \qquad \alpha = \frac{\pi}{4}$$

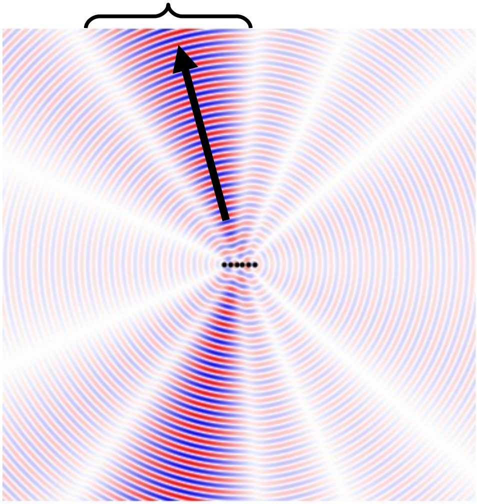
$$N = 25$$

Increasing the number of sources, increases the number of side lobes and makes the main lobe behave more like a plane wave

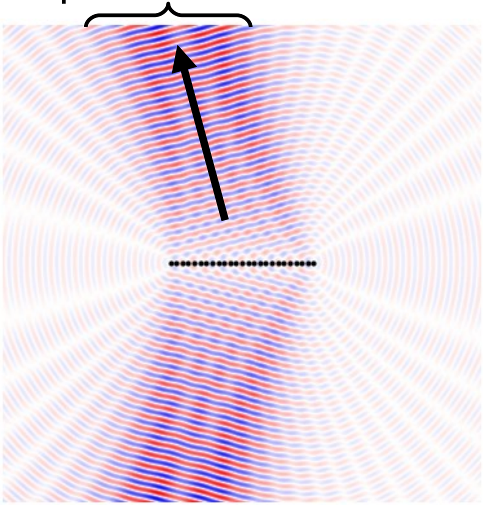


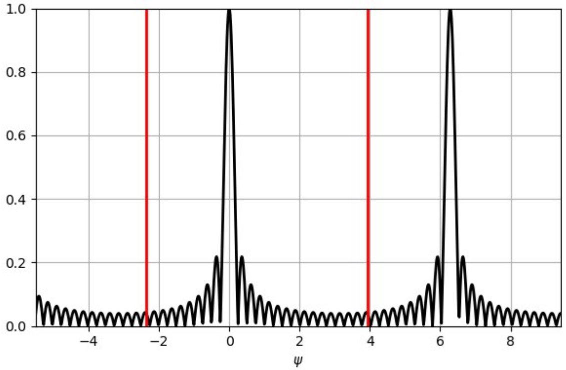


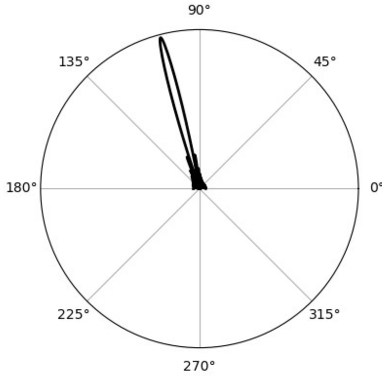
circular wave



plane wave

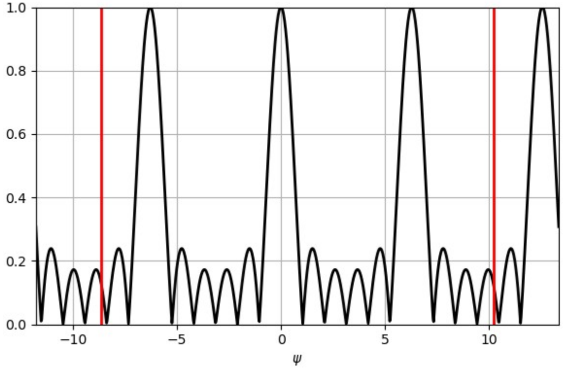






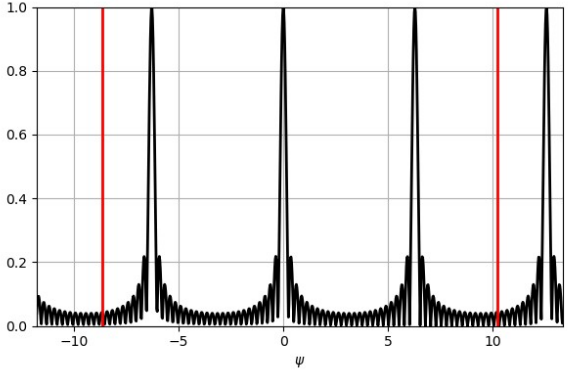
What if the Source Distance is Larger than the Wavelength

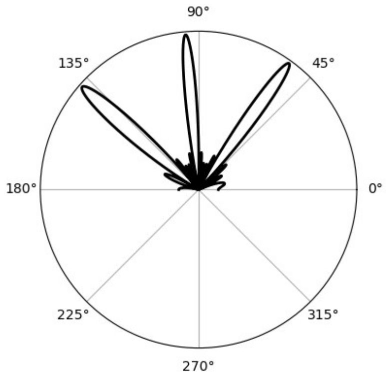
$$N = 6 \qquad d = \frac{3}{2}\lambda \qquad \alpha = \frac{\pi}{4} \qquad N = 25$$

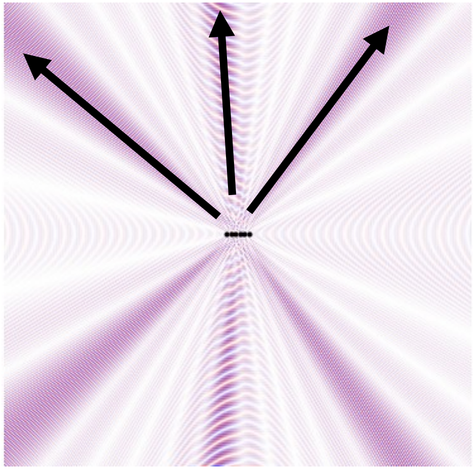


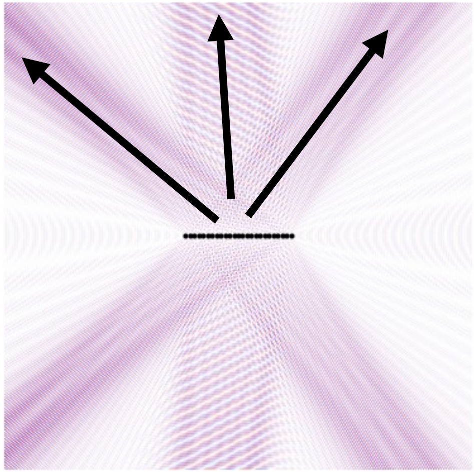
Diffraction orders (new main lobes) appear within the visible region.

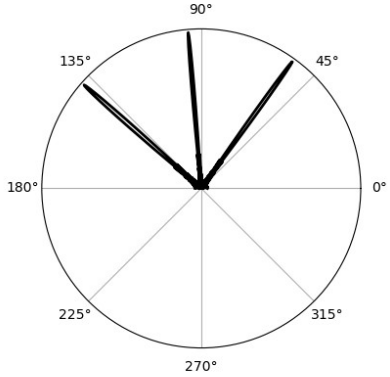
When *N* is increased the side lobes disappear leaving only the multiple main lobes that get narrower











Controlling the Radiation Angle

We want the radiated beam (main lobe) to be at the specified angle θ_{spec}

$$\psi = \beta d \cos\theta_{\text{spec}} + \alpha$$

The main lobe is when $\Psi = 0$

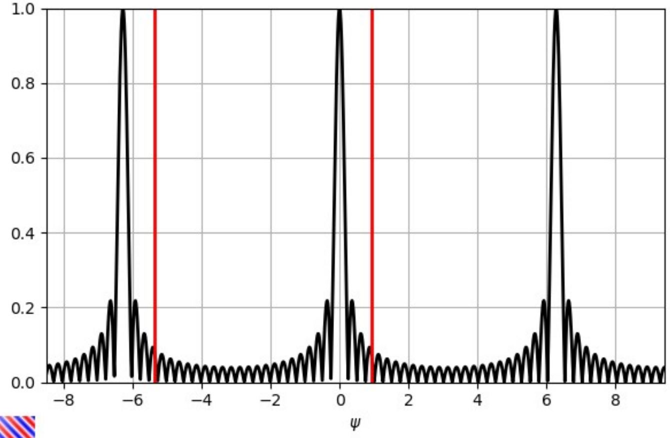
↓

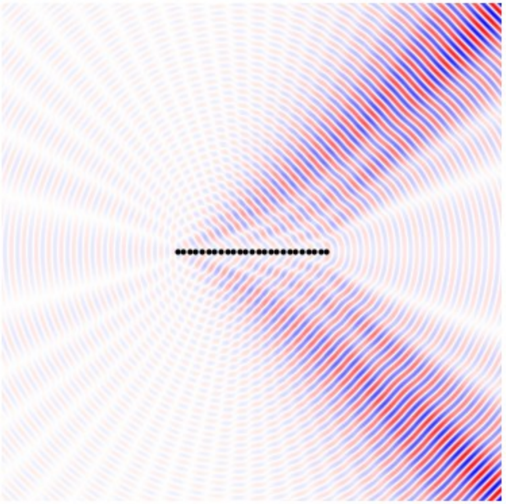
$$\alpha = -\beta d \cos\theta_{\text{spec}}$$

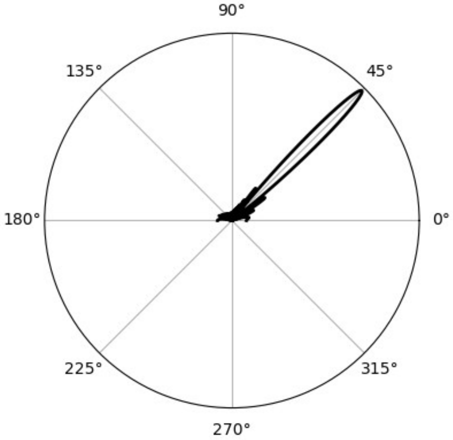
<u>Example</u>: d = λ/2 and θ_{spec} = *45°*

$$\alpha = -\frac{\pi}{\sqrt{2}}$$

$N = 25$



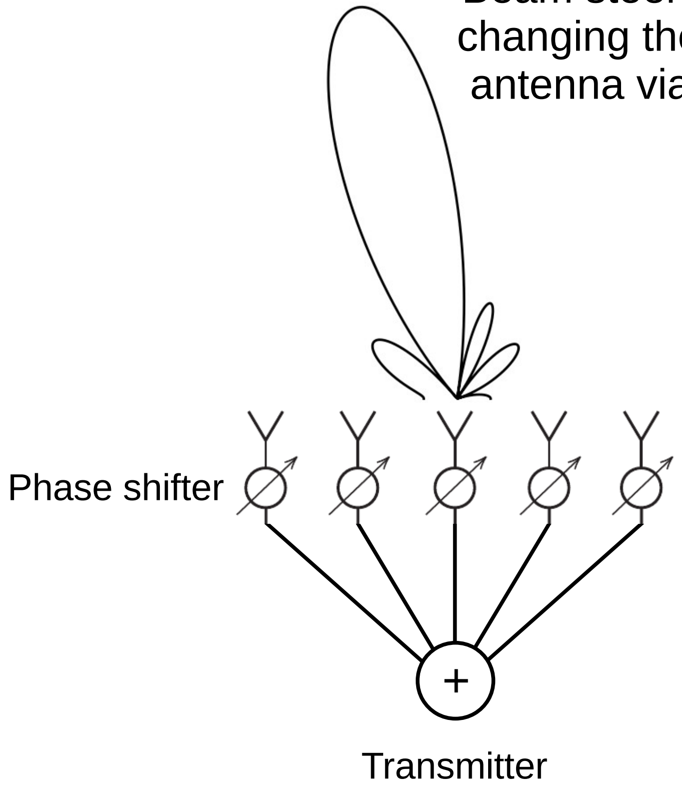




From Antenna array to Metasurface

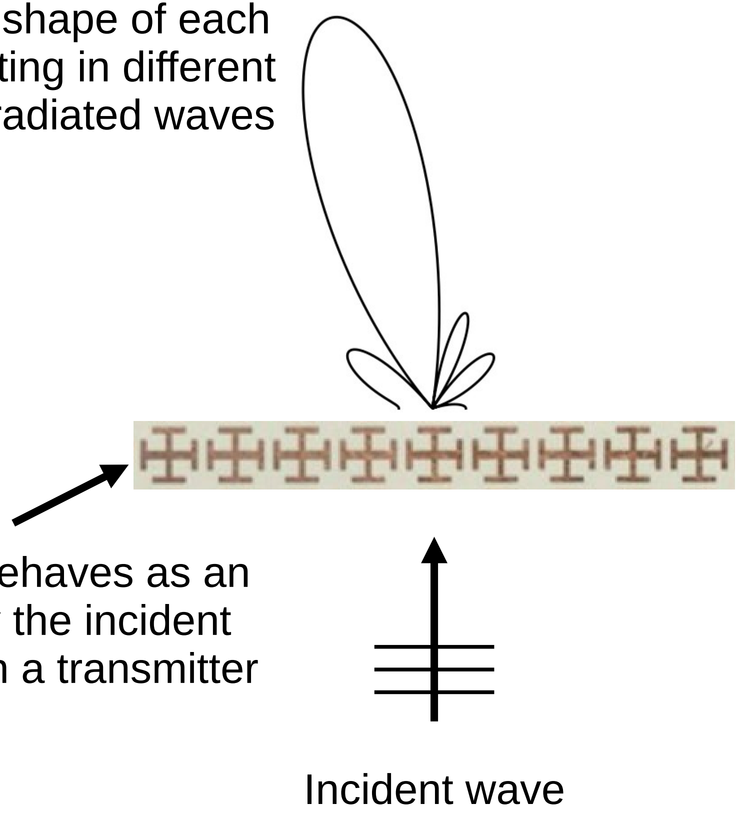
Antenna array

Beam steering achieved by changing the phase of each antenna via phase shifters



Metasurface

Beam steering achieved by changing the shape of each element resulting in different phase of the radiated waves



Each element behaves as an antenna fed by the incident wave rather than a transmitter

What Have We Learned So Far….

- An antenna array is a 1D or 2D finite array of antennas
- The way it radiates in space (assuming isotropic sources) can be assessed with the concept of the Array Factor
- An antenna array exhibits at least one main lobe for array spacing smaller than $\lambda/2$. For larger spacing, more main lobes typically appear
- Due to the finite size of the array, several side lobes are present. They increase in number but decrease in amplitude as the number of array elements increases.
- The more array elements, the more the main lobes behave as ideal plane waves rather than circular waves
- The direction of the radiation may be controlled by introducing linear phase shifts in the array
- Metasurface are antenna arrays. They control the direction of light propagation by having spatially varying resonators (antennas) that scatter light with different phase shifts.