

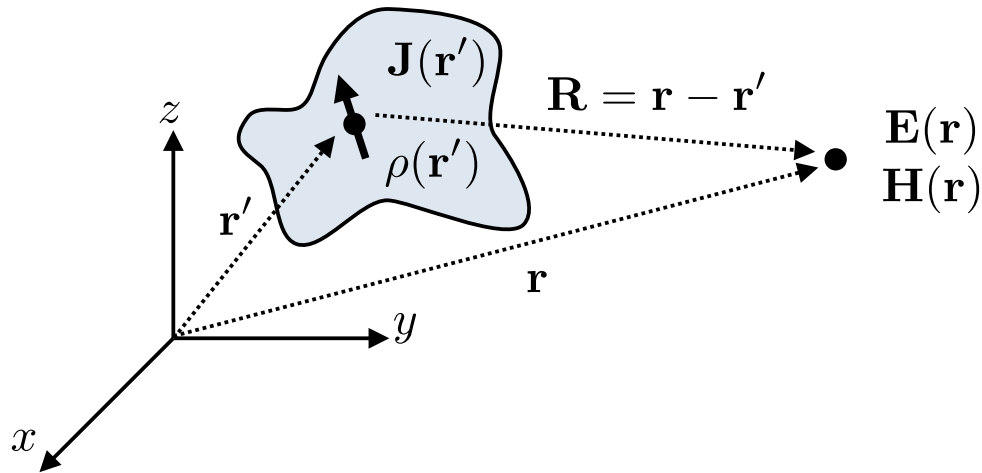
Lecture 4

Scattering

Fields and Potentials

Fields from Electric Charges and Currents

How to find the fields scattered by an arbitrary distribution of charges and currents?



Scalar and Vector Potentials

Poisson equation

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Electric potential

$$\mathbf{E} = -\nabla V$$

Valid in an electrostatic case

In the absence of magnetic charges, the magnetic field is purely solenoidal.
It can thus be expressed in terms of an arbitrary vector potential \mathbf{A}

Maxwell equations

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0$$

$$\mathbf{E} = -\nabla V - j\omega\mathbf{A}$$

Remember the identities

$$\nabla \cdot (\nabla \times \mathbf{f}) = 0$$

$$\nabla \times (\nabla f) = 0$$

Potentials and Gauge Invariance

$$\mathbf{E} = -\nabla V - j\omega \mathbf{A}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

Maxwell Eqs

$$\nabla \times \mathbf{E} = -j\omega \mu \mathbf{H}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \epsilon \mathbf{E}$$

We are free to define

$$\nabla \cdot \mathbf{A}$$

the way we want

Lorenz gauge

$$\nabla \cdot \mathbf{A} = -j\omega \mu \epsilon V$$

“Freedom of the gauge”

$$\frac{1}{\mu} \nabla \times \nabla \times \mathbf{A} = \mathbf{J} + j\omega \epsilon \mathbf{E}$$



$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} + j\omega \epsilon \mu \mathbf{E}$$



$$-j\omega \mu \epsilon \nabla V - \nabla^2 \mathbf{A} = \mu \mathbf{J} - j\omega \epsilon \mu \nabla V + \omega^2 \epsilon \mu \mathbf{A}$$



$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

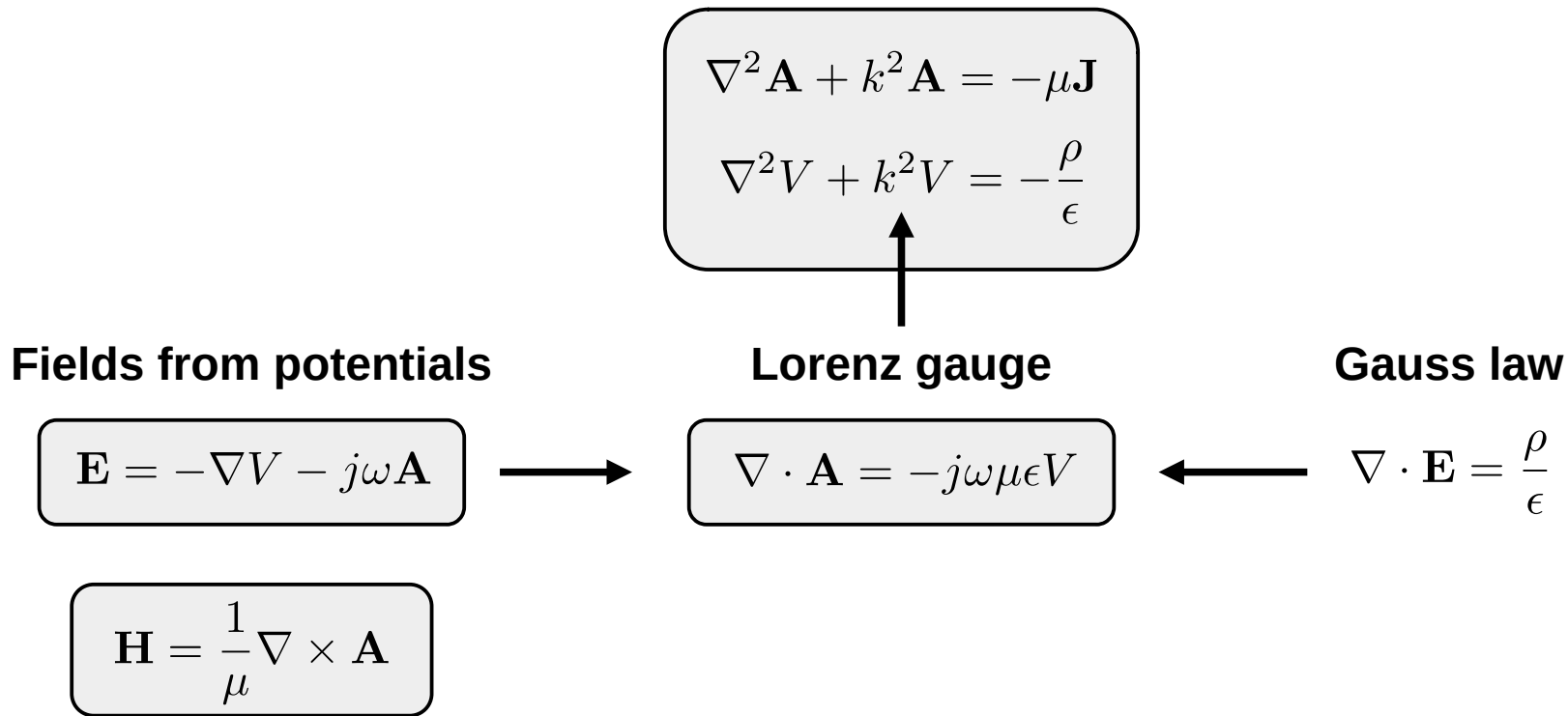
Remember the identities

$$\nabla \cdot (\nabla \times \mathbf{f}) = 0$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Helmholtz Equations of the Potentials



Full Solution for Electric Charges and Currents

$$\begin{aligned}\nabla^2 \mathbf{A} + k^2 \mathbf{A} &= -\mu \mathbf{J} \\ \nabla^2 V + k^2 V &= -\frac{\rho}{\epsilon}\end{aligned}$$

Solutions to these differential equations

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dV' \\ V(\mathbf{r}) &= \frac{1}{4\pi\epsilon} \int_V \rho(\mathbf{r}') \frac{e^{-jkR}}{R} dV'\end{aligned}$$

$$R = |\mathbf{r} - \mathbf{r}'|$$

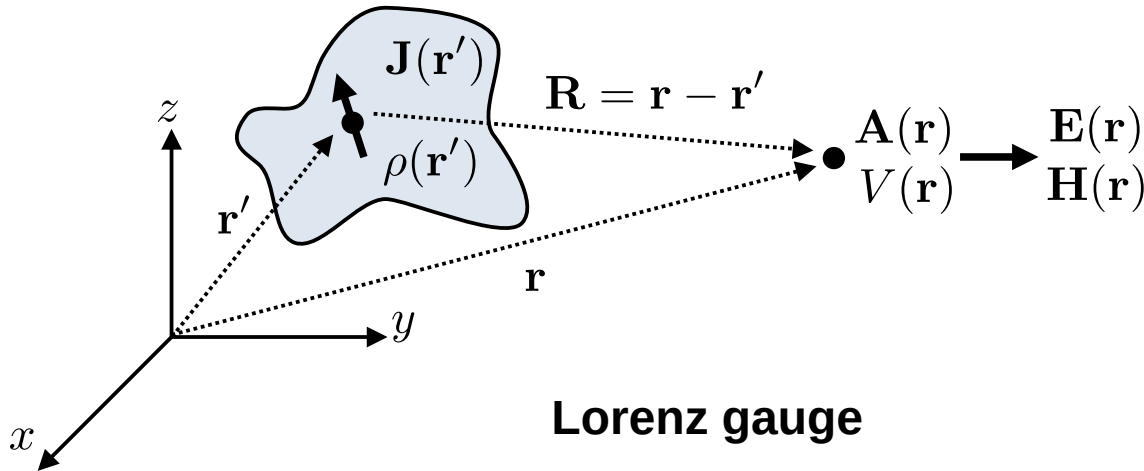
Fields from potentials

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla V - j\omega \mathbf{A}$$



$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{A}) - j\omega \mathbf{A}$$



Lorenz gauge

$$\nabla \cdot \mathbf{A} = -j\omega\mu\epsilon V$$



Generalization to Electric and Magnetic Currents

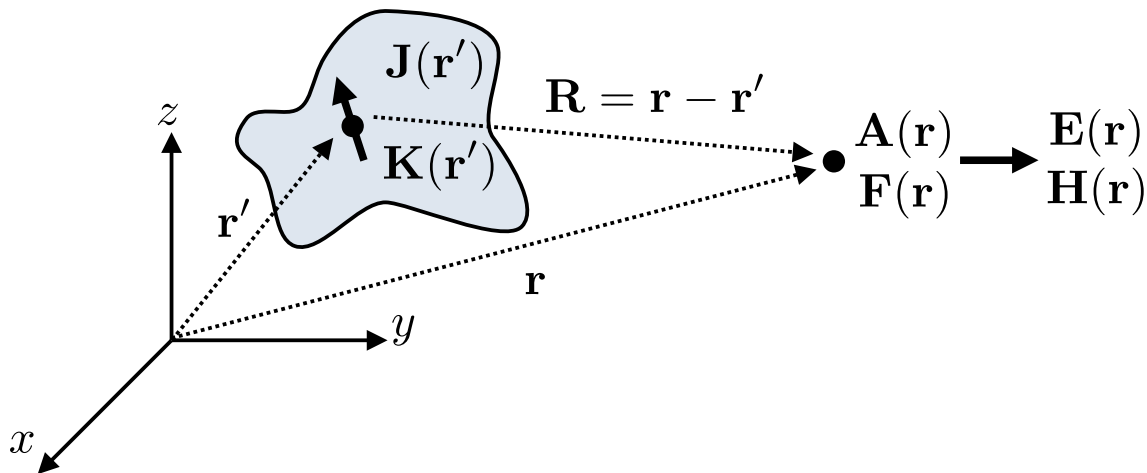
$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dV'$$

$$\mathbf{F}(\mathbf{r}) = \frac{\epsilon}{4\pi} \int_V \mathbf{K}(\mathbf{r}') \frac{e^{-jkR}}{R} dV'$$



$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{A}) - j\omega\mathbf{A} - \frac{1}{\epsilon} \nabla \times \mathbf{F}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} - j\omega\mathbf{F} + \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{F})$$



Remember duality

$$\mathbf{E} \rightarrow \mathbf{H}$$

$$\mathbf{H} \rightarrow -\mathbf{E}$$

$$\mathbf{J} \rightarrow \mathbf{K}$$

$$\epsilon \leftrightarrow \mu$$

Formulation of the Electric Green Function

Assuming no magnetic currents ($\mathbf{F} = \mathbf{0}$)

$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{A}) - j\omega\mathbf{A}$$

$$\mathbf{E} = -j\omega \left(\frac{1}{\omega^2\mu\epsilon} \nabla (\nabla \cdot \mathbf{A}) + \mathbf{A} \right)$$

$$\mathbf{E} = -j\omega \left(\mathbf{A} + \frac{\nabla\nabla}{k^2} \cdot \mathbf{A} \right)$$

$$\mathbf{E} = -j\omega \left(\bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \cdot \mathbf{A} \quad \longleftarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dV'$$

We still have $\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$

Electric Dyadic Green Function

$$\mathbf{E} = -j\omega \left(\bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \cdot \mathbf{A} \quad \longleftarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dV'$$

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \left(\bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \cdot \int \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{4\pi R} dV'$$

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int \left(\bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \frac{e^{-jkR}}{4\pi R} \cdot \mathbf{J}(\mathbf{r}') dV'$$

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int \bar{\bar{\mathbf{G}}}_e(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dV'$$

$$\bar{\bar{\mathbf{G}}}_e(\mathbf{r}, \mathbf{r}') = \left(\bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \frac{e^{-jkR}}{4\pi R}$$

Symmetry

$$\bar{\bar{\mathbf{G}}}_e(\mathbf{r}, \mathbf{r}') = \bar{\bar{\mathbf{G}}}_e(\mathbf{r}', \mathbf{r})^T$$

$$R = |\mathbf{r} - \mathbf{r}'|$$

Reciprocity

$$\bar{\bar{\mathbf{G}}}_e(\mathbf{r}, \mathbf{r}') = \bar{\bar{\mathbf{G}}}_e(\mathbf{r}', \mathbf{r})$$

Electric and Magnetic Dyadic Green Functions

$$\mathbf{E}(\mathbf{r}) = - \int \left[j\omega\mu\overline{\overline{G}}_e(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') + \nabla \times \overline{\overline{G}}_e(\mathbf{r}, \mathbf{r}') \cdot \mathbf{K}(\mathbf{r}') \right] dV'$$

$$\mathbf{H}(\mathbf{r}) = \int \left[\nabla \times \overline{\overline{G}}_e(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') - j\omega\epsilon\overline{\overline{G}}_e(\mathbf{r}, \mathbf{r}') \cdot \mathbf{K}(\mathbf{r}') \right] dV'$$

$$\overline{\overline{G}}_e(\mathbf{r}, \mathbf{r}') = \left(\overline{\overline{I}} + \frac{\nabla\nabla}{k^2} \right) \frac{e^{-jkR}}{4\pi R}$$

Sometimes we also define a magnetic Green function

$$\overline{\overline{G}}_m(\mathbf{r}, \mathbf{r}') = \nabla \times \overline{\overline{G}}_e(\mathbf{r}, \mathbf{r}')$$

Note that the cross product between the vector \mathbf{v} and the matrix M is

$$\mathbf{v} \times \overline{\overline{M}} = \mathbf{v} \times \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \times \mathbf{M}_1 & \mathbf{v} \times \mathbf{M}_2 & \mathbf{v} \times \mathbf{M}_3 \end{bmatrix}$$

Cartesian Representation of the Dyadic Green Function

$$\overline{\overline{\mathbf{G}}}_e(\mathbf{r}, \mathbf{r}') = \left(\overline{\overline{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \frac{e^{-jkR}}{4\pi R}$$

To transform this equation into Cartesian coordinates, consider the relations...

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$

$$R = |\mathbf{R}| = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\frac{\partial R}{\partial x} = \frac{x - x'}{R}, \dots \quad \frac{\partial \hat{\mathbf{R}}}{\partial x} = \frac{R\hat{\mathbf{x}} - (x - x')\hat{\mathbf{R}}}{R^2}, \dots$$

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R} \quad \nabla R = \hat{\mathbf{R}} \quad \nabla \hat{\mathbf{R}} = \frac{1}{R} \left(\overline{\overline{\mathbf{I}}} - \hat{\mathbf{R}}\hat{\mathbf{R}} \right)$$

Cartesian Representation of the Dyadic Green Function

$$\overline{\overline{\mathbf{G}}}_e(\mathbf{r}, \mathbf{r}') = \left(\overline{\overline{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \frac{e^{-jkR}}{4\pi R}$$

In Cartesian coordinates

$$\overline{\overline{\mathbf{G}}}_e(\mathbf{r}, \mathbf{r}') = \left[\left(1 - \frac{j}{kR} - \frac{1}{k^2 R^2} \right) \overline{\overline{\mathbf{I}}} - \left(1 - \frac{3j}{kR} - \frac{3}{k^2 R^2} \right) \hat{\mathbf{R}}\hat{\mathbf{R}} \right] \frac{e^{-jkR}}{4\pi R}$$

Separation into near-field, intermediate-field and far-field Green functions

Retaining only $(kR)^{-3} \rightarrow \overline{\overline{\mathbf{G}}}_{e,\text{NF}}(\mathbf{r}, \mathbf{r}') = \frac{1}{k^2 R^2} \left(-\overline{\overline{\mathbf{I}}} + 3\hat{\mathbf{R}}\hat{\mathbf{R}} \right) \frac{e^{-jkR}}{4\pi R}$

Retaining only $(kR)^{-2} \rightarrow \overline{\overline{\mathbf{G}}}_{e,\text{IF}}(\mathbf{r}, \mathbf{r}') = \frac{j}{kR} \left(-\overline{\overline{\mathbf{I}}} + 3\hat{\mathbf{R}}\hat{\mathbf{R}} \right) \frac{e^{-jkR}}{4\pi R}$

Retaining only $(kR)^{-1} \rightarrow \overline{\overline{\mathbf{G}}}_{e,\text{FF}}(\mathbf{r}, \mathbf{r}') = \left(\overline{\overline{\mathbf{I}}} - \hat{\mathbf{R}}\hat{\mathbf{R}} \right) \frac{e^{-jkR}}{4\pi R}$

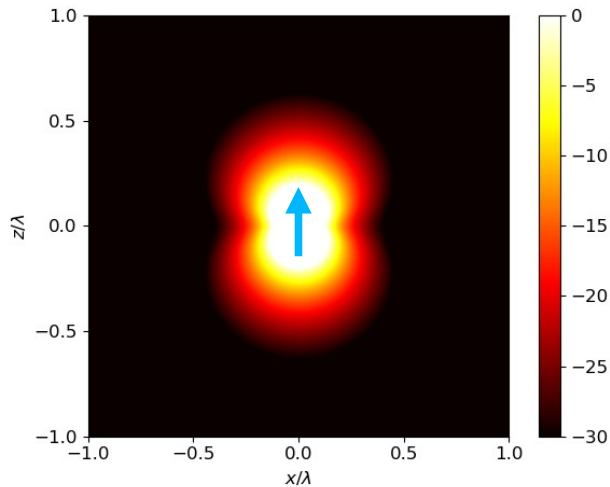
Illustration of Near, Intermediate and Far Fields

For a z-oriented point source $\mathbf{J}(\mathbf{r}') = \hat{\mathbf{z}}\delta(\mathbf{r}')$

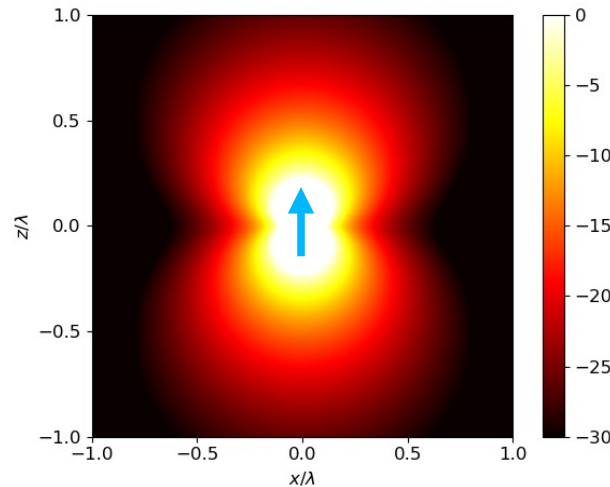
$$\mathbf{E}(\mathbf{r}) = -\omega\mu \int \left[\overline{\overline{\mathbf{G}}}_{e,\text{NF}}(\mathbf{r}, \mathbf{r}') + \overline{\overline{\mathbf{G}}}_{e,\text{IF}}(\mathbf{r}, \mathbf{r}') + \overline{\overline{\mathbf{G}}}_{e,\text{FF}}(\mathbf{r}, \mathbf{r}') \right] \cdot \mathbf{J}(\mathbf{r}') dV'$$

$$\mathbf{E}(\mathbf{r}) = -\omega\mu \left[\overline{\overline{\mathbf{G}}}_{e,\text{NF}}(\mathbf{r}, \mathbf{0}) + \overline{\overline{\mathbf{G}}}_{e,\text{IF}}(\mathbf{r}, \mathbf{0}) + \overline{\overline{\mathbf{G}}}_{e,\text{FF}}(\mathbf{r}, \mathbf{0}) \right] \cdot \hat{\mathbf{z}}$$

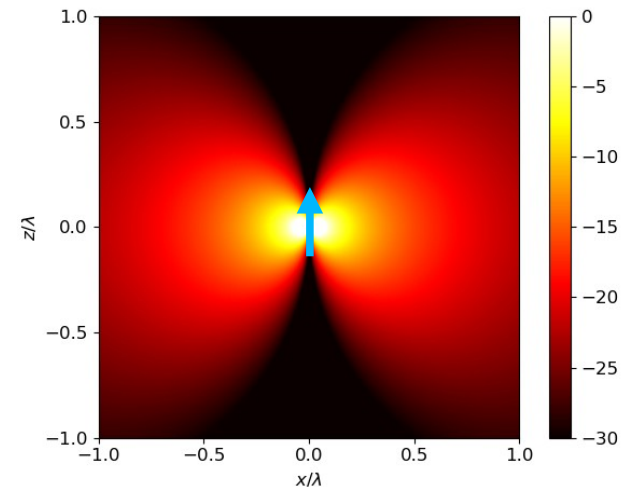
$10 \log_{10}(|\mathbf{E}_{\text{NF}}|^2)$



$10 \log_{10}(|\mathbf{E}_{\text{IF}}|^2)$



$10 \log_{10}(|\mathbf{E}_{\text{FF}}|^2)$



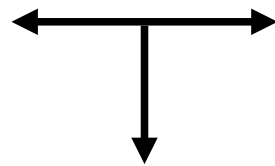
Current Point Source and Electric Dipole

Maxwell Eq

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D}$$

Constitutive relation

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

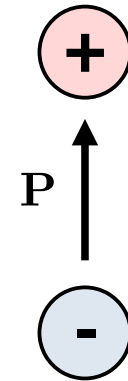


$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \epsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P}$$

$$\mathbf{J} = \frac{\partial}{\partial t} \mathbf{P}$$

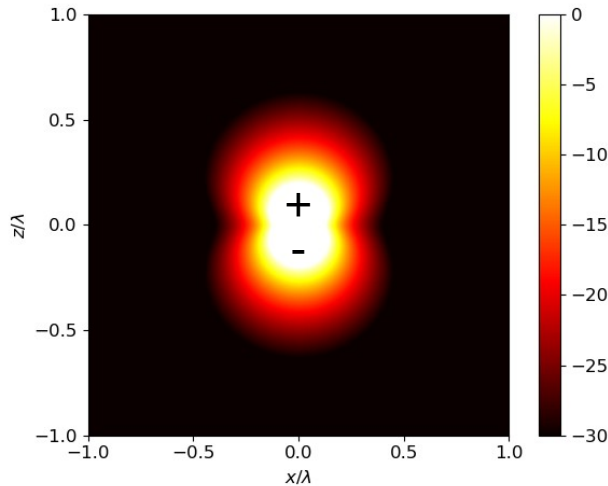
An electric current is equivalent to a time-varying dipole. In the frequency domain, we have

$$\mathbf{J} = j\omega \mathbf{P}$$



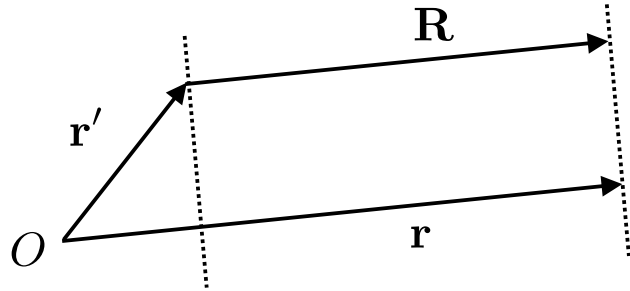
In a static case, only \mathbf{P} but no current

$10 \log_{10}(|\mathbf{E}_{\text{NF}}|^2)$



Far-Field Approximations of the Potentials

In the far-field ($r \gg r'$) \mathbf{R} is almost parallel to \mathbf{r}



$$R = |\mathbf{r} - \mathbf{r}'| = \sqrt{(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')} = \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'}$$

$$= r \sqrt{1 + \underbrace{\left(\frac{r'}{r}\right)^2}_{\approx 0} - \frac{2}{r^2} \mathbf{r} \cdot \mathbf{r}'} \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' \quad \text{where } \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dV' \approx \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_V \mathbf{J}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

$$\mathbf{F}(\mathbf{r}) = \frac{\epsilon}{4\pi} \int_V \mathbf{K}(\mathbf{r}') \frac{e^{-jkR}}{R} dV' \approx \frac{\epsilon}{4\pi} \frac{e^{-jkr}}{r} \int_V \mathbf{K}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

Equivalent to Fourier transforms

Remember that

$$\sqrt{1+x} \approx 1 + \frac{x}{2} + \dots$$

$$R \approx r$$

Far-Field Approximations of the Fields

In general the fields may be found using

$$\mathbf{A}(\mathbf{r}) \approx \mu \frac{e^{-jkr}}{4\pi r} \int_V \mathbf{J}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

$$\mathbf{F}(\mathbf{r}) \approx \epsilon \frac{e^{-jkr}}{4\pi r} \int_V \mathbf{K}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{A}) - j\omega\mathbf{A} - \frac{1}{\epsilon} \nabla \times \mathbf{F}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} - j\omega\mathbf{F} + \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{F})$$

We may derive approximations for the fields

$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{A}) - j\omega\mathbf{A} = -j\omega \left(\bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k^2} \right) \cdot \mathbf{A} \approx -j\omega \left(\bar{\bar{\mathbf{I}}} - \hat{\mathbf{r}}\hat{\mathbf{r}} \right) \cdot \mathbf{A} \approx j\omega\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{A}$$

Similar to far-field Green function

Far-Field Approximations of the Fields

$$\mathbf{A}(\mathbf{r}) \approx \mu \frac{e^{-jkr}}{4\pi r} \int_V \mathbf{J}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

The far fields are

$$\mathbf{E}_A \approx j\omega \hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{A}$$
$$\mathbf{F}(\mathbf{r}) \approx \epsilon \frac{e^{-jkr}}{4\pi r} \int_V \mathbf{K}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$
$$\mathbf{H}_F \approx j\omega \hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{F}$$

Since in the far-field, \mathbf{E} and \mathbf{H} are orthogonal, we have

$$\mathbf{E}_F = \eta \mathbf{H}_F \times \hat{\mathbf{r}} = j\omega \eta \hat{\mathbf{r}} \times \mathbf{F}$$

The total electric field is thus

$$\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = j\omega (\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{A} + \eta \hat{\mathbf{r}} \times \mathbf{F})$$

Far-Field Approximations of the Fields

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dV'$$

$$\mathbf{F}(\mathbf{r}) = \frac{\epsilon}{4\pi} \int_V \mathbf{K}(\mathbf{r}') \frac{e^{-jkR}}{R} dV'$$



$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{A}) - j\omega\mathbf{A} - \frac{1}{\epsilon} \nabla \times \mathbf{F}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} - j\omega\mathbf{F} + \frac{1}{j\omega\mu\epsilon} \nabla (\nabla \cdot \mathbf{F})$$

In the far-field, we have

$$\mathbf{A}(\mathbf{r}) \approx \mu \frac{e^{-jkr}}{4\pi r} \int_V \mathbf{J}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

$$\mathbf{F}(\mathbf{r}) \approx \epsilon \frac{e^{-jkr}}{4\pi r} \int_V \mathbf{K}(\mathbf{r}') e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$



$$\mathbf{E}_{\text{FF}} = j\omega [(\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{A}) + \eta (\hat{\mathbf{r}} \times \mathbf{F})]$$

$$\mathbf{H}_{\text{FF}} = j\omega \left[(\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{F}) - \frac{1}{\eta} (\hat{\mathbf{r}} \times \mathbf{A}) \right]$$

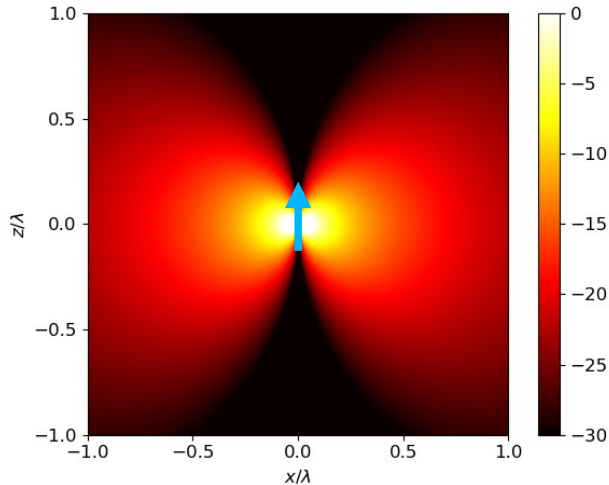
Far-Field Approximations of the Fields

$$\mathbf{P}(\mathbf{r}') = \mathbf{p}\delta(\mathbf{r}') \longrightarrow \mathbf{J}(\mathbf{r}') = j\omega\mathbf{P} = j\omega\mathbf{p}\delta(\mathbf{r}')$$

$$\mathbf{A}(\mathbf{r}) \approx \mu \frac{e^{-jkr}}{4\pi r} \int_V \mathbf{J}(\mathbf{r}') e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'} dV' = j\omega\mu\mathbf{p} \frac{e^{-jkr}}{4\pi r}$$

z-oriented dipole

$$10 \log_{10}(|\mathbf{E}_{\text{FF}}|^2)$$



$$\mathbf{E}_{\text{FF}} = j\omega (\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{A}) = \omega^2 \mu \frac{e^{-jkr}}{4\pi r} (\hat{\mathbf{r}} \times \mathbf{p} \times \hat{\mathbf{r}})$$

$$\mathbf{E}_{\text{FF}} = \omega^2 \mu \frac{e^{-jkr}}{4\pi r} (\hat{\mathbf{r}} \times \mathbf{p} \times \hat{\mathbf{r}})$$

$$\mathbf{H}_{\text{FF}} = \hat{\mathbf{r}} \times \mathbf{E}_{\text{FF}} / \eta = \frac{\omega^2 \mu e^{-jkr}}{\eta 4\pi r} (\hat{\mathbf{r}} \times \mathbf{p})$$

Far Fields for Electric and Magnetic Dipoles

Electric dipole far fields

$$\mathbf{E}_{\text{FF}} = \omega^2 \mu \frac{e^{-jkr}}{4\pi r} (\hat{\mathbf{r}} \times \mathbf{p} \times \hat{\mathbf{r}})$$

$$\mathbf{H}_{\text{FF}} = \frac{\omega^2 \mu}{\eta} \frac{e^{-jkr}}{4\pi r} (\hat{\mathbf{r}} \times \mathbf{p})$$

By duality



$$\mathbf{E} \rightarrow \mathbf{H}$$

$$\mathbf{H} \rightarrow -\mathbf{E}$$

$$\mathbf{p} \rightarrow \mu \mathbf{m}$$

$$\epsilon \leftrightarrow \mu$$

Magnetic dipole far fields

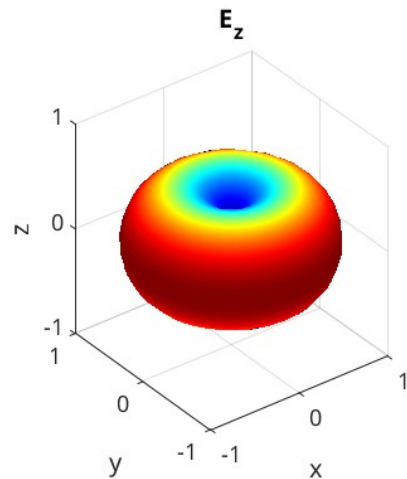
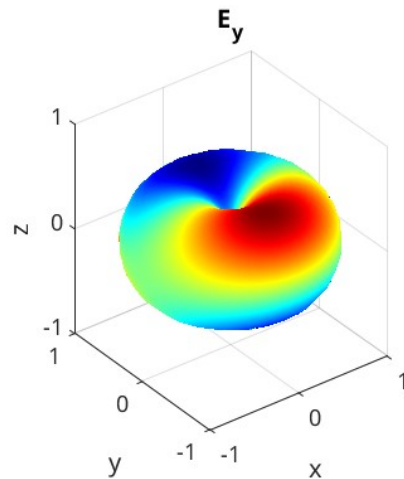
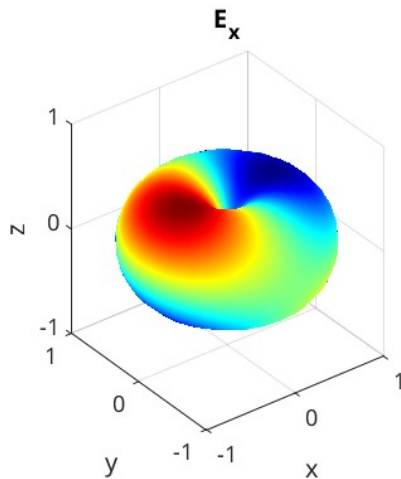
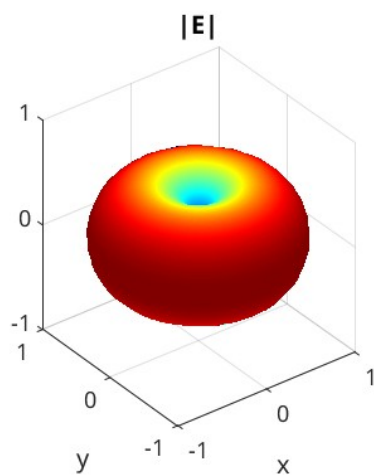
$$\mathbf{E}_{\text{FF}} = \frac{\omega^2 \mu}{c} \frac{e^{-jkr}}{4\pi r} (\mathbf{m} \times \hat{\mathbf{r}})$$

$$\mathbf{H}_{\text{FF}} = k^2 \frac{e^{-jkr}}{4\pi r} (\hat{\mathbf{r}} \times \mathbf{m} \times \hat{\mathbf{r}})$$

Illustrations for z-Oriented Dipoles

$$\mathbf{E}_{\text{FF}} \propto \hat{\mathbf{r}} \times \mathbf{p} \times \hat{\mathbf{r}}$$

$$\text{with } \mathbf{p} = \hat{\mathbf{z}}$$



$$\mathbf{E}_{\text{FF}} \propto \mathbf{m} \times \hat{\mathbf{r}}$$

$$\text{with } \mathbf{m} = \hat{\mathbf{z}}$$

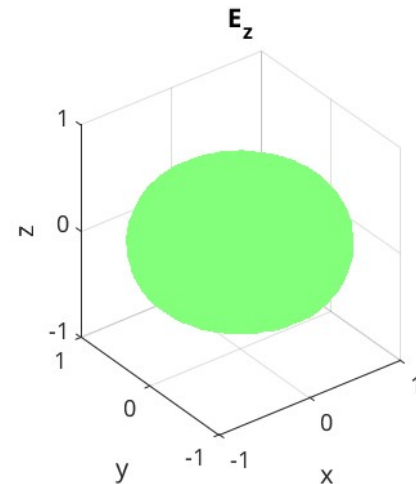
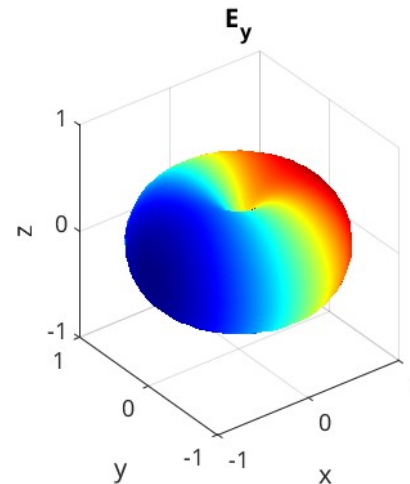
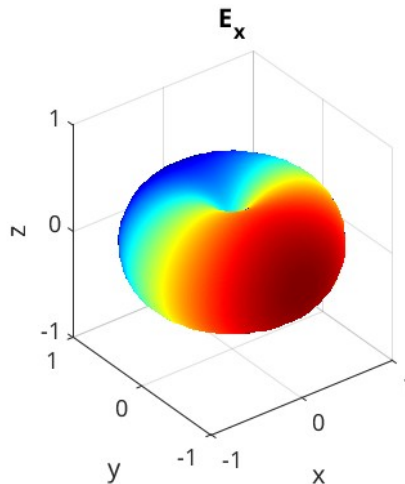
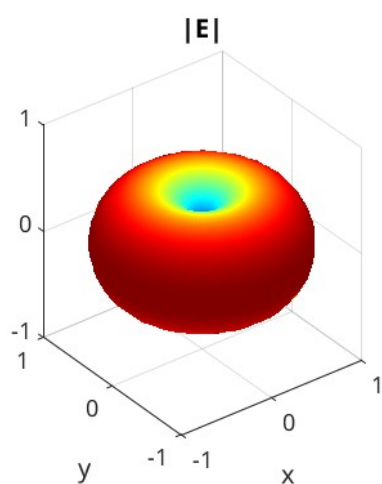
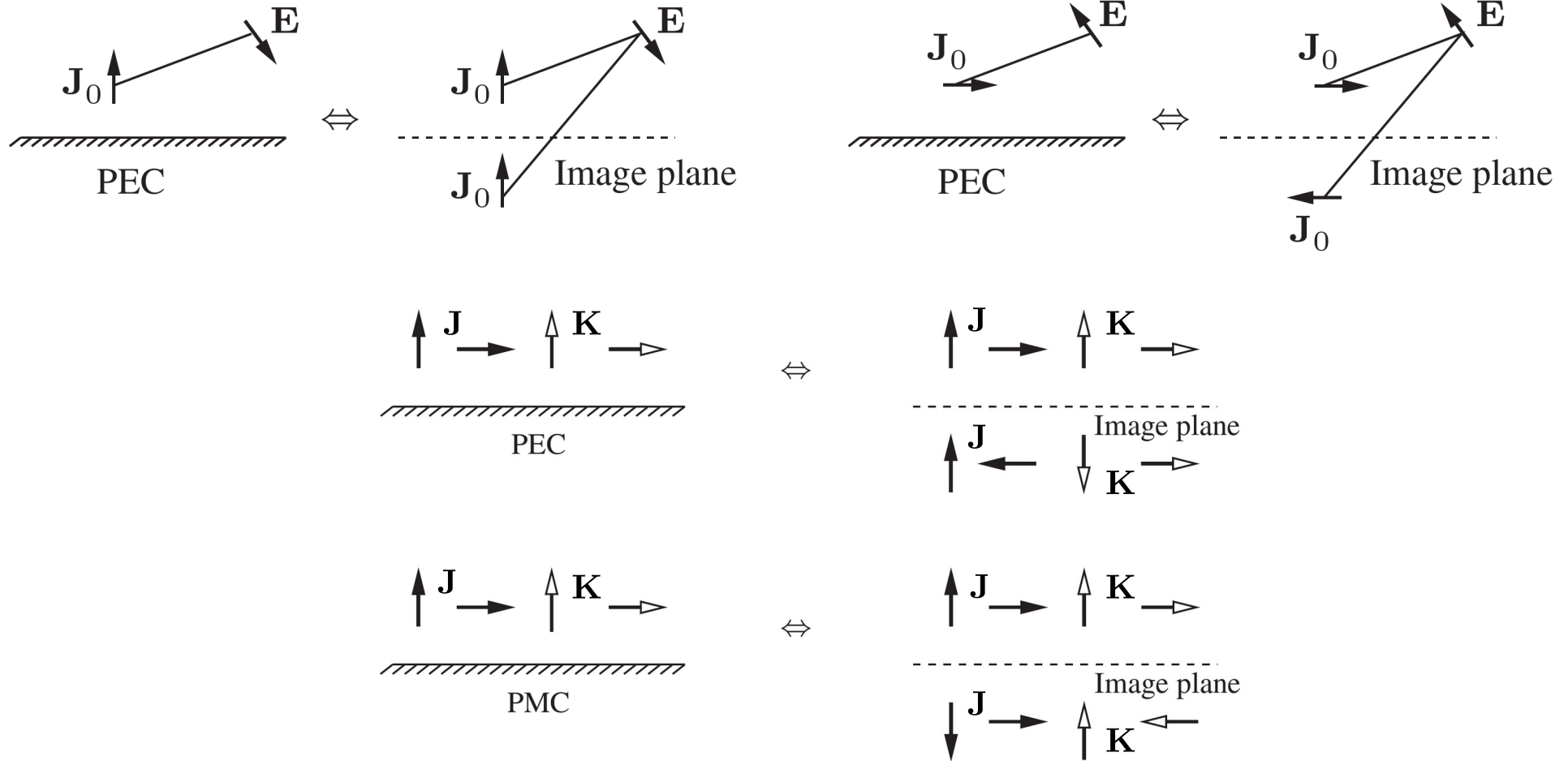
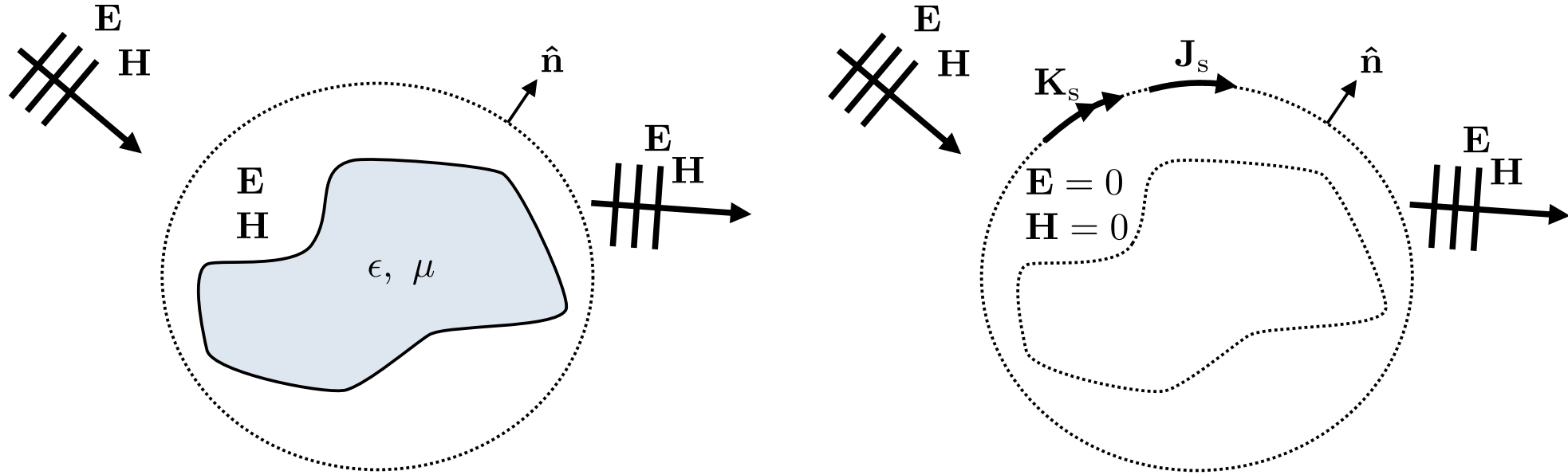


Image Theory



Surface Equivalence Principle

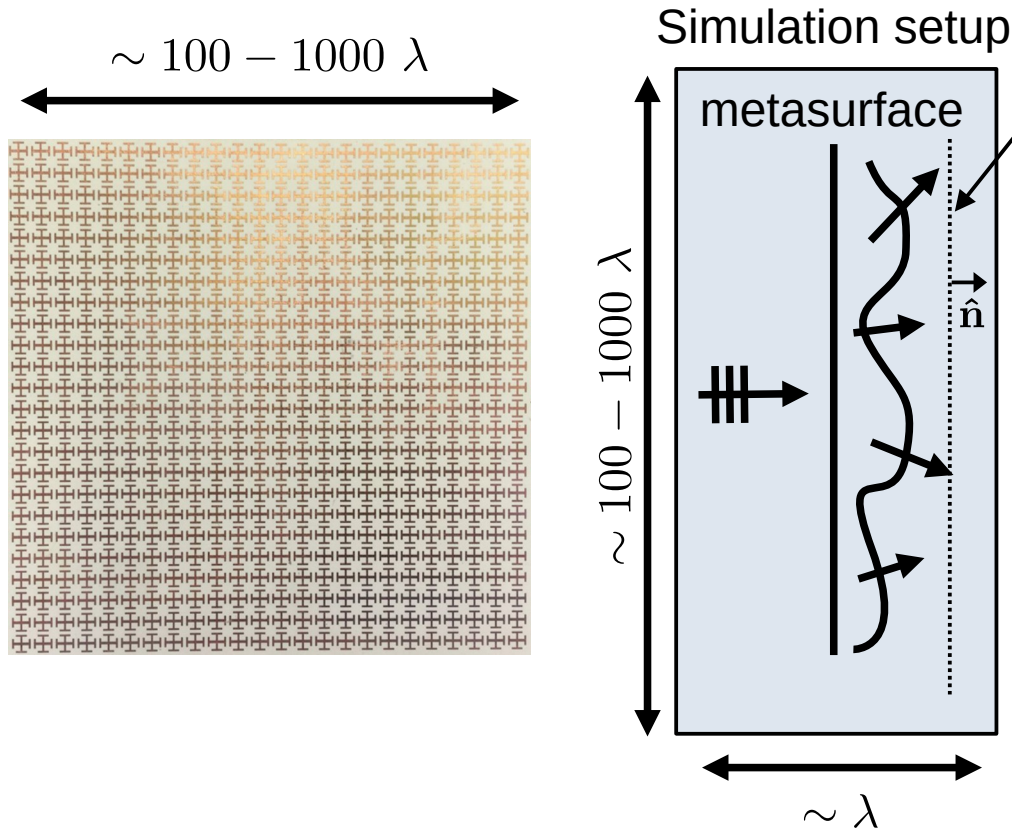


$$\mathbf{J}_s = \hat{\mathbf{n}} \times (\mathbf{H} - 0)$$

$$\mathbf{K}_s = (\mathbf{E} - 0) \times \hat{\mathbf{n}}$$

Example of Application: Simulations

How to simulate a very large metasurface?



- 1) Find “aperture” fields on near-field plane
- 2) Transform fields into equivalent currents

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}_a \quad \mathbf{K}_s = \mathbf{E}_a \times \hat{\mathbf{n}}$$

- 3) Find vector potentials

$$\mathbf{A}(\mathbf{r}) \approx \mu \frac{e^{-jk r}}{4\pi r} \int_V \mathbf{J}(\mathbf{r}') e^{jk \hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

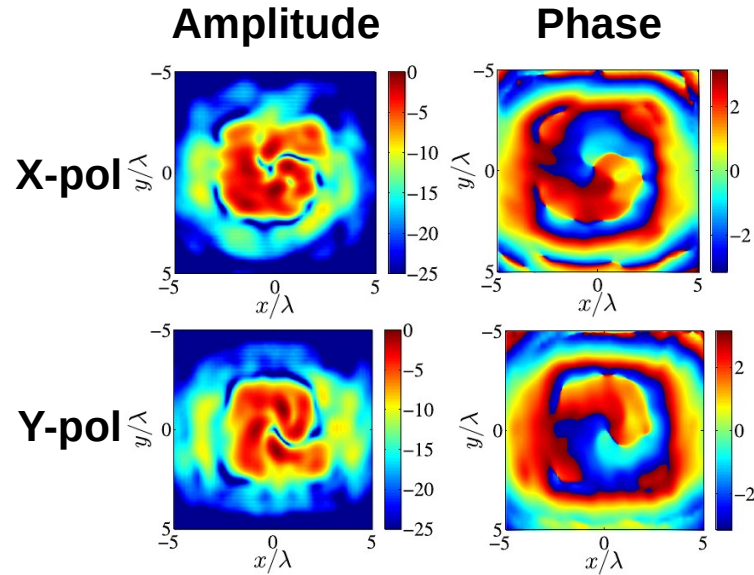
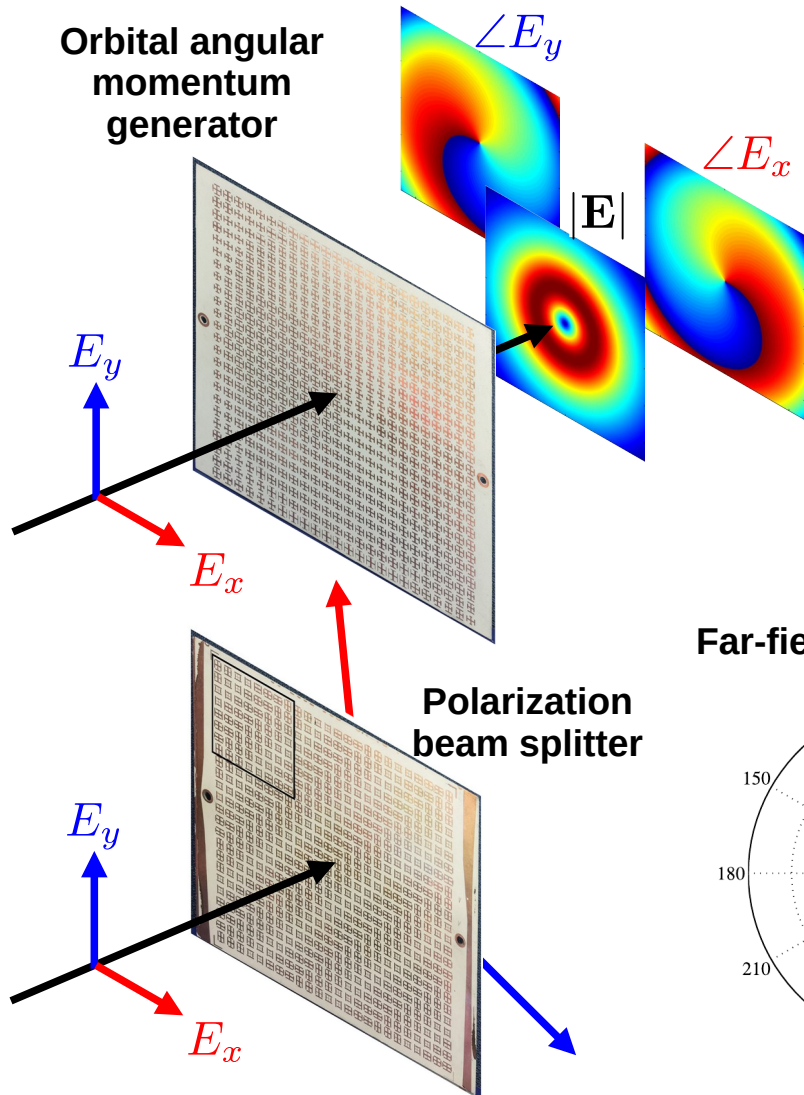
$$\mathbf{F}(\mathbf{r}) \approx \epsilon \frac{e^{-jk r}}{4\pi r} \int_V \mathbf{K}(\mathbf{r}') e^{jk \hat{\mathbf{r}} \cdot \mathbf{r}'} dV'$$

- 4) Compute far fields

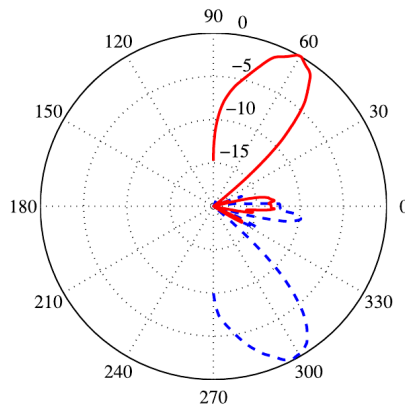
$$\mathbf{E}_{\text{FF}} = j\omega [(\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{A}) + \eta (\hat{\mathbf{r}} \times \mathbf{F})]$$

$$\mathbf{H}_{\text{FF}} = j\omega \left[(\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{F}) - \frac{1}{\eta} (\hat{\mathbf{r}} \times \mathbf{A}) \right]$$

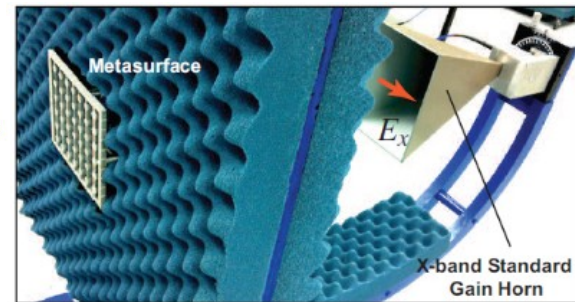
Example of Application: Measurements



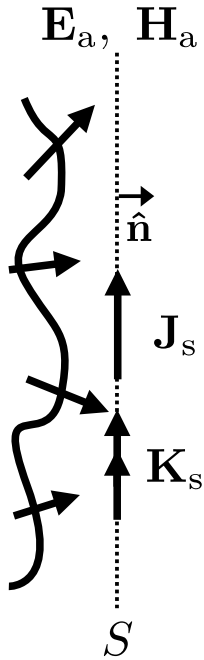
Far-field radiation pattern



A probe antenna is used to scan the near field



Example of Application: Measurements



Equivalent currents

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}_a$$

$$\mathbf{K}_s = \mathbf{E}_a \times \hat{\mathbf{n}}$$



$$\mathbf{A}(\mathbf{r}) \approx \mu \frac{e^{-jkr}}{4\pi r} \hat{\mathbf{n}} \times \int_S \mathbf{H}_a e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dS'$$

$$\mathbf{F}(\mathbf{r}) \approx -\epsilon \frac{e^{-jkr}}{4\pi r} \hat{\mathbf{n}} \times \int_S \mathbf{E}_a e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dS'$$

Equivalent to 2D Fourier transforms



$$\mathbf{E}_{\text{FF}} = j\omega [(\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{A}) + \eta (\hat{\mathbf{r}} \times \mathbf{F})]$$

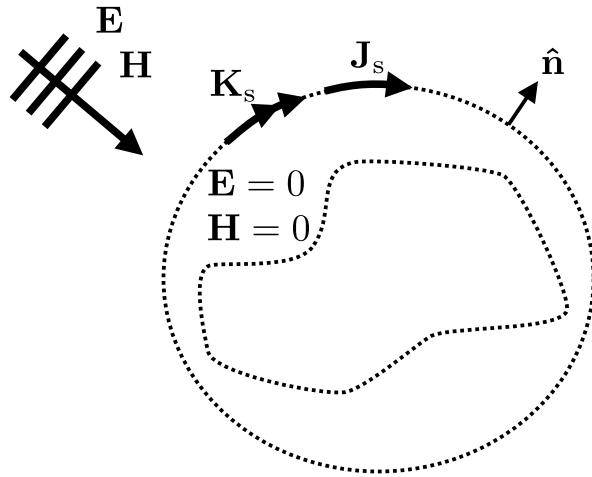


$$\mathbf{E}_{\text{FF}} = -jk \frac{e^{-jkr}}{4\pi r} \hat{\mathbf{r}} \times \int_S [\hat{\mathbf{n}} \times \mathbf{E}_a - \eta \hat{\mathbf{r}} \times (\hat{\mathbf{n}} \times \mathbf{H}_a)] e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dS'$$

The far field is a Fourier transform of the aperture fields

Alternative Formulations

What if we do not know either E_a or H_a ?



Since the fields inside the volume defined by S are null, we are free to place a PEC or PMC wall inside of it, which cancels out one of the two currents by image theory.

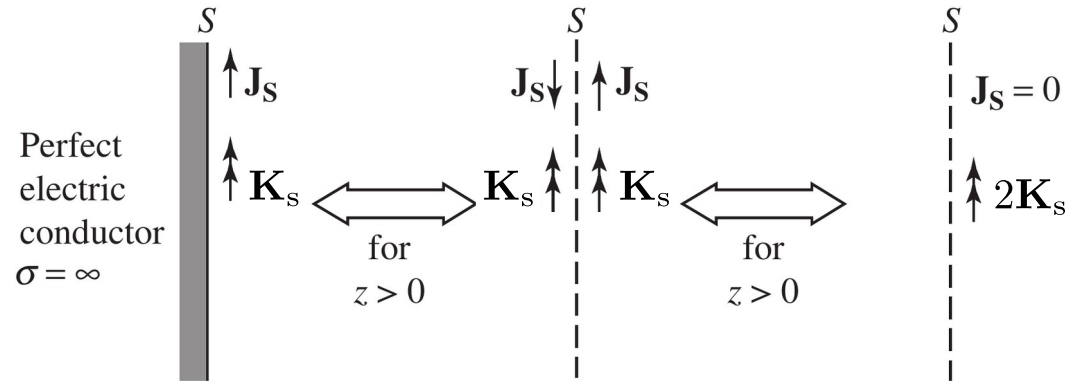
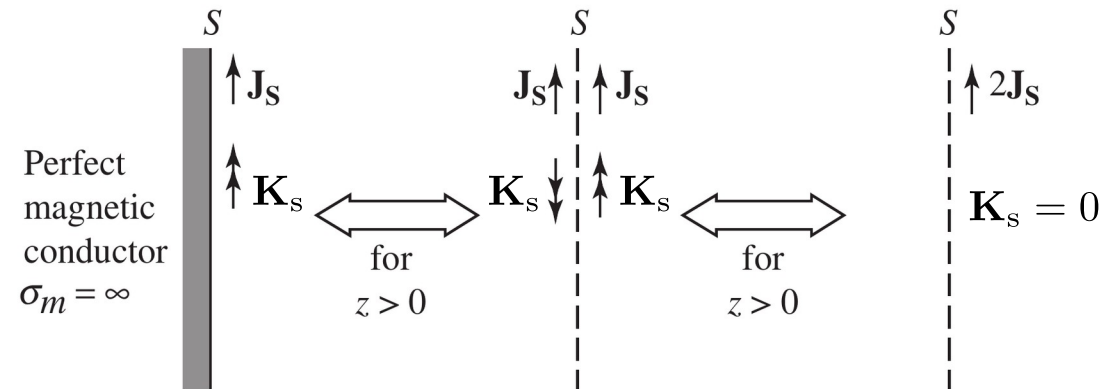


Image theory



Alternative Formulations

If both the electric and magnetic fields are known

$$\mathbf{E}_{\text{FF}} = -jk \frac{e^{-jkr}}{4\pi r} \hat{\mathbf{r}} \times \int_S [\hat{\mathbf{n}} \times \mathbf{E}_a - \eta \hat{\mathbf{r}} \times (\hat{\mathbf{n}} \times \mathbf{H}_a)] e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dS'$$

If only the electric field is known

$$\mathbf{E}_{\text{FF}} = -jk \frac{e^{-jkr}}{2\pi r} \hat{\mathbf{r}} \times \hat{\mathbf{n}} \times \int_S \mathbf{E}_a e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dS'$$

If only the magnetic field is known

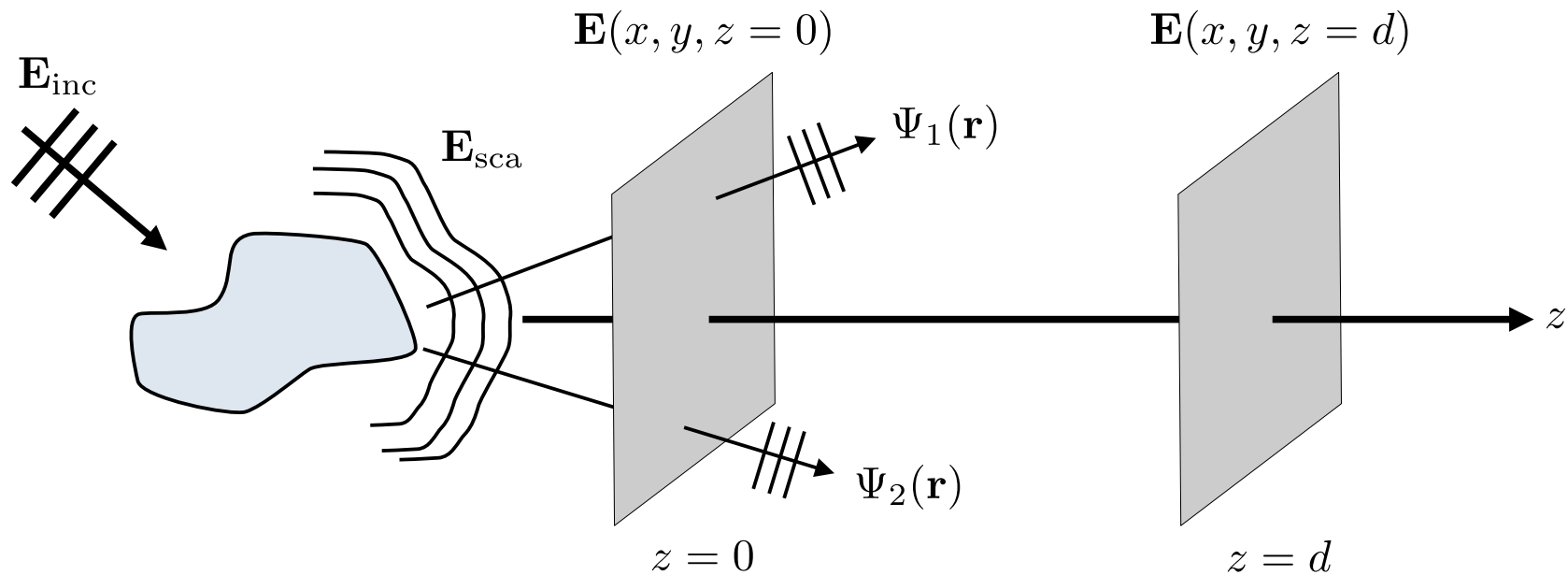
$$\mathbf{E}_{\text{FF}} = jk\eta \frac{e^{-jkr}}{2\pi r} \hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \hat{\mathbf{n}} \times \int_S \mathbf{H}_a e^{jk\hat{\mathbf{r}} \cdot \mathbf{r}'} dS'$$

What Have We Learned So Far....

- To decouple scalar and vector potentials, we consider the Lorenz gauge. This leads to the vector potentials being only related to the induced currents and not the charges anymore
- To find the fields scattered by a system, we typically consider the induced electric and magnetic currents. Integrating these currents gives us the vector potentials, which allows us to compute the fields
- We can skip the consideration of the potentials by directly using the dyadic Green function
- In Cartesian coordinates, the dyadic Green function can be easily split into near-field and far-field terms by retaining the $1/r^3$ and $1/r$ terms. These terms present very different radiation pattern and characteristics
- In the far-field, the scattered fields are proportional to Fourier transforms of the currents
- You should be able to plot the radiation pattern of an electric/magnetic dipole including the polarization of the fields
- Image theory is derived by considering the boundary conditions at the interface with a PEC (tangential electric field must be zero). On a PMC the tangential magnetic fields are zeros
- The surface equivalence principle allows us to transform fields into currents by setting the fields inside the volume defined by the surface to zero
- Combining the far-field radiation integrals with image theory and the surface equivalence principle allows us to define near-to-far field transformations that are useful in both simulation and experimental situations

Angular Spectrum Representation

Angular Spectrum Representation



$$\Psi_1(\mathbf{r}) = \mathbf{A}_1 e^{-j(k_{1,x}x + k_{1,y}y + k_{1,z}z)}$$

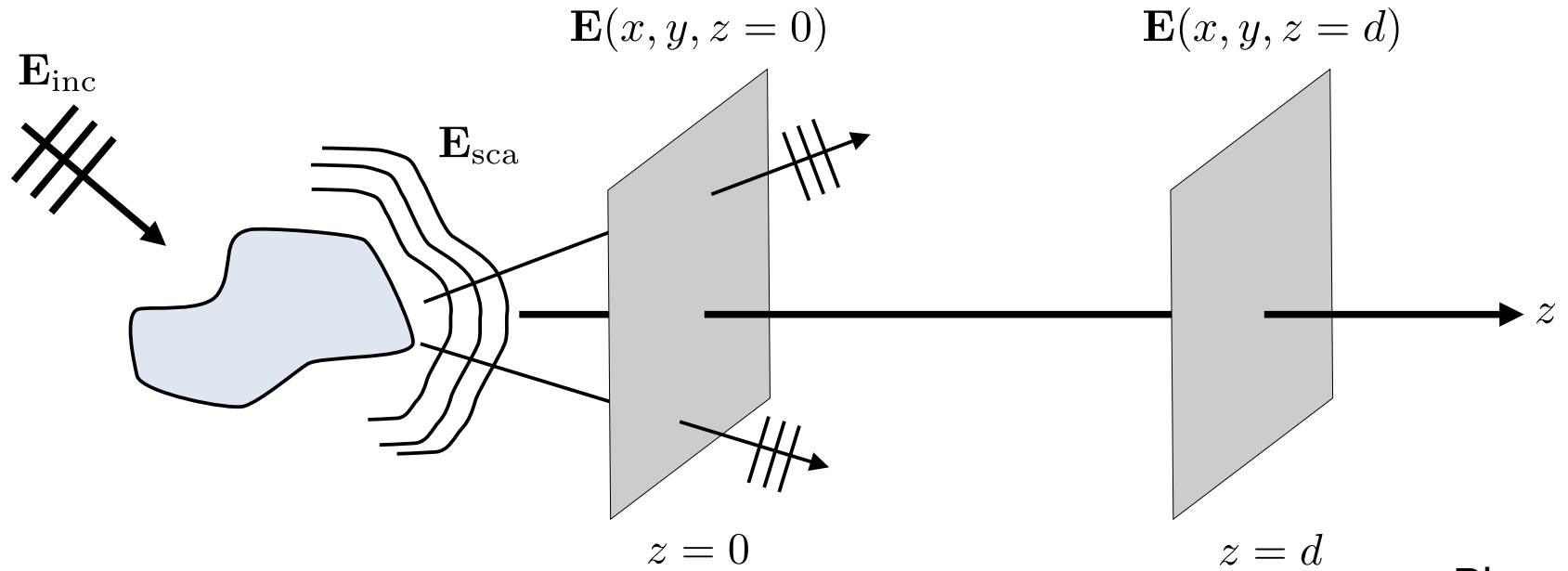
$$\mathbf{E}(x, y, z=0) = \Psi_1(z=0) + \Psi_2(z=0) + \dots$$

$$\Psi_2(\mathbf{r}) = \mathbf{A}_2 e^{-j(k_{2,x}x + k_{2,y}y + k_{2,z}z)}$$

$$\mathbf{E}(x, y, z=d) = \Psi_1(z=d) + \Psi_2(z=d) + \dots$$

To find the field \mathbf{E} at $z=d$, we only need to know the z -accumulated phase of each waves Ψ .

Plane Wave Expansion in Fourier Space



Fourier transform

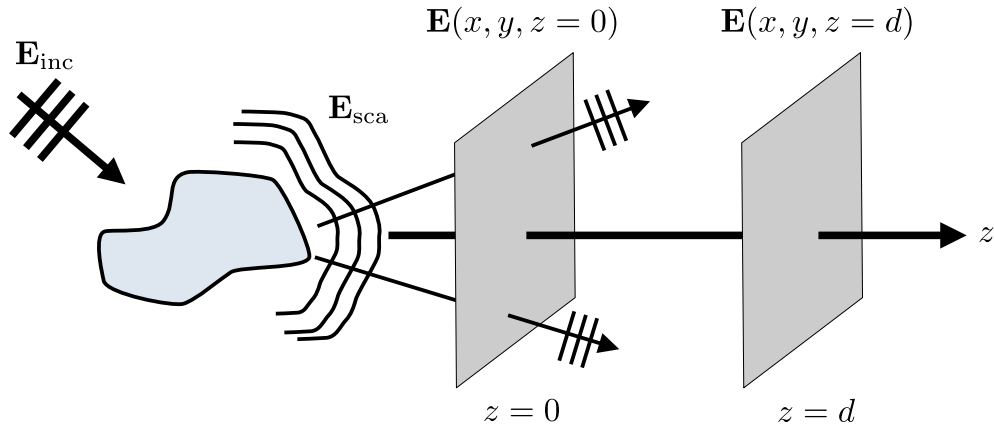
$$\mathbf{E}(x, y, z) = \iint_{-\infty}^{\infty} \tilde{\mathbf{E}}(k_x, k_y, z) e^{-j(k_x x + k_y y)} dk_x dk_y$$

Plane wave expansion

Inverse Fourier transform

$$\tilde{\mathbf{E}}(k_x, k_y, z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \mathbf{E}(x, y, z) e^{j(k_x x + k_y y)} dx dy$$

Angular Spectrum Representation



The field amplitudes at $z=0$ is

$$\tilde{\mathbf{E}}(k_x, k_y, 0) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \mathbf{E}(x, y, 0) e^{j(k_x x + k_y y)} dx dy$$

known

For each plane wave, the k_z is simply

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

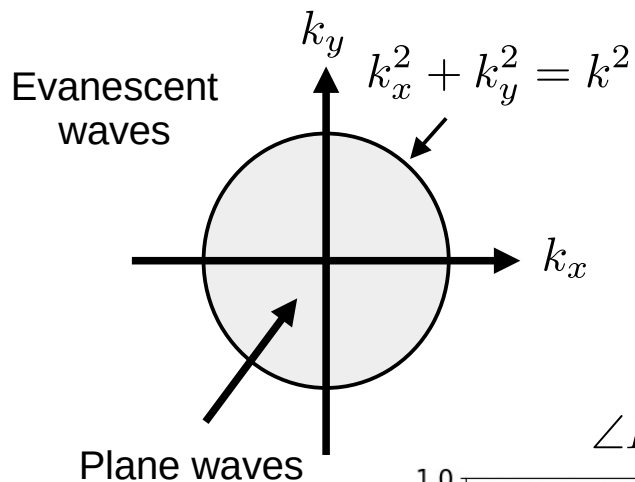
The field amplitudes at $z=d$ is

$$\tilde{\mathbf{E}}(k_x, k_y, d) = \tilde{\mathbf{E}}(k_x, k_y, 0) e^{-jk_z d}$$

The total field at $z=d$ is then

$$\mathbf{E}(x, y, d) = \iint_{-\infty}^{\infty} \tilde{\mathbf{E}}(k_x, k_y, 0) e^{-j(k_x x + k_y y + k_z d)} dk_x dk_y$$

Field Propagator

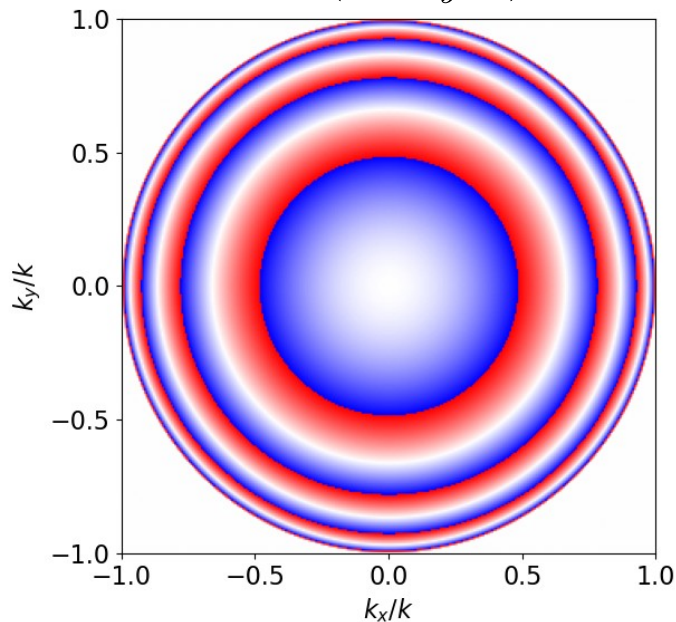


$$\mathbf{E}(x, y, d) = \iint_{-\infty}^{\infty} \tilde{\mathbf{E}}(k_x, k_y, 0) \tilde{H}(k_x, k_y, d) e^{-j(k_x x + k_y y)} dk_x dk_y$$

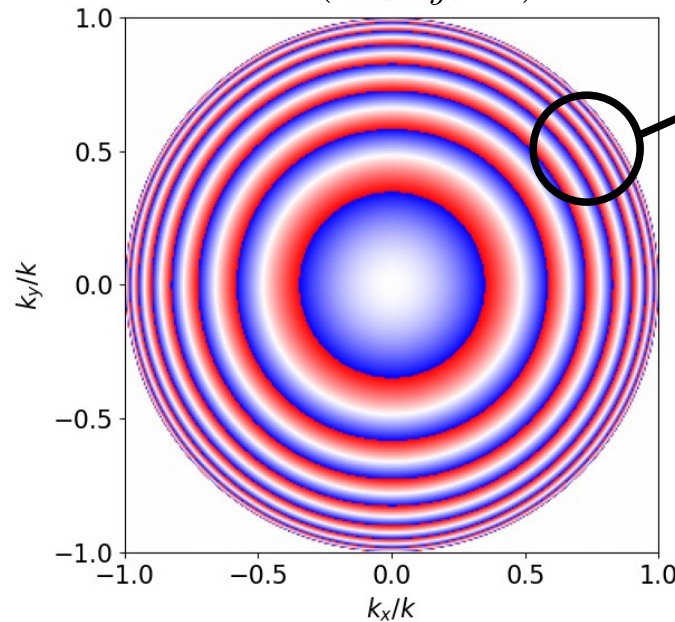
$$\tilde{H}(k_x, k_y, d) = e^{-jk_z d}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\angle \tilde{H}(k_x, k_y, d)$$



$$\angle \tilde{H}(k_x, k_y, 2d)$$



As d increases, the higher the required resolution

Field Propagator in the Far Field

$$\tilde{H}(k_x, k_y, d) = e^{-jk_z d}$$

In the paraxial approximation, we have that $k_z^2 > k_x^2 + k_y^2$

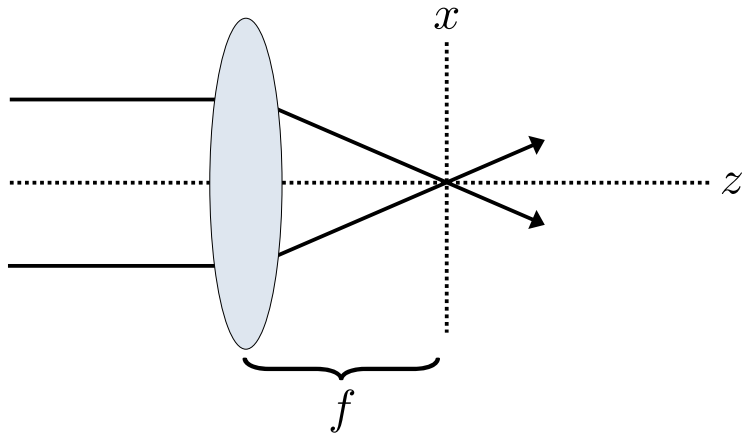
$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} = k\sqrt{1 - \frac{k_x^2 + k_y^2}{k^2}} \approx k - \frac{k_x^2 + k_y^2}{2k}$$

$$\tilde{H}(k_x, k_y, z) \approx e^{-jkz} e^{j\frac{k_x^2 + k_y^2}{2k}z}$$



$$\mathbf{E}(x, y, d) \approx e^{-jkz} \iint_{-\infty}^{\infty} \tilde{\mathbf{E}}(k_x, k_y, 0) e^{-j\left(k_x x + k_y y - \frac{k_x^2 + k_y^2}{2k}d\right)} dk_x dk_y$$

Transfer Function of a Lens



What is the transmission function of a lens?

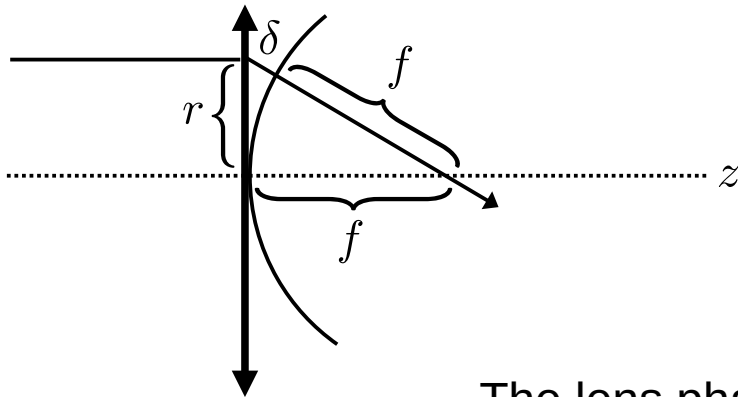
We assume it is transparent: $|T(x, y)| = 1$

What about its phase?

$$\delta + f = \sqrt{f^2 + r^2}$$

$$\delta = \sqrt{f^2 + r^2} - f = f \sqrt{1 + \frac{r^2}{f^2}} - f$$

$$\delta \approx \frac{r^2}{2f} = \frac{x^2 + y^2}{2f}$$



The lens phase shift is

$$\phi = k\delta$$

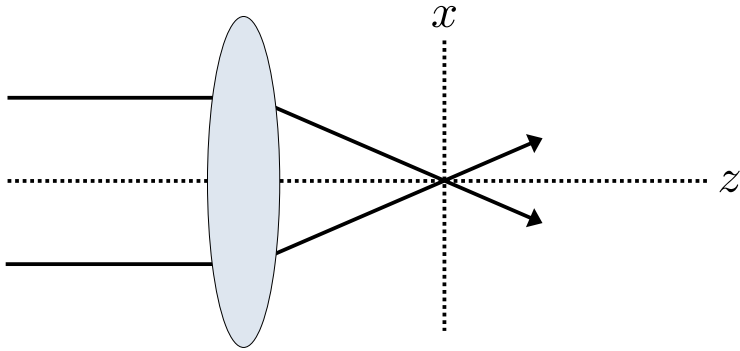


$$T(x, y) \approx e^{j\frac{k}{2f}(x^2 + y^2)}$$

Remember that

$$\sqrt{1+x} \approx 1 + \frac{x}{2} + \dots$$

Fourier Transform Properties of a Lens



Transfer function of a lens

$$T(x, y) \approx e^{j \frac{k}{2f} (x^2 + y^2)}$$

The field transmitted by the lens is

$$\mathbf{E}_t(x, y, 0) = T(x, y) \mathbf{E}_i(x, y, 0) = \mathbf{E}_0 e^{j \frac{k}{2f} (x^2 + y^2)} e^{-j(k'_x x + k'_y y)}$$

Incident field at the lens

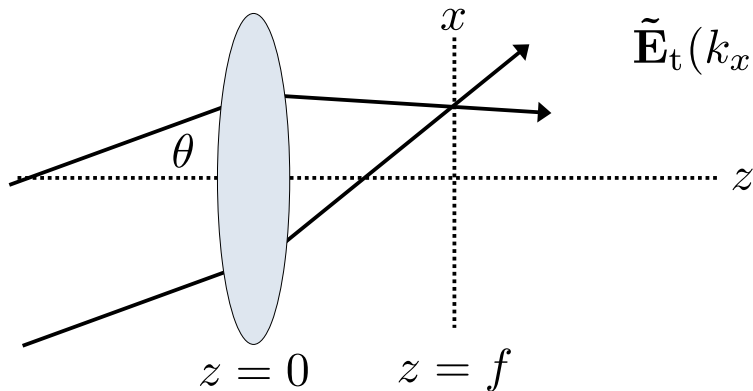
$$\mathbf{E}_i(x, y, 0) = \mathbf{E}_0 e^{-j(k'_x x + k'_y y)}$$

The corresponding inverse Fourier transform is

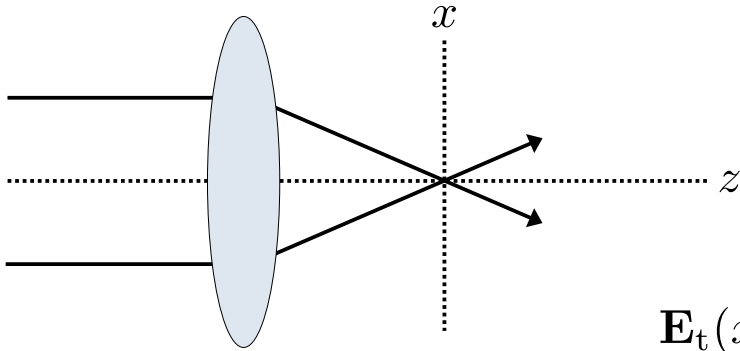
$$\tilde{\mathbf{E}}_t(k_x, k_y, 0) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \mathbf{E}_0 e^{j \frac{k}{2f} (x^2 + y^2)} e^{-j(k'_x x + k'_y y)} e^{j(k_x x + k_y y)} dx dy$$



$$\tilde{\mathbf{E}}_t(k_x, k_y, 0) = \mathbf{E}_0 \frac{jf}{2k\pi} e^{-\frac{jf}{2k} [(k_x - k'_x)^2 + (k_y - k'_y)^2]}$$

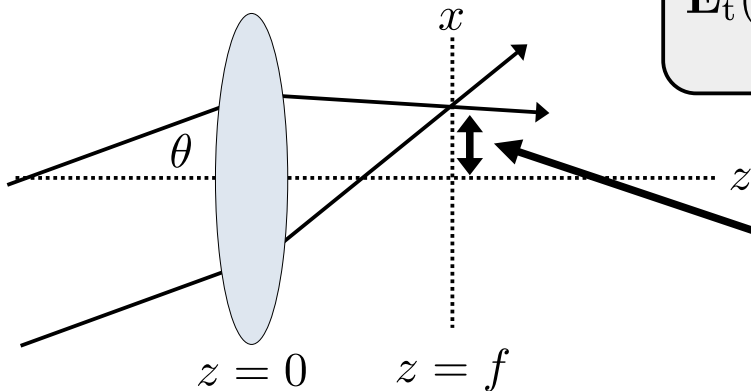


Fourier Transform Properties of a Lens



Incident field at the lens

$$\mathbf{E}_i(x, y, 0) = \mathbf{E}_0 e^{-j(k'_x x + k'_y y)}$$



$$\tilde{\mathbf{E}}_t(k_x, k_y, 0) = \mathbf{E}_0 \frac{jf}{2k\pi} e^{-\frac{jf}{2k} [(k_x - k'_x)^2 + (k_y - k'_y)^2]}$$

The transmitted field at the focal plane is

$$\mathbf{E}_t(x, y, f) = e^{-jkz} \iint_{-\infty}^{\infty} \tilde{\mathbf{E}}_t(k_x, k_y, 0) e^{-j\left(k_x x + k_y y - \frac{k_x^2 + k_y^2}{2k} f\right)} dk_x dk_y$$

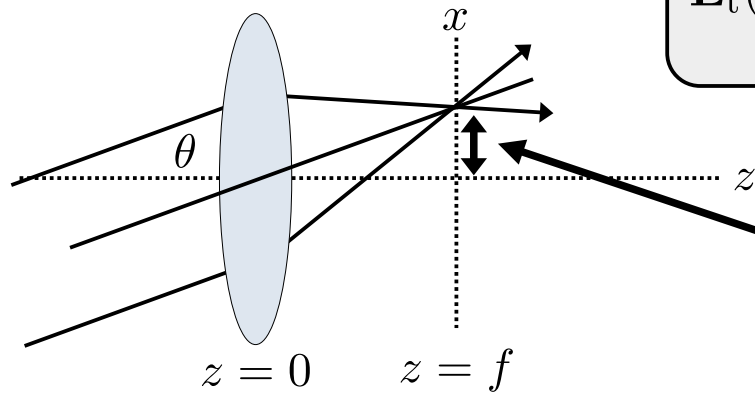


$$\mathbf{E}_t(x, y, f) = jf\lambda\mathbf{E}_0 e^{-j\frac{f}{2k}(2k^2 + k_x'^2 + k_y'^2)} \delta\left(x - \frac{f}{k}k'_x\right) \delta\left(y - \frac{f}{k}k'_y\right)$$

So a lens maps the spatial frequencies of the incident wave to the focal plane positions

$$x = \frac{f}{k}k'_x \quad y = \frac{f}{k}k'_y$$

Fourier Transform Properties of a Lens



$$\mathbf{E}_t(x, y, f) = j f \lambda \mathbf{E}_0 e^{-j \frac{f}{2k} (2k^2 + k_x'^2 + k_y'^2)} \delta \left(x - \frac{f}{k} k_x' \right) \delta \left(y - \frac{f}{k} k_y' \right)$$

So a lens maps the spatial frequencies of the incident wave to the focal plane positions

$$x = \frac{f}{k} k_x' \quad y = \frac{f}{k} k_y'$$

By trigonometry, we find

$$x = f \tan \theta$$



$$x \approx f \theta$$



$$x = \frac{f}{k} k_x' = f \sin \theta$$

If $k_y = 0$

In the small angle approximation, we have that

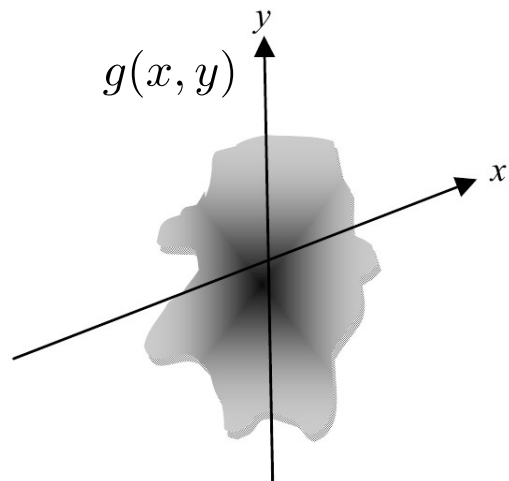
$$\tan \theta \approx \sin \theta \approx \theta$$

What Have We Learned So Far....

- The fields scattered by an object may be decomposed in terms of plane waves. This decomposition is simply a Fourier transform
- To propagate a field along the z -direction by a distance d , we just have to phase shift each of the plane wave composing (Fourier components) the original field by $k_z d$, where k_z is obtained from k_x and k_y via the dispersion relation
- The propagated field at position d is then reconstructed by taking the inverse Fourier transform
- This type of field propagation (angular spectrum method) is exact. This is not an approximation
- It can nonetheless be approximated (paraxial approximation), which allows us to find analytical expressions of the propagated fields
- An application example is that of a lens, which behaves as an optical Fourier transformer that maps the (k_x, k_y) components of the incident wave to (x, y) positions in the focal plane (also called Fourier plane)
- This result is obtained by using the approximated propagation method with the transfer function of lens (parabolic phase profile)

How to Numerically Implement Fourier Propagation

2D analytical function



Shannon-Nyquist sampling theorem

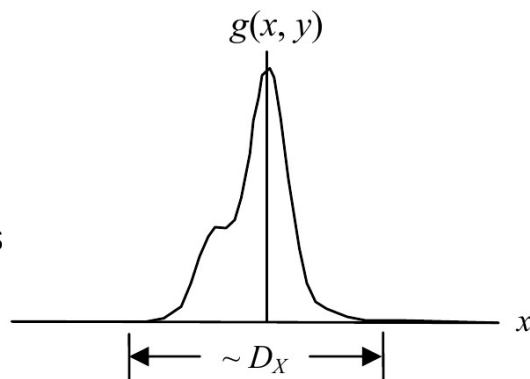
$$\Delta x < \frac{1}{2B_X}$$

$$\Delta y < \frac{1}{2B_Y}$$

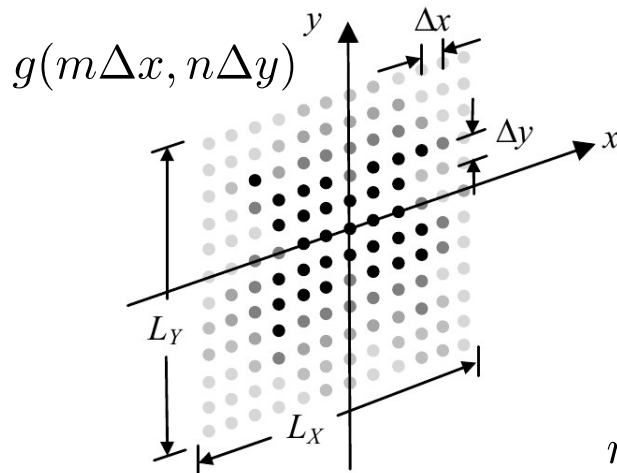
Support (D)
span of significant values

$$D_X < L_X$$

$$D_Y < L_Y$$



2D sampled function

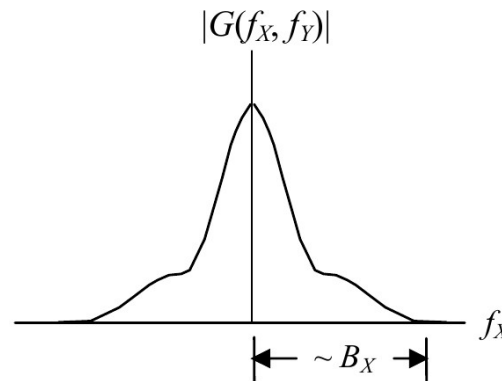


$$L_X = M\Delta x$$

$$L_Y = N\Delta y$$

$$m = \left[-\frac{M}{2}, \dots, \frac{M}{2} \right]$$

$$n = \left[-\frac{N}{2}, \dots, \frac{N}{2} \right]$$

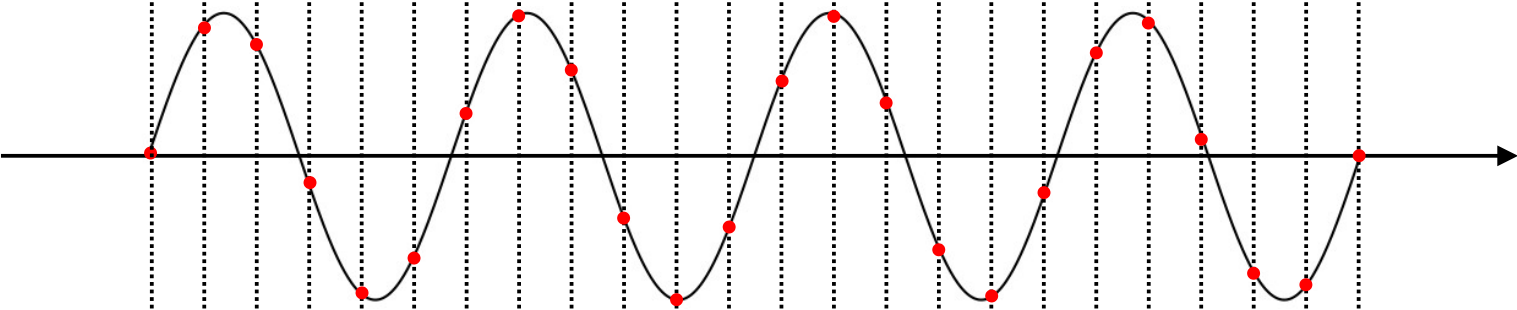


I prefer to use odd values for N and M

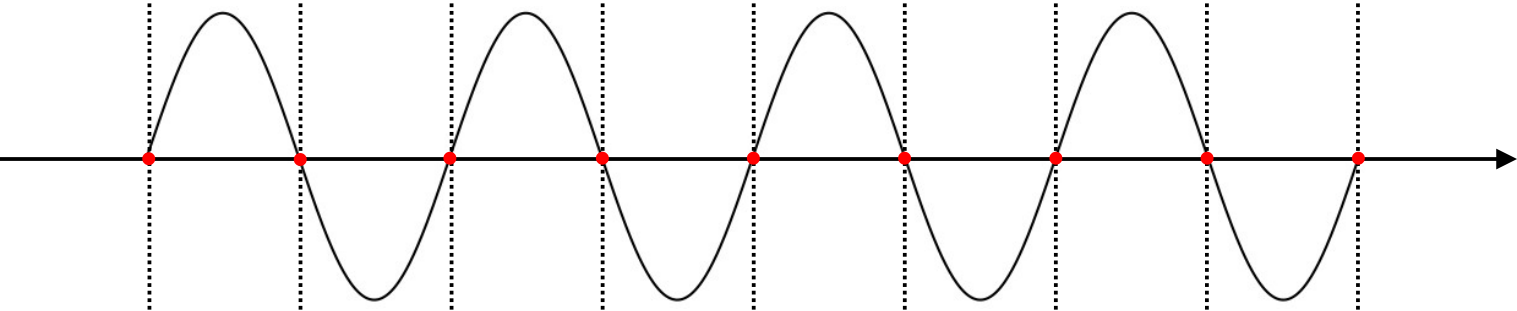
$$\lfloor x \rfloor = \text{floor}(x)$$

Under-Sampling Issues

Properly sampled function



Under-sampled function



Python Code for Fourier Propagation

$$\mathbf{E}(x, y, d) = \iint_{-\infty}^{\infty} \tilde{\mathbf{E}}(k_x, k_y, 0) e^{-j(k_x x + k_y y + k_z d)} dk_x dk_y$$



$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\tilde{\mathbf{E}}(k_x, k_y, 0) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \mathbf{E}(x, y, 0) e^{j(k_x x + k_y y)} dx dy$$

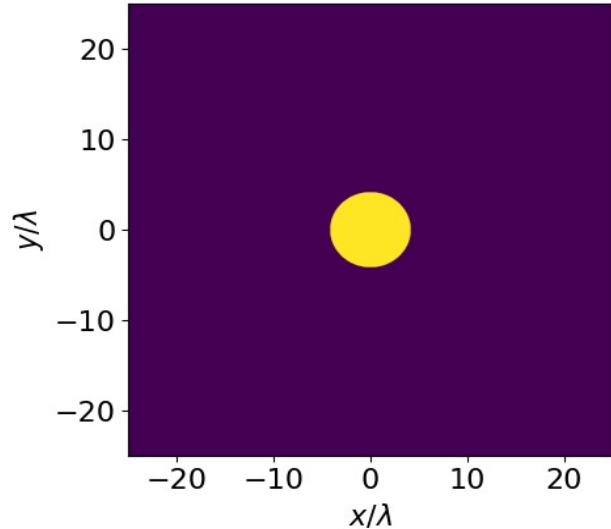
known

In what follows, python packages are loaded like this

```
from numpy import *  
from matplotlib.pyplot import *
```

Create an Object in Direct Space

Simple circular aperture



```
L = 50 # window dimension in wavelength  
N = 1001 # number of pixels (odd value)  
R = 5 # aperture radius in wavelength
```

```
x = linspace(-L/2,L/2,N)  
X,Y = meshgrid(x,x)
```

```
f = ones((N,N))  
f[ X**2 + Y**2 > R**2 ] = 0
```

This defines the field at $z=0$

$$\mathbf{E}(x, y, 0)$$

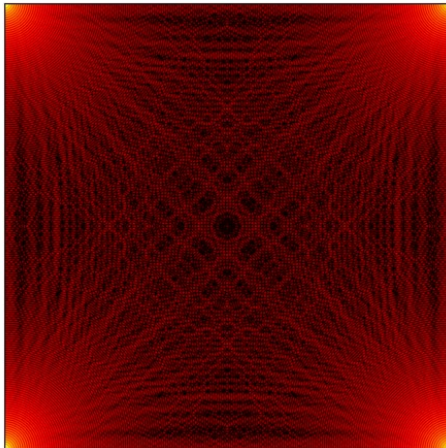
Taking the Inverse Fourier Transform

Knowing $\mathbf{E}(x, y, 0)$ we want to compute its inverse Fourier transform

$$\tilde{\mathbf{E}}(k_x, k_y, 0) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \mathbf{E}(x, y, 0) e^{j(k_x x + k_y y)} dx dy$$

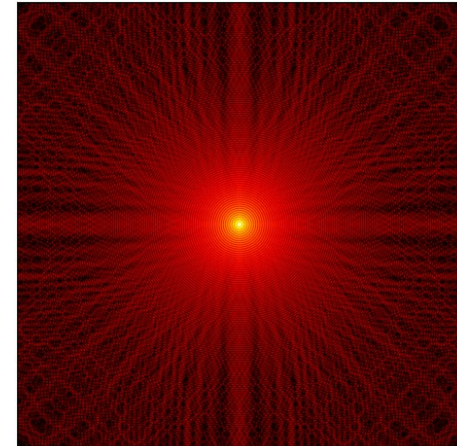
Wrong way

```
fft.fft2(f)
```



Correct way

```
fft.fftshift(fft.fft2(fft.ifftshift(f)))
```



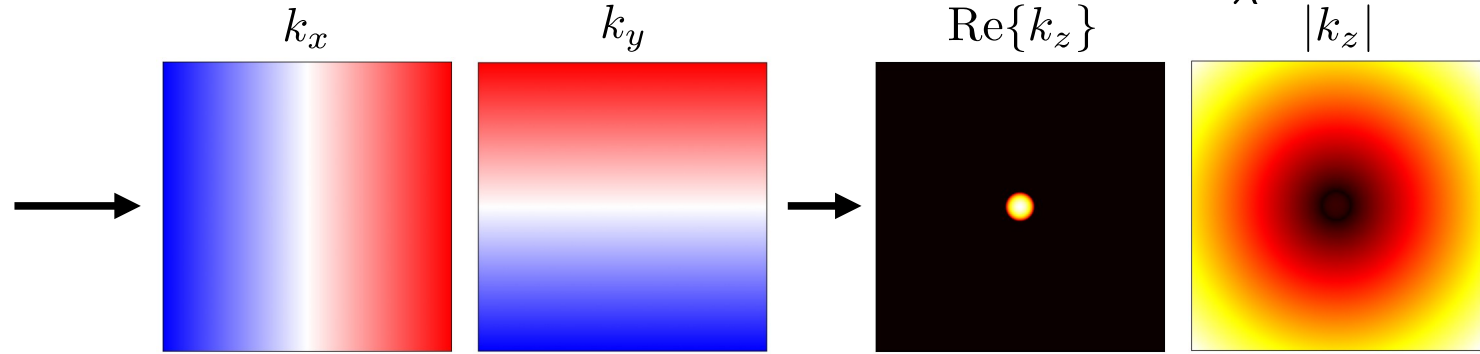
The data must be shifted such that it is centered

Defining the K-Vectors and the Fourier Propagator

We need to define k_z for each wave $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ where $k = \frac{2\pi}{\lambda}$

$$k_x = \left[-\frac{M}{2}, \dots, \frac{M}{2} \right] \frac{2\pi}{L_X}$$

$$k_y = \left[-\frac{N}{2}, \dots, \frac{N}{2} \right] \frac{2\pi}{L_Y}$$



M, N : number of pixels along x, y
 L_X, L_Y : window physical dimensions

$$\tilde{\mathbf{E}}(k_x, k_y, d) = \tilde{\mathbf{E}}(k_x, k_y, 0) e^{-jk_z d}$$

We assume that $\lambda=1$

```

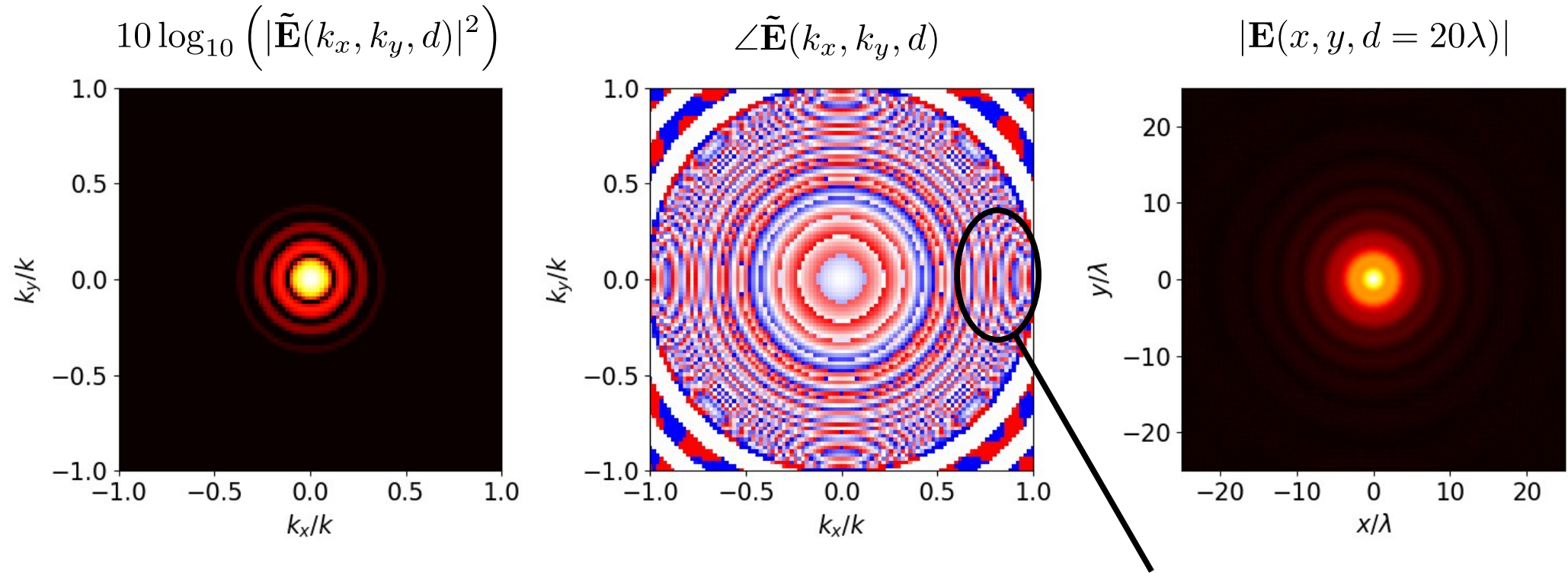
k = 2*pi
kx = arange(-floor(N/2), floor(N/2)+1)*2*pi/L
Kx, Ky = meshgrid(kx,kx)
Kz = sqrt(k**2 - Kx**2 - Ky**2, dtype=complex128)

F = fft.fftshift(fft.fft2(fft.ifftshift(f))) * exp(1j*Kz*Lz)
fz = fft.fftshift(fft.ifft2(fft.ifftshift(F)))
    
```

$$\mathbf{E}(x, y, d) = \iint_{-\infty}^{\infty} \tilde{\mathbf{E}}(k_x, k_y, d) e^{-j(k_x x + k_y y)} dk_x dk_y$$

Propagation distance

Plotting the Propagated Field



where $\tilde{\mathbf{E}}(k_x, k_y, d) = \tilde{\mathbf{E}}(k_x, k_y, 0)e^{-jk_z d}$

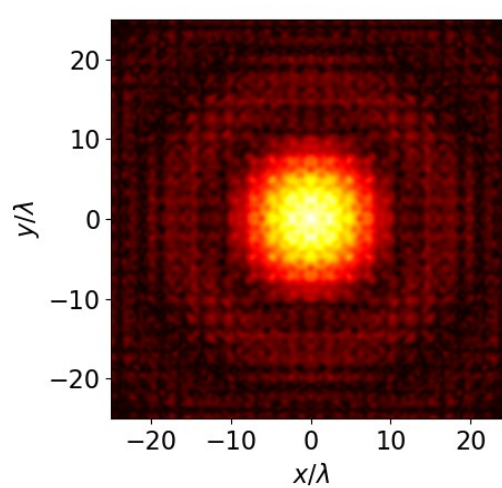
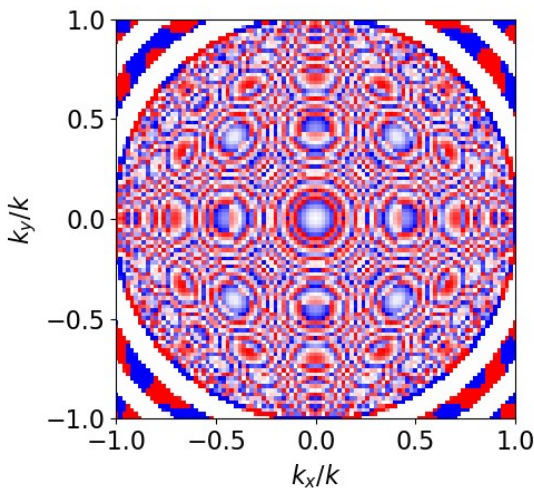
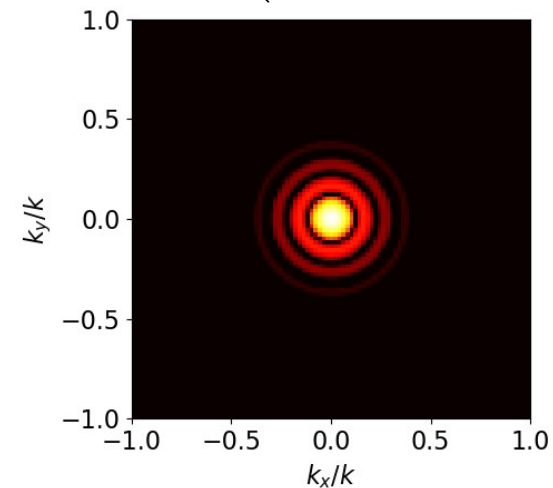
Aliasing !

Highlighting Aliasing Issues

$$10 \log_{10} \left(|\tilde{\mathbf{E}}(k_x, k_y, d)|^2 \right)$$

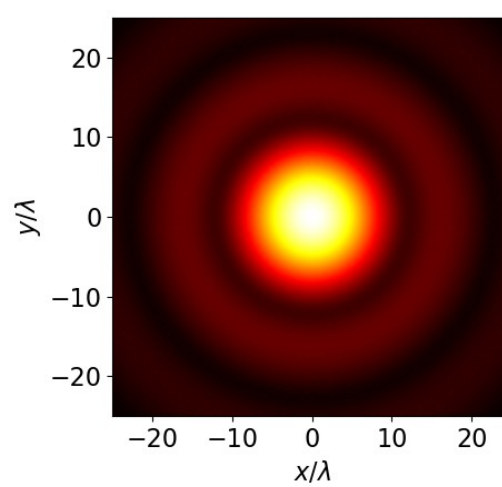
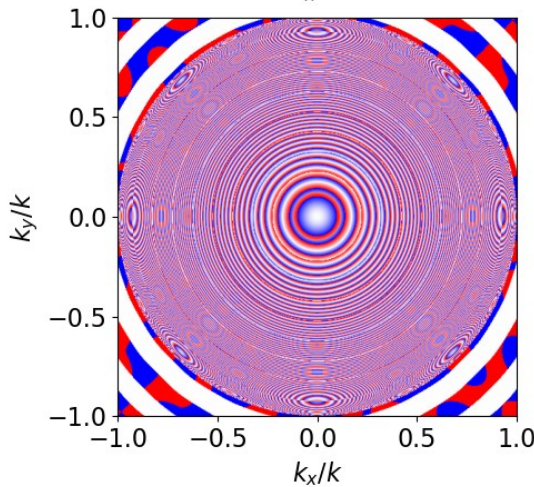
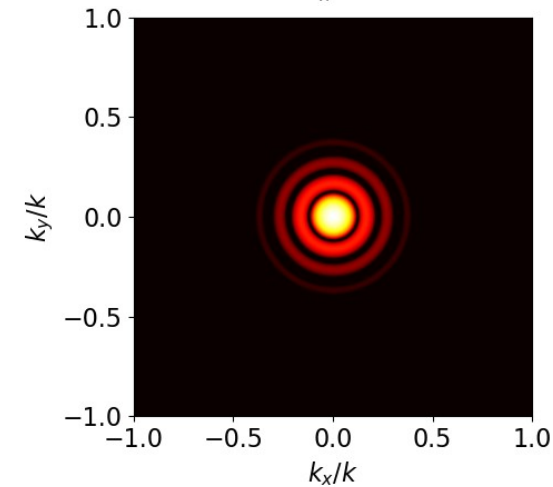
$$\angle \tilde{\mathbf{E}}(k_x, k_y, d)$$

$$|\mathbf{E}(x, y, d = 100\lambda)|$$



$$L_X = L_Y = 50\lambda$$

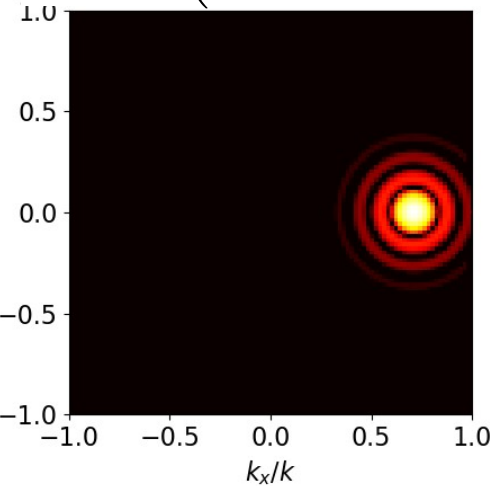
Increase resolution
by increasing
window physical size



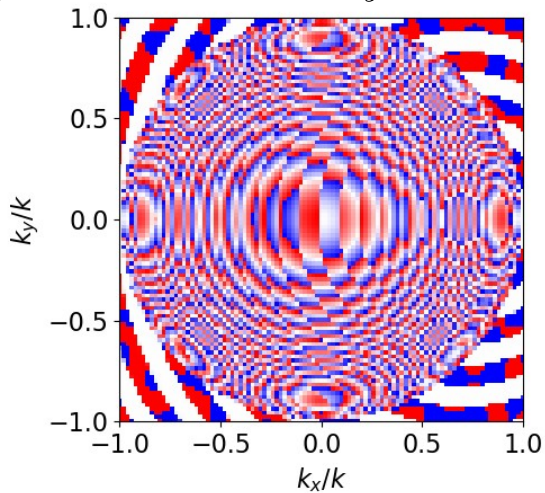
$$L_X = L_Y = 250\lambda$$

Highlighting Aliasing Issues

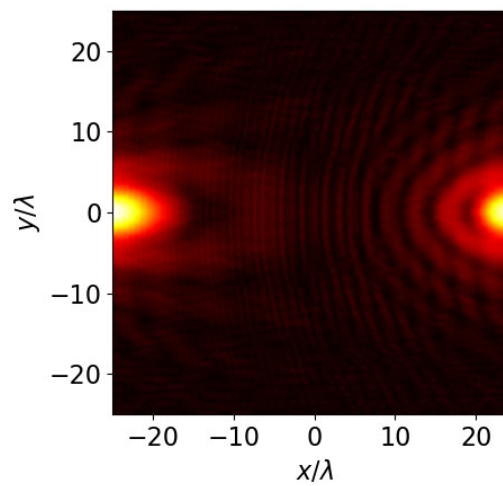
$$10 \log_{10} \left(|\tilde{\mathbf{E}}(k_x, k_y, d)|^2 \right)$$



$$\angle \tilde{\mathbf{E}}(k_x, k_y, d)$$



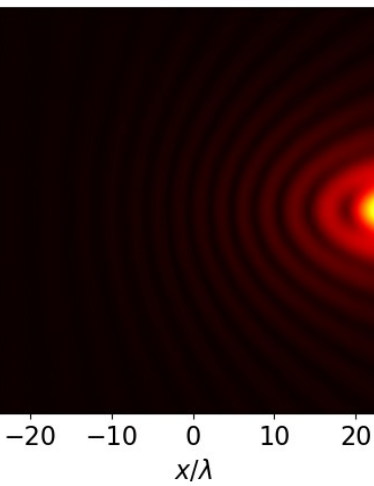
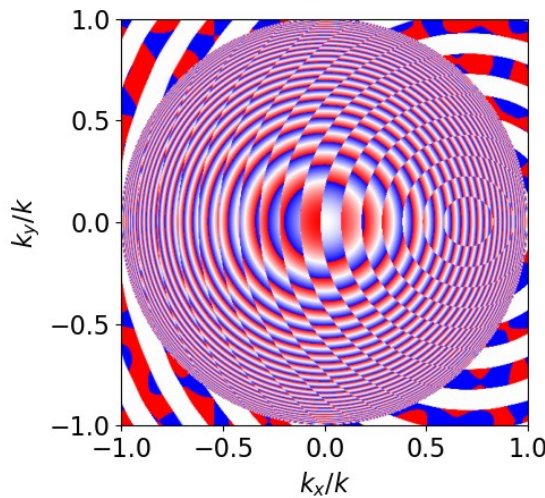
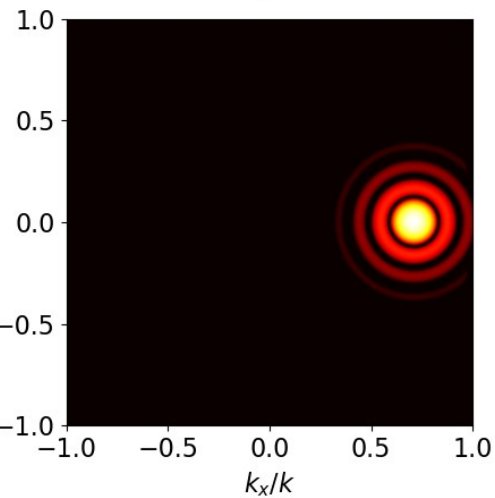
$$|\mathbf{E}(x, y, d = 25\lambda)|$$



Adding a phase shift along x

$$\mathbf{E}(x, y, 0) e^{-jk \frac{\sqrt{2}}{2} x}$$

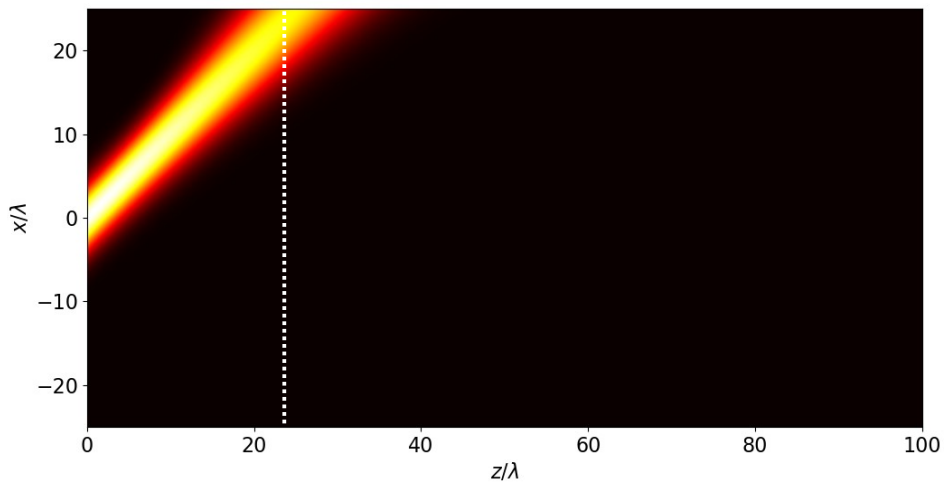
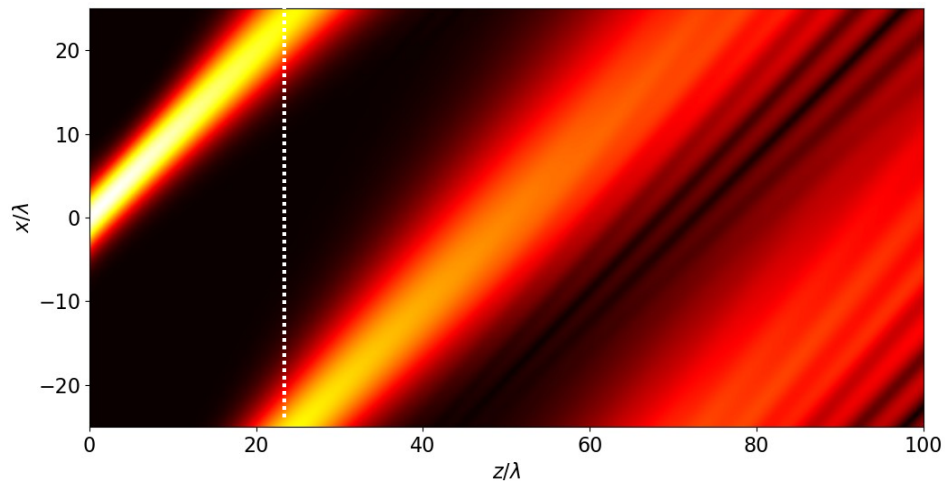
$$L_X = L_Y = 50\lambda$$



$$L_X = L_Y = 250\lambda$$

Highlighting Aliasing Issues

$$|\mathbf{E}(x, y, d)|$$



Adding a phase
shift along x

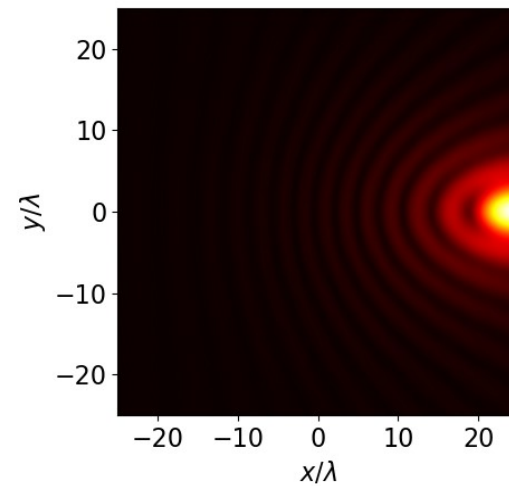
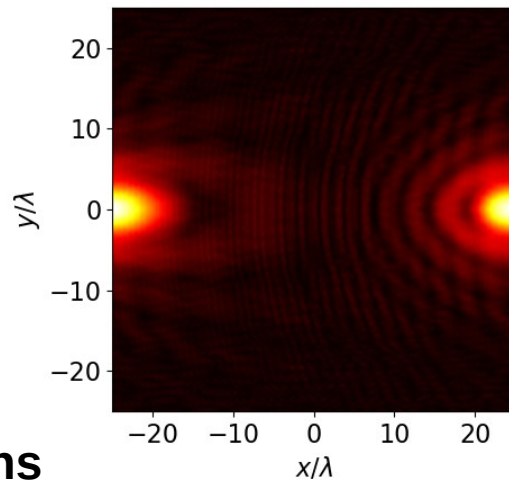
$$\mathbf{E}(x, y, 0)e^{-jk\frac{\sqrt{2}}{2}x}$$

$$L_X = L_Y = 50\lambda$$

Plotting cross-sections
in the xz -plane

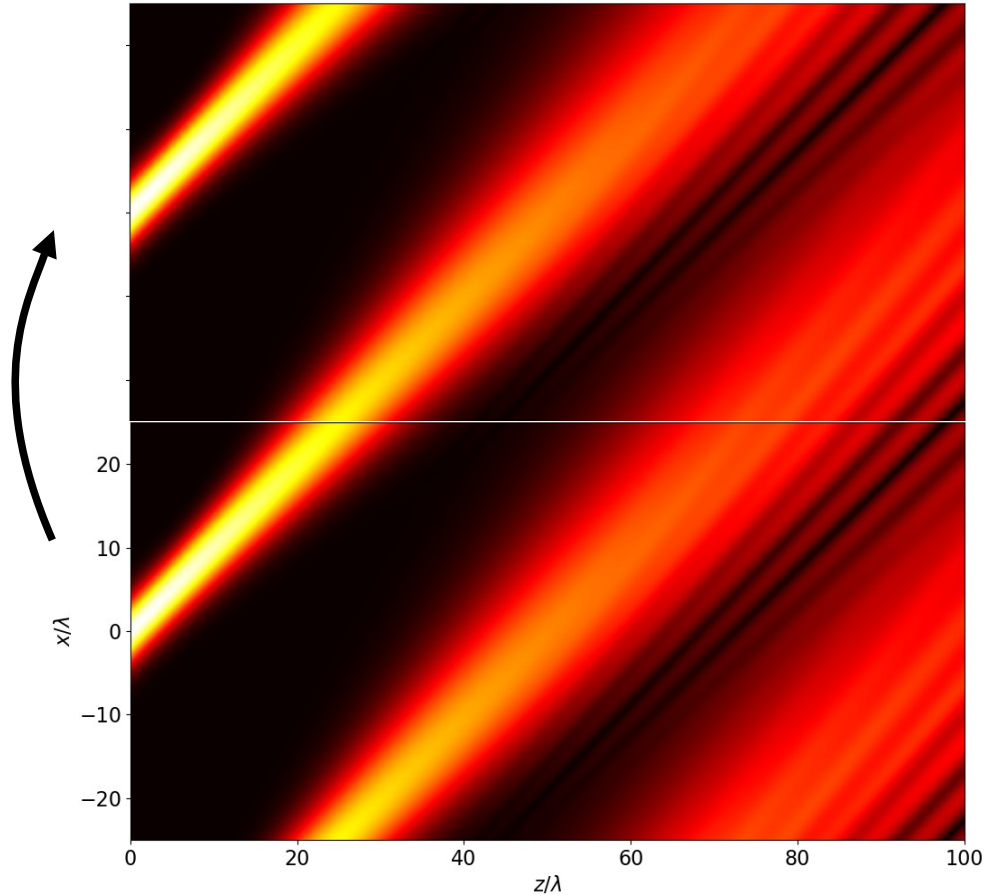
$$L_X = L_Y = 250\lambda$$

$$|\mathbf{E}(x, y, d = 25\lambda)|$$



Highlighting Aliasing Issues

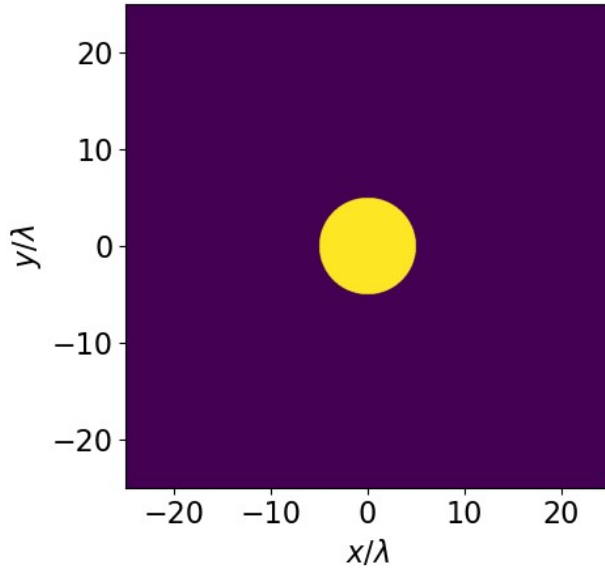
Copy of the original field



Remember that the FFT algorithm creates a periodic extension of the original function

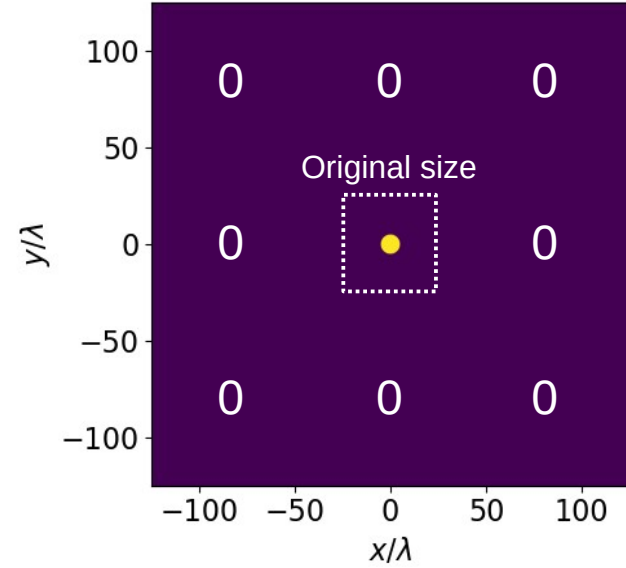
Zero-Padding to Avoid Aliasing Issues

$$L_X = L_Y = 50\lambda$$



zero-padding

$$L_X = L_Y = 250\lambda$$



Redefine physical size

```
d = 100 # size of zero padding arrays (even value)
f = pad(f,d,constant_values=0)

N = N + 2*d
dx = x[1] - x[0]
L = (N - 1)*dx
```

Putting Everything Together

```
from numpy import *
from matplotlib.pyplot import *
# -----
# dashboard
L = 50 # window dimension in wavelength
N = 1001 # number of pixels (odd value)
R = 5 # aperture radius
Lz = 20 # propagation distance in wavelength
d = 100 # size of zero padding arrays (even value)
# -----
# define direct space parameters
x = linspace(-L/2,L/2,N)
X,Y = meshgrid(x,x)

f = ones((N,N))
f[ X**2 + Y**2 > R**2 ] = 0
# -----
# add padding
f = pad(f,d,constant_values=0)
N = N + 2*d
dx = x[1] - x[0]
L0, L = L, (N - 1)*dx
mid = round(N/2)
Wp, Wm = int(mid + round(L0/2/dx)), int(mid - round(L0/2/dx))
# -----
# define Fourier space parameters
k = 2*pi
kx = arange(-floor(N/2), floor(N/2)+1)*2*pi/L
Kx, Ky = meshgrid(kx,kx)
Kz = sqrt(k**2 - Kx**2 - Ky**2, dtype=complex128)

F = fft.fftshift(fft.fft2(fft.ifftshift(f))) * exp(1j*Kz*Lz)
fz = fft.fftshift(fft.ifft2(fft.ifftshift(F)))
# -----
# plot
imshow(abs(fz[Wm:Wp,Wm:Wp]), cmap='hot', extent = [-L0/2, L0/2, -L0/2, L0/2])
xlabel('$x/\lambda$')
ylabel('$y/\lambda$')
tight_layout()
show()
```

What Have We Learned So Far....

- The angular spectrum method may be numerically implemented in just a few lines of code using the FFT algorithm
- This is simple and fast but comes with some issues such as aliasing effects and the fact the FFT algorithm considers the function to be periodic, which leads to weird distortions
- These issues may be generally mitigated by increasing the physical size of the simulation window, which leads to an increase of the resolution in Fourier space