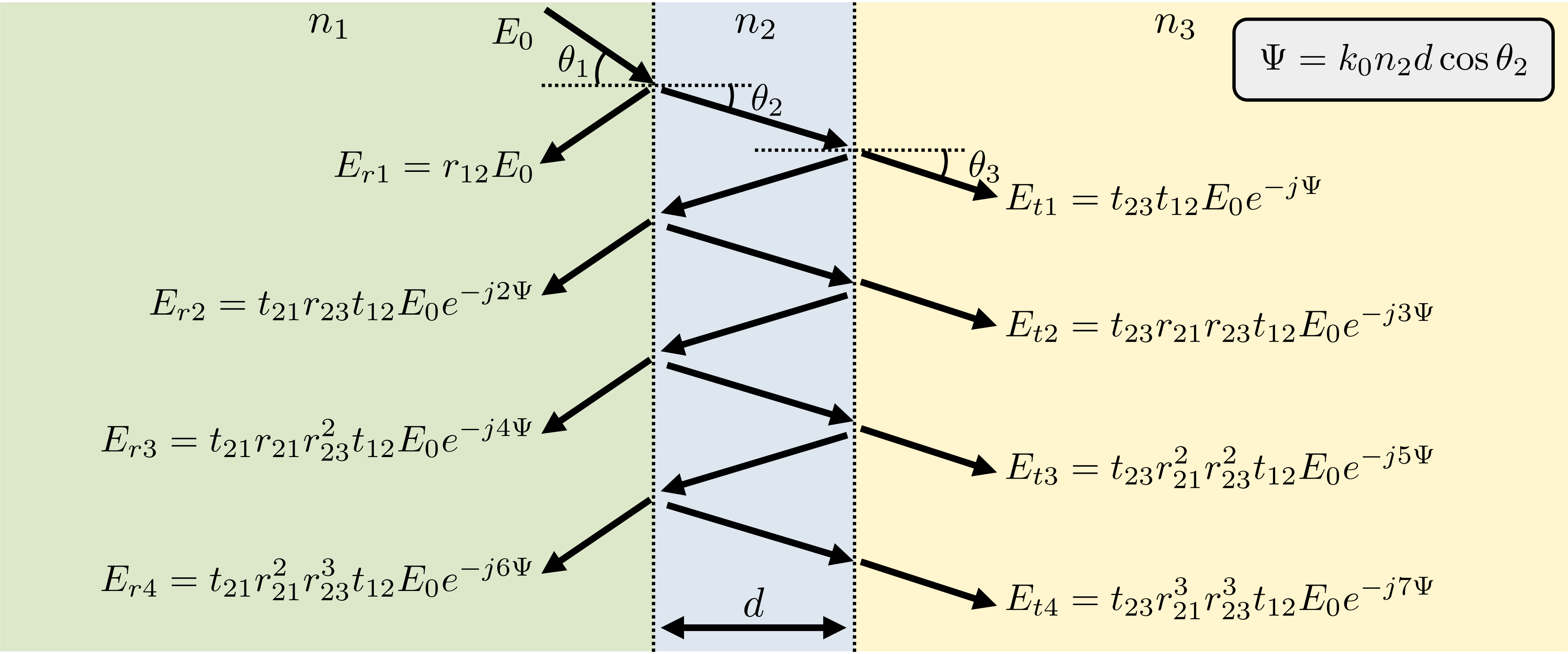
Lecture 3

Scattering From Multilayer Structures

Scattering from a Slab

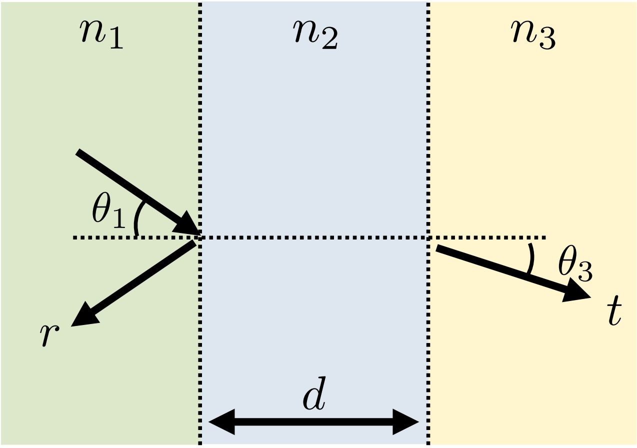
Scattering from a Slab



The r_{ab} and t_{ab} terms are the TE and TM Fresnel coefficients

$$\begin{cases} r = r_{12} + t_{12}t_{21}\displaystyle\sum_{n=0}^{\infty} r_{21}^n r_{23}^{n+1} e^{-j2(n+1)\Psi} = r_{12} + t_{12}t_{21}r_{23}e^{-j2\Psi}\sum_{n=0}^{\infty}\left(r_{21}r_{23}e^{-j2\Psi}\right)^n \\ t = t_{12}t_{23}\displaystyle\sum_{n=0}^{\infty}(r_{21}r_{23})^n e^{-j(2n+1)\Psi} = t_{12}t_{23}e^{-j\Psi}\sum_{n=0}^{\infty}\left(r_{21}r_{23}e^{-j2\Psi}\right)^n \end{cases}$$

Simplification Using Geometric Series



$$\Psi = k_0 n_2 d \cos\theta_2$$

$$\begin{cases} r = r_{12} + t_{12} t_{21} r_{23} e^{-j2\Psi} \sum_{n=0}^{\infty} \left(r_{21} r_{23} e^{-j2\Psi} \right)^n \\ t = t_{12} t_{23} e^{-j\Psi} \sum_{n=0}^{\infty} \left(r_{21} r_{23} e^{-j2\Psi} \right)^n \end{cases}$$

The infinite series may be simplified using

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x} \qquad |x| < 1$$

$$\begin{cases} r = r_{12} + \dfrac{t_{12} t_{21} r_{23} e^{-j2\Psi}}{1 - r_{21} r_{23} e^{-j2\Psi}} \\ t = \dfrac{t_{12} t_{23} e^{-j\Psi}}{1 - r_{21} r_{23} e^{-j2\Psi}} \end{cases}$$

Fresnel coefficients

$$t_{\mathrm{TM}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1} \qquad r_{\mathrm{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$t_{\mathrm{TE}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2} \qquad r_{\mathrm{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

Some Useful Relations

General properties of the Fresnel coefficients

$$r_{21} = -r_{12}$$

$$t_{\mathrm{TE},12} = 1 + r_{\mathrm{TE},12}$$

$$t_{\mathrm{TE},21} = 1 + r_{\mathrm{TE},21}$$

$$t_{\mathrm{TM},21} = \frac{\cos\theta_2}{\cos\theta_1}\left(1 + r_{\mathrm{TM},21}\right)$$

$$t_{\mathrm{TM},12} = \frac{\cos\theta_1}{\cos\theta_2}\left(1 + r_{\mathrm{TM},12}\right)$$

Other useful relations

$$t_{12}t_{21} = \left(1 + r_{12}\right)\left(1 - r_{12}\right)$$

$$t_{\mathrm{TE},12}t_{\mathrm{TE},23} = \left(1 + r_{\mathrm{TE},12}\right)\left(1 + r_{\mathrm{TE},23}\right)$$

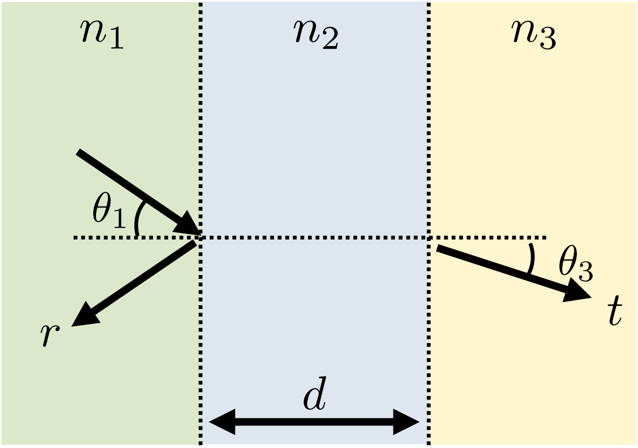
$$t_{\mathrm{TM},12}t_{\mathrm{TM},23} = \frac{\cos\theta_1}{\cos\theta_3}\left(1 + r_{\mathrm{TM},12}\right)\left(1 + r_{\mathrm{TM},23}\right)$$

Fresnel coefficients

$$t_{\mathrm{TM}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} \qquad r_{\mathrm{TM}} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1}$$

$$t_{\mathrm{TE}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} \qquad r_{\mathrm{TE}} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}$$

Total Field and Power Scattering Parameters



$$\Psi = k_0 n_2 d \cos\theta_2$$

Fresnel reflection coefficients

$$r_{\mathrm{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$r_{\mathrm{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

Total field scattering parameters

$$r = \frac{r_{12} + r_{23} e^{-j2\Psi}}{1 + r_{12} r_{23} e^{-j2\Psi}}$$

$$t = C \frac{(1 + r_{12})(1 + r_{23}) e^{-j\Psi}}{1 + r_{12} r_{23} e^{-j2\Psi}}$$

For TE waves

$$C = 1$$

For TM waves

$$C = \frac{\cos\theta_1}{\cos\theta_3}$$

The power coefficients are found using

$$T = |t|^2 \mathrm{Re}\left\{\frac{n_3 \cos\theta_3}{\mu_{\mathrm{r},3}}\right\} \mathrm{Re}\left\{\frac{\mu_{\mathrm{r},1}}{n_1 \cos\theta_1}\right\}$$

$$R = |r|^2$$

Phase matching

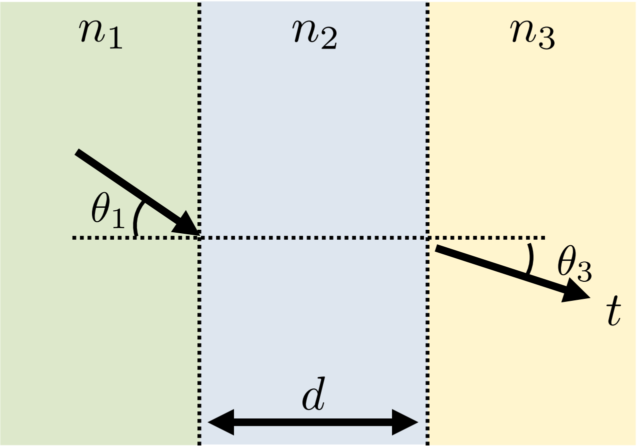
$$n_1 \sin\theta_1 = n_2 \sin\theta_2 = n_3 \sin\theta_3$$

What Have We Learned So Far….

- A slab of dielectric leads to an infinite number of reflected and transmitted waves that may be conveniently summed together as geometric series
- This allows us to find the slab scattering parameters directly in terms of the Fresnel reflection coefficients at the two interfaces of the slab

Anti-Reflection Coating

Anti-Reflection Coating



$$\Psi = k_0 n_2 d \cos\theta_2$$

Total reflection coefficient

$$r = \frac{r_{12} + r_{23} e^{-j2\Psi}}{1 + r_{12} r_{23} e^{-j2\Psi}}$$

$$r = 0$$

$$e^{-j2\Psi} = -\frac{r_{12}}{r_{23}}$$

Fresnel reflection coefficients

$$r_{\mathrm{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

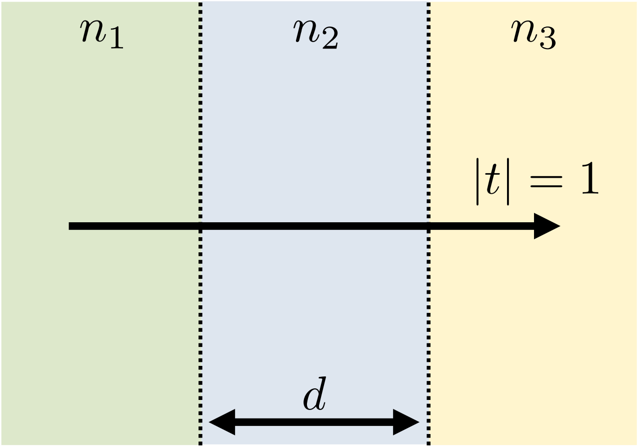
$$r_{\mathrm{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

General expression to solve for *d* and n_2

$$\ln\left(e^{-j2\Psi}\right) = \ln\left(-\frac{r_{12}}{r_{23}}\right) \longrightarrow j\left(2\Psi + 2\pi m\right) = \ln\left(\frac{r_{23}}{r_{12}}\right) \longrightarrow \Psi = \frac{1}{2j}\ln\left(\frac{r_{23}}{r_{12}}\right) - \pi m$$

At normal incidence, we have $r_{\mathrm{TE}} = r_{\mathrm{TM}}$ $\quad r_{12} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ $\quad r_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$

Anti-Reflection Coating at Normal Incidence



At normal incidence

$$r_{12} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$r_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

$$\Psi = k_0 n_2 d = \frac{2\pi}{\lambda_0} n_2 d$$

$$\Psi = \frac{1}{2j} \ln\left(\frac{r_{23}}{r_{12}}\right) - \pi m$$

Solve for *d*

$$d = \frac{\lambda_0}{j4\pi n_2} \ln\left[\frac{(\eta_2 + \eta_1)(\eta_2 - \eta_3)}{(\eta_2 - \eta_1)(\eta_2 + \eta_3)}\right] - m\frac{\lambda_0}{2n_2}$$

is generally complex

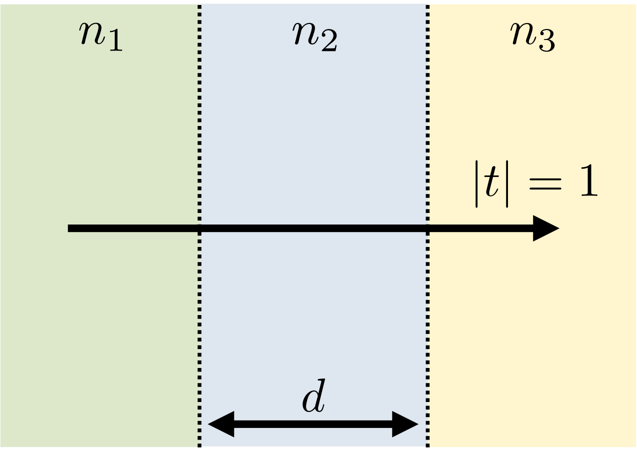
Can be significantly simplified by choosing

$$\eta_2 = \sqrt{\eta_1 \eta_3}$$

$$d = \frac{\lambda_0}{4n_2}(1 + 2m) \qquad m = 0, 1, 2, \ldots$$

Corresponds to a quarter wave slab for *m = 0*

Dielectric Anti-Reflection Coating at Normal Incidence



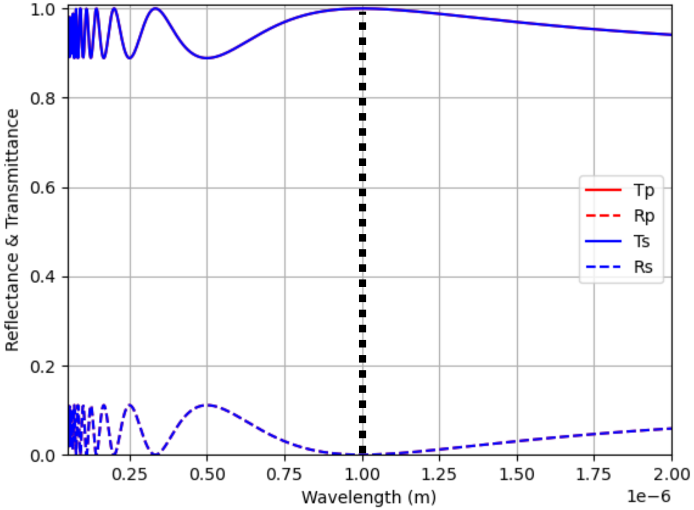
$$d = \frac{\lambda_0}{4n_2}(1 + 2m)$$

In a dielectric material $\mu_r = 1 \rightarrow \eta = \frac{\eta_0}{n}$

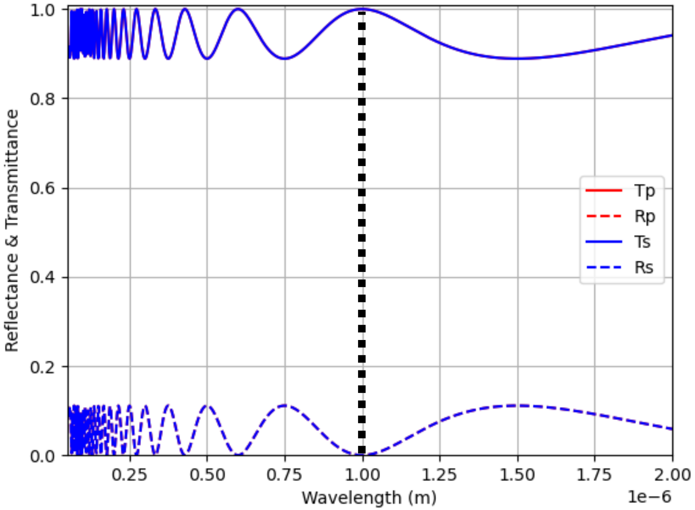
$$\eta_2 = \sqrt{\eta_1 \eta_3} \rightarrow n_2 = \sqrt{n_1 n_3} \rightarrow \epsilon_{r,2} = \sqrt{\epsilon_{r,1}\epsilon_{r,3}}$$

<u>Example:</u> $\lambda_0 = 1\ \mu m$ $\quad \epsilon_{r,1} = 1,\ \epsilon_{r,3} = 4 \rightarrow \epsilon_{r,2} = \sqrt{\epsilon_{r,1}\epsilon_{r,3}} = 2$

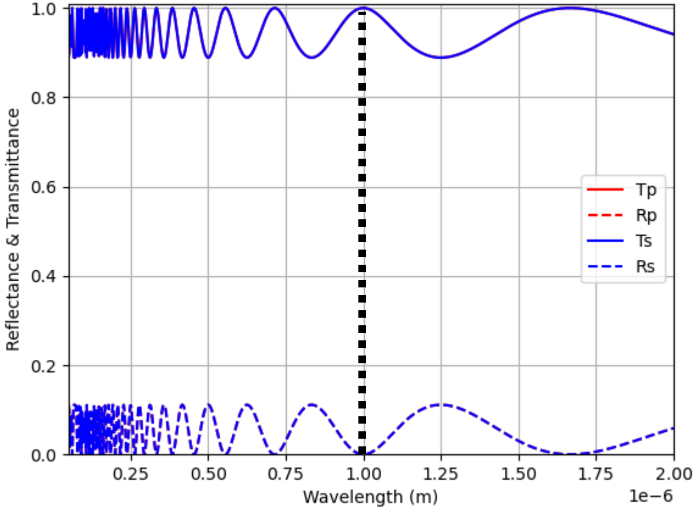
$m = 0 \rightarrow d = 176$ nm



$m = 1 \rightarrow d = 530$ nm

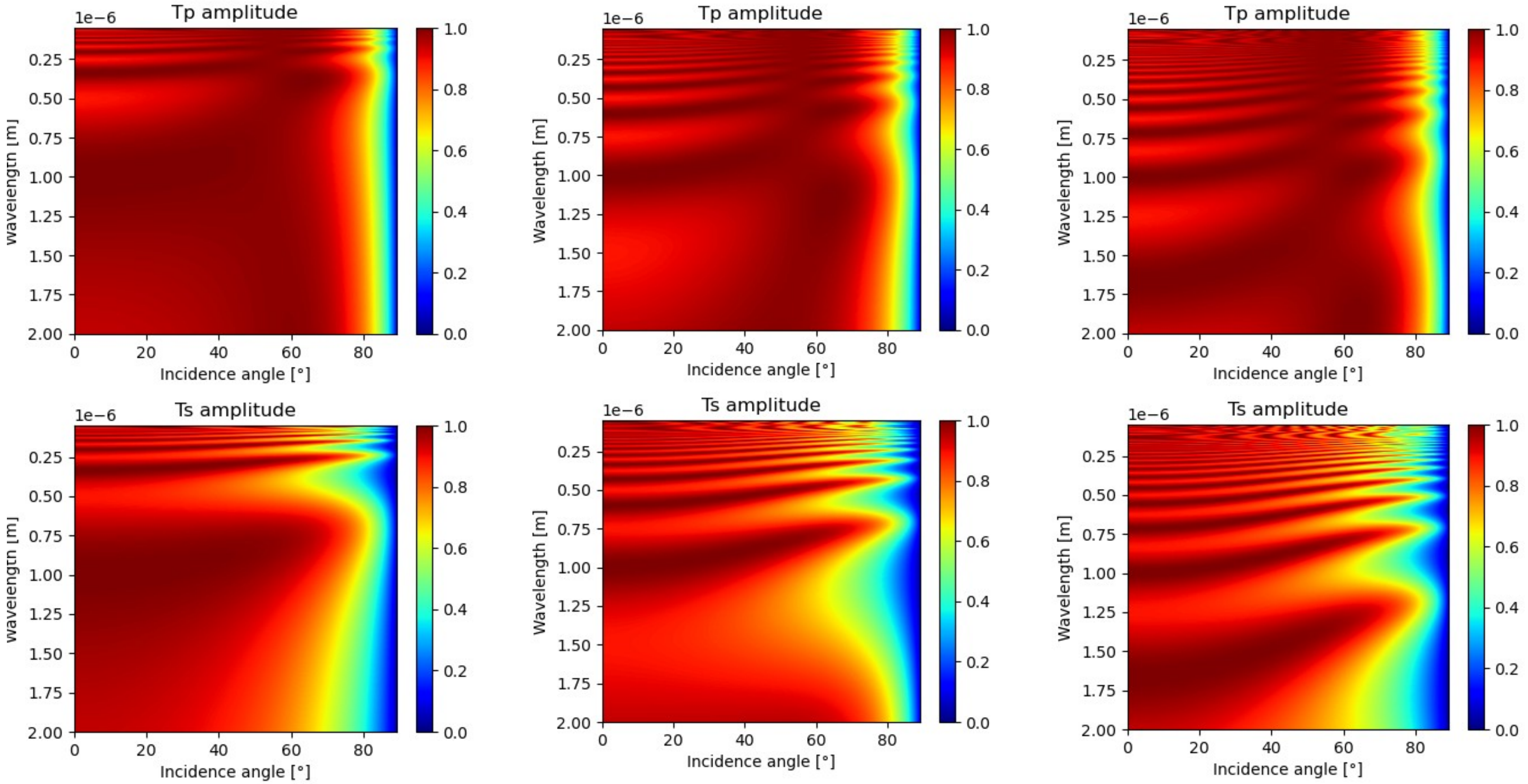


$m = 2 \rightarrow d = 884$ nm



Angular Behavior

$m = 0 \rightarrow d = 176\ \mathrm{nm}$ | $m = 1 \rightarrow d = 530\ \mathrm{nm}$ | $m = 2 \rightarrow d = 884\ \mathrm{nm}$

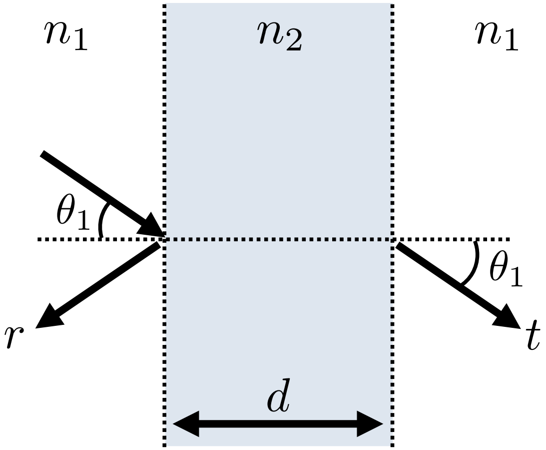


What Have We Learned So Far….

- A dielectric slab may be used as an anti-reflection coating
- Its dimension is typically a quarter wavelength long and its refractive index is the square root of the multiplication of the refractive indices of both media surrounding the slab
- If the thickness of the slab increases, the transmission bandwidth decreases
- Anti-reflection effects only really work at normal incidence with reduced performance for oblique propagation

Scattering from a Symmetric Slab

Scattering from a Slab Surrounded by Identical Media



$$\Psi = k_0 n_2 d \cos\theta_2$$

Fresnel reflection coefficients

$$r_{\text{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$r_{\text{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

We simplify the general result with

$$r_{23} = r_{21} = -r_{12} \qquad n_1 = n_3 \qquad \theta_1 = \theta_3$$

Total field scattering parameters

$$r = \frac{r_{12}\left(1 - e^{-j2\Psi}\right)}{1 - r_{12}^2 e^{-j2\Psi}}$$

$$t = \frac{\left(1 - r_{12}^2\right) e^{-j\Psi}}{1 - r_{12}^2 e^{-j2\Psi}}$$

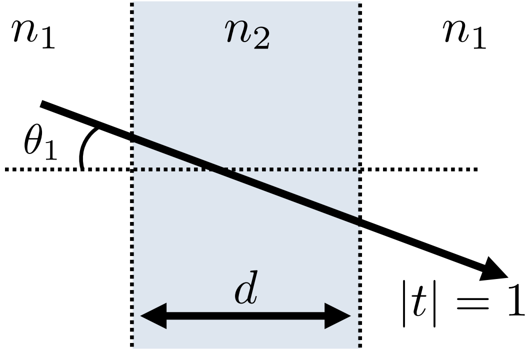
The power coefficients are found using

$$R = |r|^2 \qquad T = |t|^2$$

Phase matching

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Invisible Slab



$$\Psi = k_0 n_2 d \cos\theta_2$$

Fresnel reflection coefficients

$$r_{\mathrm{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$r_{\mathrm{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

Total field scattering parameters

$$r = \frac{r_{12}\left(1 - e^{-j2\Psi}\right)}{1 - r_{12}^2 e^{-j2\Psi}}$$

$$t = \frac{\left(1 - r_{12}^2\right) e^{-j\Psi}}{1 - r_{12}^2 e^{-j2\Psi}}$$

How can we get *r = 0* ?

$$e^{-j2\Psi} = 1$$

Periodic function of λ and slow variation w.r.t the angle

$$r_{12} = 0$$

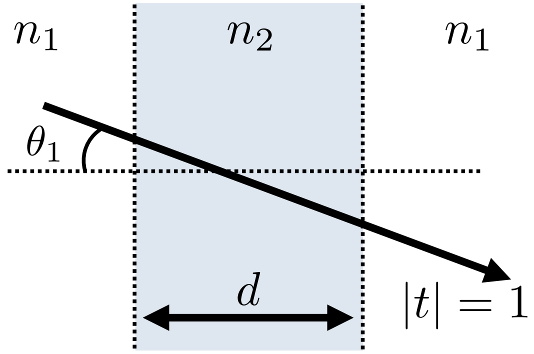
Possible with a Brewster angle.
Little to no dependence on λ !

If either conditions is satisfied

$$r = 0$$
$$t = e^{-j\Psi}$$

Full phase coverage for varying λ

Example of Invisible Slab



<u>Example:</u>

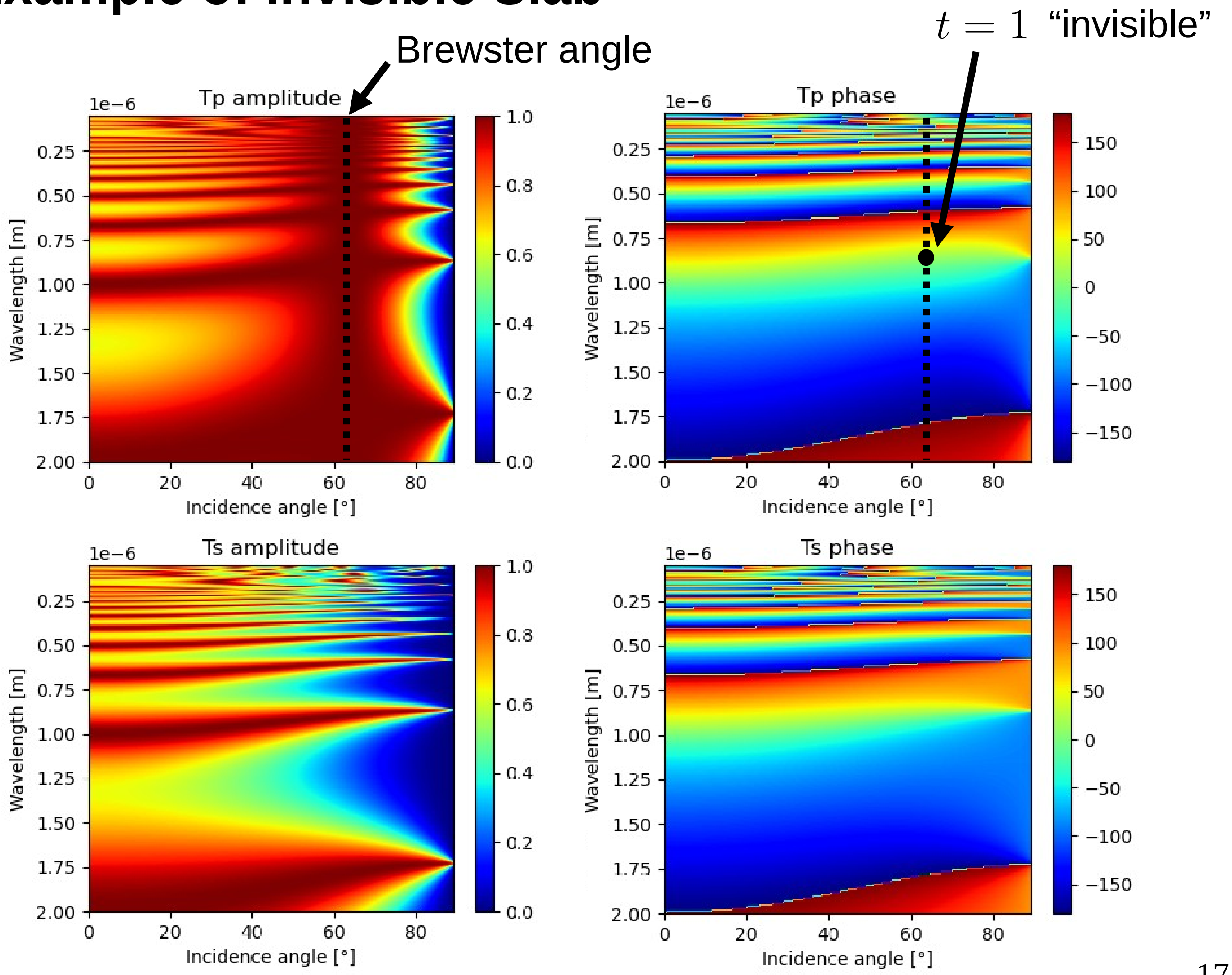
$$n_1 = 1$$

$$n_2 = 2$$

$$d = 500 \text{ nm}$$

$$r = 0$$
$$t = e^{-j\Psi}$$

TM Brewster angle at 63.43°



The Finesse Coefficient

Field transmission

$$t = \frac{\left(1 - r_{12}^2\right) e^{-j\Psi}}{1 - r_{12}^2 e^{-j2\Psi}}$$

Power transmission

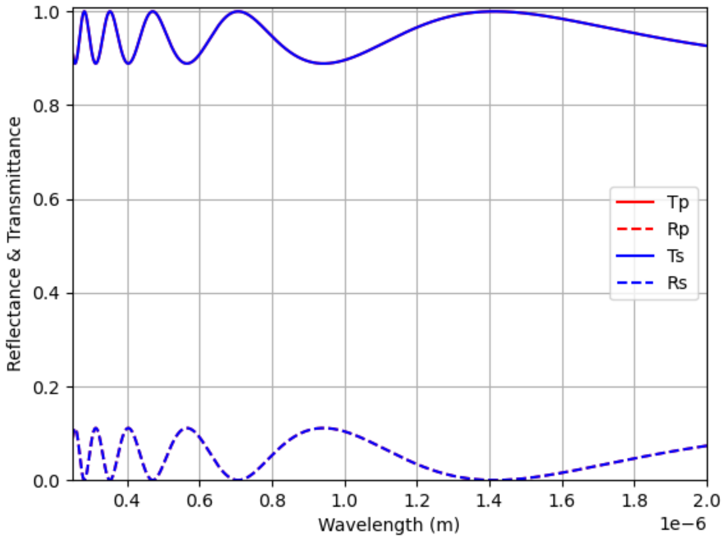
$$T = |t|^2 = \frac{\left(1 - r_{12}^2\right)}{1 - 2r_{12}\cos\left(2\Psi\right) + r_{12}^2} = \frac{1}{1 + F\sin^2(\Psi)}$$

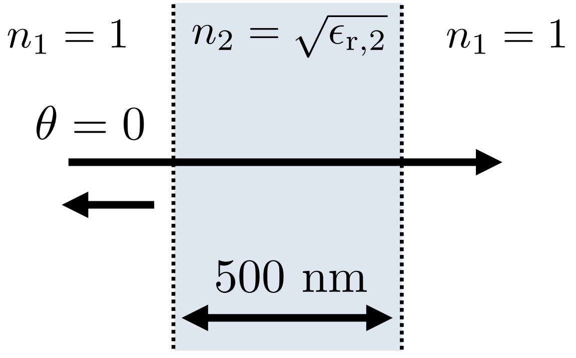
Finesse coefficient

$$F = \frac{4r_{12}^2}{1 - r_{12}^2}$$

Low finesse = low reflectivity r_{12}

$\epsilon_{\mathrm{r},2} = 2$

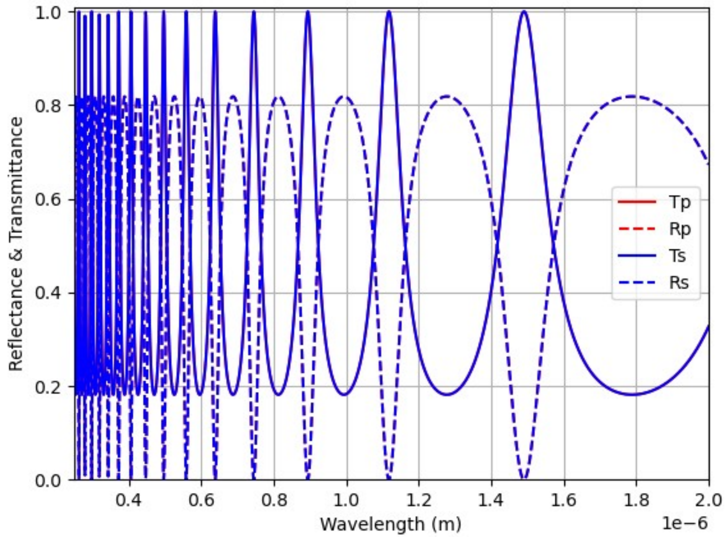




$$\Psi = k_0 n_2 d$$

High finesse = high reflectivity r_{12}

$\epsilon_{\mathrm{r},2} = 20$

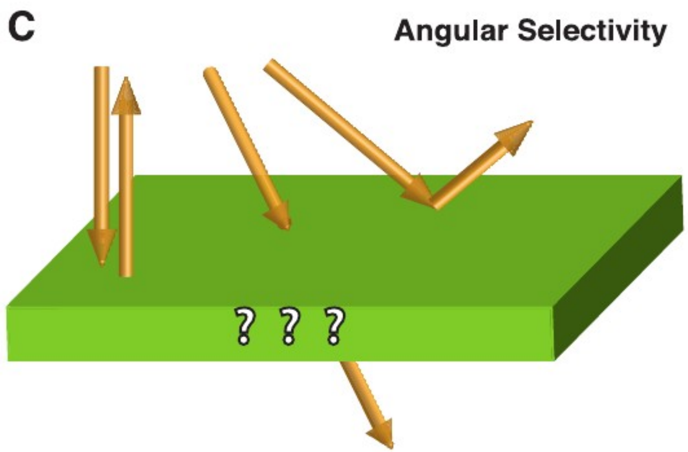


What Have We Learned So Far….

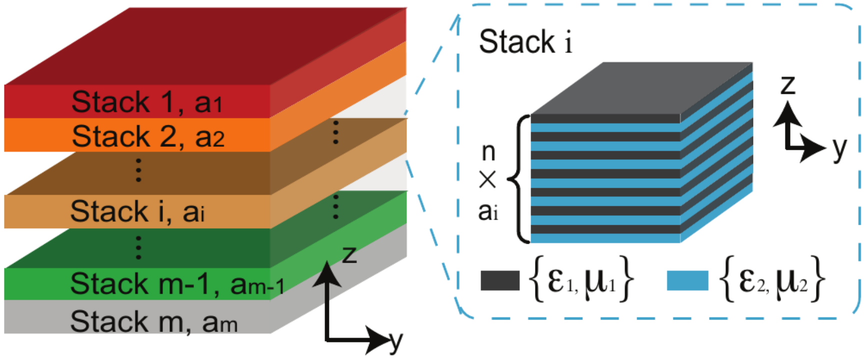
- Two mechanisms can be used to cancel the reflection of a symmetric slab: Brewster angle and destructive interferences
- Both mechanisms have very different angular and spectral behavior
- The Brewster angle is typically weakly wavelength dependent in contrast to interference effects that are periodic w.r.t the wavelength
- The Finesse is a measure of the narrowness of resonances in a slab or a cavity
- The higher the reflection, the higher the Finesse, the more narrow band the resonances
- A high Finesse is desired for creating good filters
- The reflection (and thus the Finesse) may be increased by increasing the permittivity contrast (for a slab) or by making a cavity with highly reflective films on both sides

Application: Broadband Angular Selectivity

Angular Selective Using Bragg Reflector and Brewster Angle

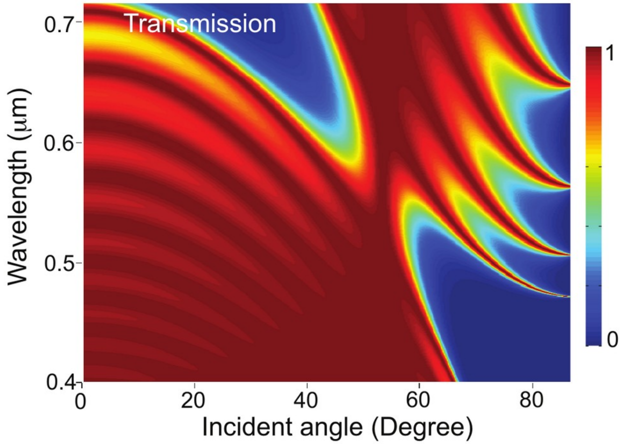


Bragg reflector (quarter-wave stacks) with a Brewster angle

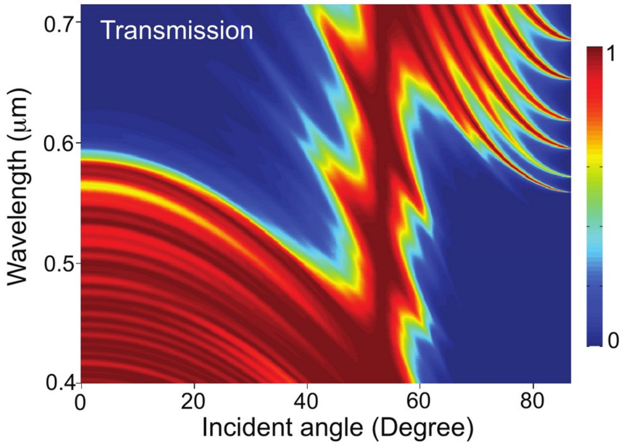


$$\begin{cases} \epsilon_{r,1} = 1 \\ \epsilon_{r,2} = 2 \\ \text{Periodicity} \\ a_i = a_0 r^{i-1} \\ r = 1.0212 \\ a_0 = 200\ \text{nm} \end{cases}$$

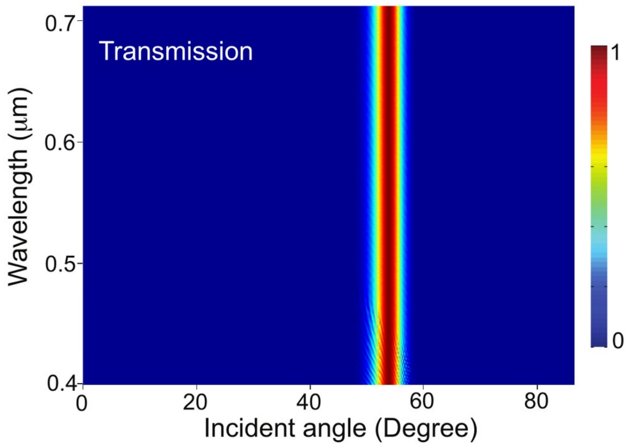
m=1, n=10



m=3, n=10



m=50, n=10



https://doi.org/10.1126/science.1249799

Experimental Demonstration

$$\epsilon_{\mathrm{r},1} \approx 2.18$$
$$\epsilon_{\mathrm{r},2} \approx 4.33$$

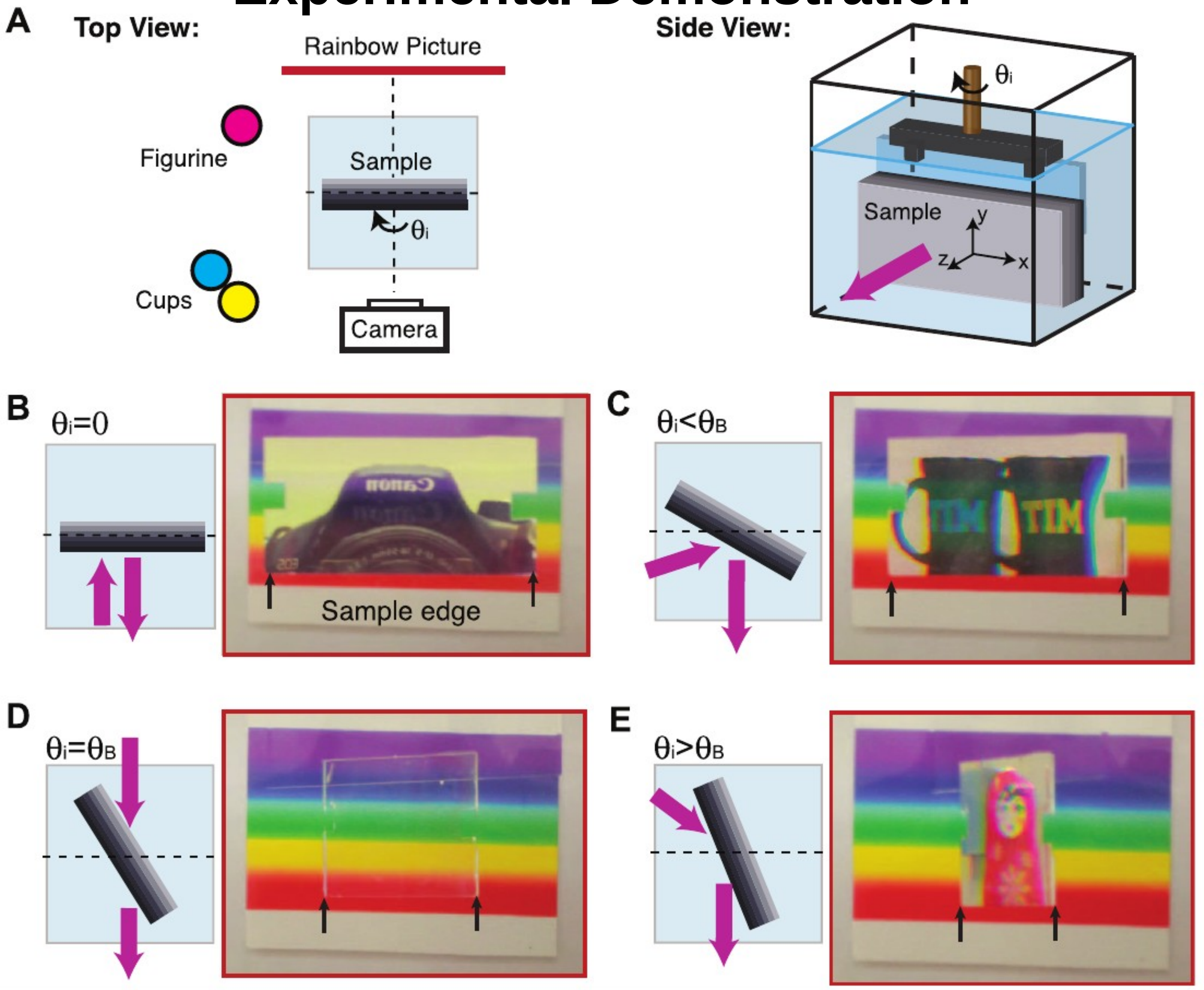
$$m = 6, \; n = 7$$

Periodicity

$$a_i = a_0 r^{i-1}$$

$$a_0 = 140 \text{ nm}$$

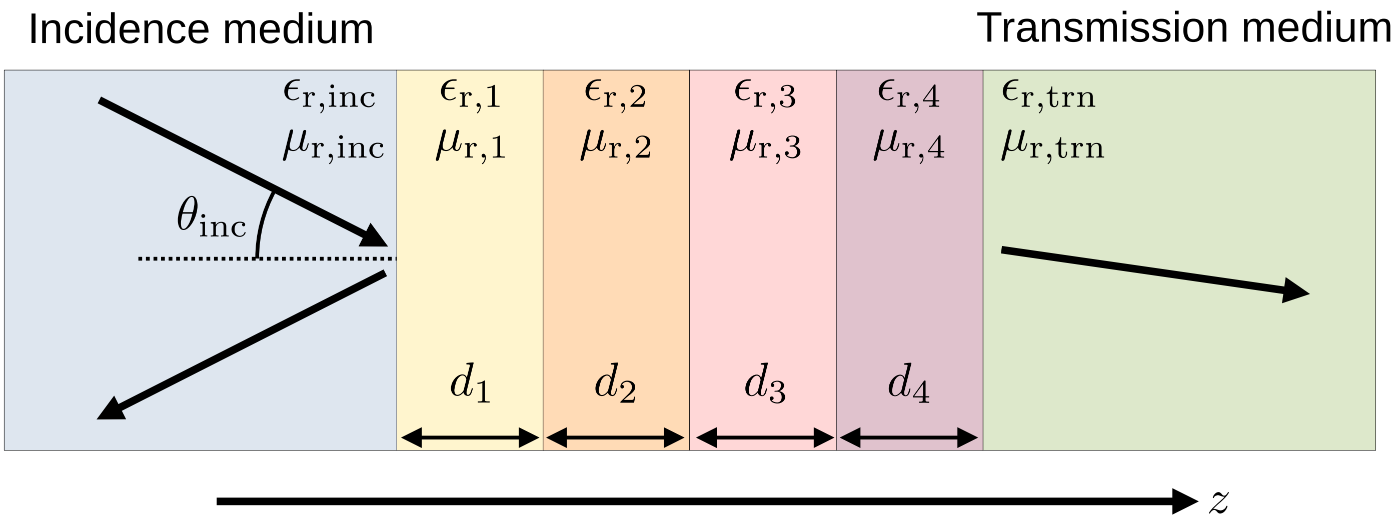
$$r = 1.165$$



https://doi.org/10.1126/science.1249799

Transfer Matrix Method

Scattering from a Multilayer System



The system is invariant along *x* and *y* and only varies in the *z* direction

From the boundary conditions, we know that the tangential field and wave-vector components are all equal at each interface

How to find the total reflection and transmission coefficients?

Normalizing Maxwell Equations

No variations along *x* and *y*

$$\partial_y \to -jk_y \qquad \partial_x \to -jk_x$$

Normalization of the **H** field

$$\tilde{\mathbf{H}} = -j\eta_0 \mathbf{H}$$

Maxwell equations

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

We express and simplify Maxwell equations in each layer of the system

$$-jk_y\tilde{H}_z - \frac{\partial \tilde{H}_y}{\partial z} = k_0\epsilon_\mathrm{r}E_x$$

$$\frac{\partial \tilde{H}_x}{\partial z} + jk_x\tilde{H}_z = k_0\epsilon_\mathrm{r}E_y$$

$$-jk_x\tilde{H}_y + jk_y\tilde{H}_x = k_0\epsilon_\mathrm{r}E_z$$

$$\frac{\partial E_y}{\partial z} + jk_yE_z = -k_0\mu_\mathrm{r}\tilde{H}_x$$

$$-jk_xE_z - \frac{\partial E_x}{\partial z} = -k_0\mu_\mathrm{r}\tilde{H}_y$$

$$-jk_yE_x + jk_xE_y = -k_0\mu_\mathrm{r}\tilde{H}_z$$

Additional normalizations

$$z' = k_0 z$$

$$\tilde{k}_i = k_i/k_0$$

$$-j\tilde{k}_y\tilde{H}_z - \frac{\partial \tilde{H}_y}{\partial z'} = \epsilon_\mathrm{r}E_x$$

$$\frac{\partial \tilde{H}_x}{\partial z'} + j\tilde{k}_x\tilde{H}_z = \epsilon_\mathrm{r}E_y$$

$$-j\tilde{k}_x\tilde{H}_y + j\tilde{k}_y\tilde{H}_x = \epsilon_\mathrm{r}E_z$$

$$-\frac{\partial E_y}{\partial z'} - j\tilde{k}_yE_z = \mu_\mathrm{r}\tilde{H}_x$$

$$j\tilde{k}_xE_z + \frac{\partial E_x}{\partial z'} = \mu_\mathrm{r}\tilde{H}_y$$

$$j\tilde{k}_yE_x - j\tilde{k}_xE_y = \mu_\mathrm{r}\tilde{H}_z$$

Eliminating the Normal Field Components

$$-j\tilde{k}_y\tilde{H}_z - \frac{\partial \tilde{H}_y}{\partial z'} = \epsilon_{\mathrm{r}} E_x$$

$$\frac{\partial \tilde{H}_x}{\partial z'} + j\tilde{k}_x\tilde{H}_z = \epsilon_{\mathrm{r}} E_y$$

$$-j\tilde{k}_x\tilde{H}_y + j\tilde{k}_y\tilde{H}_x = \epsilon_{\mathrm{r}} E_z$$

$$-\frac{\partial E_y}{\partial z'} - j\tilde{k}_y E_z = \mu_{\mathrm{r}}\tilde{H}_x$$

$$j\tilde{k}_x E_z + \frac{\partial E_x}{\partial z'} = \mu_{\mathrm{r}}\tilde{H}_y$$

$$j\tilde{k}_y E_x - j\tilde{k}_x E_y = \mu_{\mathrm{r}}\tilde{H}_z$$

$$E_z = \frac{j}{\epsilon_{\mathrm{r}}}\left(\tilde{k}_y\tilde{H}_x - \tilde{k}_x\tilde{H}_y\right)$$

$$\tilde{H}_z = \frac{j}{\mu_{\mathrm{r}}}\left(\tilde{k}_y E_x - \tilde{k}_x E_y\right)$$

After substitution and simplification

$$\tilde{k}_y\left(\tilde{k}_y E_x - \tilde{k}_x E_y\right) - \mu_{\mathrm{r}}\frac{\partial \tilde{H}_y}{\partial z'} = \epsilon_{\mathrm{r}}\mu_{\mathrm{r}} E_x$$

$$\tilde{k}_x\left(\tilde{k}_x E_y - \tilde{k}_y E_x\right) + \mu_{\mathrm{r}}\frac{\partial \tilde{H}_x}{\partial z'} = \epsilon_{\mathrm{r}}\mu_{\mathrm{r}} E_y$$

$$\tilde{k}_y\left(\tilde{k}_y\tilde{H}_x - \tilde{k}_x\tilde{H}_y\right) - \epsilon_{\mathrm{r}}\frac{\partial E_y}{\partial z'} = \epsilon_{\mathrm{r}}\mu_{\mathrm{r}}\tilde{H}_x$$

$$\tilde{k}_x\left(\tilde{k}_x\tilde{H}_y - \tilde{k}_y\tilde{H}_x\right) + \epsilon_{\mathrm{r}}\frac{\partial E_x}{\partial z'} = \epsilon_{\mathrm{r}}\mu_{\mathrm{r}}\tilde{H}_y$$

Writing the System as a Matrix Equation

After rearranging the terms, we get

$$\frac{\partial E_x}{\partial z'} = \frac{\tilde{k}_x \tilde{k}_y}{\epsilon_\mathrm{r}} \tilde{H}_x + \left(\mu_\mathrm{r} - \frac{\tilde{k}_x^2}{\epsilon_\mathrm{r}} \right) \tilde{H}_y$$

$$\frac{\partial E_y}{\partial z'} = \left(\frac{\tilde{k}_y^2}{\epsilon_\mathrm{r}} - \mu_\mathrm{r} \right) \tilde{H}_x - \frac{\tilde{k}_x \tilde{k}_y}{\epsilon_\mathrm{r}} \tilde{H}_y$$

$$\frac{\partial \tilde{H}_x}{\partial z'} = \frac{\tilde{k}_x \tilde{k}_y}{\mu_\mathrm{r}} E_x + \left(\epsilon_\mathrm{r} - \frac{\tilde{k}_x^2}{\mu_\mathrm{r}} \right) E_y$$

$$\frac{\partial \tilde{H}_y}{\partial z'} = \left(\frac{\tilde{k}_y^2}{\mu_\mathrm{r}} - \epsilon_\mathrm{r} \right) E_x - \frac{\tilde{k}_x \tilde{k}_y}{\mu_\mathrm{r}} E_y$$

We express them as a matrix system of equations

$$\frac{\partial}{\partial z'} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{\epsilon_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix} \cdot \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} = \overline{\overline{\mathrm{P}}} \cdot \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix}$$

$$\frac{\partial}{\partial z'} \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} = \frac{1}{\mu_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \overline{\overline{\mathrm{Q}}} \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Electric Field System of Equations

We now eliminate the magnetic field

$$\frac{\partial}{\partial z'}\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \overline{\overline{\mathrm{P}}} \cdot \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} = \overline{\overline{\mathrm{P}}}^{-1} \cdot \frac{\partial}{\partial z'}\begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$\frac{\partial}{\partial z'}\begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} = \overline{\overline{\mathrm{Q}}} \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad \longrightarrow \quad \overline{\overline{\mathrm{P}}}^{-1} \cdot \frac{\partial}{\partial z'}\frac{\partial}{\partial z'}\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \overline{\overline{\mathrm{Q}}} \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Second-order differential equation

$$\frac{d^2}{dz'^2}\begin{bmatrix} E_x \\ E_y \end{bmatrix} - \overline{\overline{\Omega}}^2 \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0 \qquad \text{where} \qquad \overline{\overline{\Omega}}^2 = \overline{\overline{\mathrm{P}}} \cdot \overline{\overline{\mathrm{Q}}}$$

The P and Q matrices are defined as

$$\overline{\overline{\mathrm{P}}} = \frac{1}{\epsilon_\mathrm{r}}\begin{bmatrix} \tilde{k}_x\tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x\tilde{k}_y \end{bmatrix} \qquad \overline{\overline{\mathrm{Q}}} = \frac{1}{\mu_\mathrm{r}}\begin{bmatrix} \tilde{k}_x\tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x\tilde{k}_y \end{bmatrix}$$

Solution for the Electric Field

Second-order differential equation

$$\frac{d^2}{dz'^2}\begin{bmatrix}E_x\\E_y\end{bmatrix} - \overline{\overline{\Omega}}^2 \cdot \begin{bmatrix}E_x\\E_y\end{bmatrix} = 0 \quad \text{where} \quad \overline{\overline{\Omega}}^2 = \overline{\overline{\mathrm{P}}} \cdot \overline{\overline{\mathrm{Q}}}$$

$$\begin{cases} \overline{\overline{\mathrm{P}}} = \dfrac{1}{\epsilon_\mathrm{r}}\begin{bmatrix}\tilde{k}_x\tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2\\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x\tilde{k}_y\end{bmatrix} \\[2ex] \overline{\overline{\mathrm{Q}}} = \dfrac{1}{\mu_\mathrm{r}}\begin{bmatrix}\tilde{k}_x\tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2\\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x\tilde{k}_y\end{bmatrix} \end{cases}$$

General solution

$$\begin{bmatrix}E_x(z')\\E_y(z')\end{bmatrix} = e^{-\overline{\overline{\Omega}}z'} \cdot \mathbf{a}^+ + e^{+\overline{\overline{\Omega}}z'} \cdot \mathbf{a}^-$$

Forward propagating mode | Backward propagating mode

How to take the exponential of a matrix ?

$$e^{\pm\overline{\overline{\Omega}}z'} = \overline{\overline{\mathrm{W}}} \cdot e^{\pm\overline{\overline{\lambda}}z'} \cdot \overline{\overline{\mathrm{W}}}^{-1}$$

Applying a function *f* to a matrix *A*

$$f(\overline{\overline{\mathrm{A}}}) = \overline{\overline{\mathrm{W}}} \cdot f(\overline{\overline{\lambda}}) \cdot \overline{\overline{\mathrm{W}}}^{-1}$$

$\overline{\overline{\mathrm{W}}}$: matrix of the eigen-vectors

$\overline{\overline{\lambda}}$: matrix of the eigen-values

$\overline{\overline{\lambda}}$ is a diagonal matrix and we therefore have $e^{\pm\overline{\overline{\lambda}}z'} = \begin{bmatrix}e^{\pm\lambda_1 z'} & 0\\ 0 & e^{\pm\lambda_2 z'}\end{bmatrix}$

Simplifying the Solution

$$\begin{bmatrix} E_x(z') \\ E_y(z') \end{bmatrix} = e^{-\overline{\overline{\Omega}}z'} \cdot \mathbf{a}^+ + e^{+\overline{\overline{\Omega}}z'} \cdot \mathbf{a}^-$$

where $\overline{\overline{\Omega}}^2 = \overline{\overline{\mathrm{P}}} \cdot \overline{\overline{\mathrm{Q}}}$

$$\begin{cases} \overline{\overline{\mathrm{P}}} = \dfrac{1}{\epsilon_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix} \\ \overline{\overline{\mathrm{Q}}} = \dfrac{1}{\mu_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix} \end{cases}$$

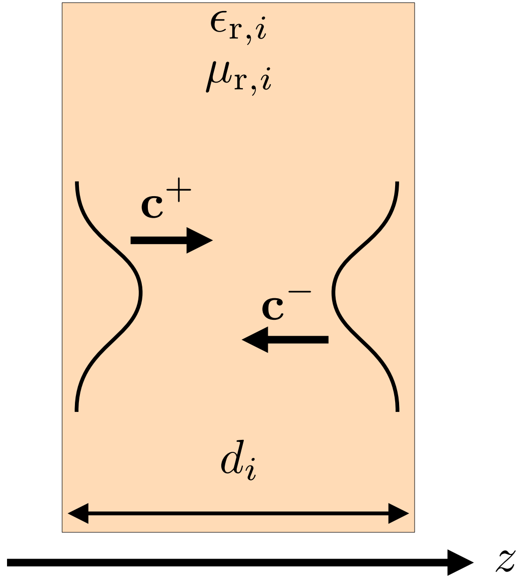
$$e^{\pm\overline{\overline{\Omega}}z'} = \overline{\overline{\mathrm{W}}} \cdot e^{\pm\overline{\overline{\lambda}}z'} \cdot \overline{\overline{\mathrm{W}}}^{-1}$$

$$\begin{bmatrix} E_x(z') \\ E_y(z') \end{bmatrix} = \overline{\overline{\mathrm{W}}} \cdot e^{-\overline{\overline{\lambda}}z'} \cdot \overline{\overline{\mathrm{W}}}^{-1} \cdot \mathbf{a}^+ + \overline{\overline{\mathrm{W}}} \cdot e^{\overline{\overline{\lambda}}z'} \cdot \overline{\overline{\mathrm{W}}}^{-1} \cdot \mathbf{a}^-$$

We re-express the mode amplitudes

$$\begin{bmatrix} E_x(z') \\ E_y(z') \end{bmatrix} = \overline{\overline{\mathrm{W}}} \cdot e^{-\overline{\overline{\lambda}}z'} \cdot \mathbf{c}^+ + \overline{\overline{\mathrm{W}}} \cdot e^{\overline{\overline{\lambda}}z'} \cdot \mathbf{c}^-$$

i^{th}-layer of the system



Solution for the Magnetic Field

Solution for the electric field

$$\begin{bmatrix} E_x(z') \\ E_y(z') \end{bmatrix} = \overline{\overline{\text{W}}} \cdot e^{-\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^+ + \overline{\overline{\text{W}}} \cdot e^{\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^-$$

$$\frac{\partial}{\partial z'} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \overline{\overline{\text{P}}} \cdot \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix}$$

$$\frac{\partial}{\partial z'} \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} = \overline{\overline{\text{Q}}} \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Let's express the magnetic field solution as

$$\begin{bmatrix} \tilde{H}_x(z') \\ \tilde{H}_y(z') \end{bmatrix} = -\overline{\overline{\text{V}}} \cdot e^{-\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^+ + \overline{\overline{\text{V}}} \cdot e^{\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^-$$

Derive along z

$$\frac{\partial}{\partial z'} \begin{bmatrix} \tilde{H}_x(z') \\ \tilde{H}_y(z') \end{bmatrix} = \overline{\overline{\text{V}}} \cdot \overline{\overline{\lambda}} \cdot e^{-\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^+ + \overline{\overline{\text{V}}} \cdot \overline{\overline{\lambda}} \cdot e^{\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^- = \overline{\overline{\text{Q}}} \cdot \left(\overline{\overline{\text{W}}} \cdot e^{-\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^+ + \overline{\overline{\text{W}}} \cdot e^{\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^- \right)$$

$$\overline{\overline{\text{V}}} \cdot \overline{\overline{\lambda}} \cdot \left(e^{-\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^+ + e^{\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^- \right) = \overline{\overline{\text{Q}}} \cdot \overline{\overline{\text{W}}} \cdot \left(e^{-\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^+ + e^{\overline{\overline{\lambda}} z'} \cdot \mathbf{c}^- \right)$$

$$\overline{\overline{\text{V}}} \cdot \overline{\overline{\lambda}} = \overline{\overline{\text{Q}}} \cdot \overline{\overline{\text{W}}}$$

$$\overline{\overline{\text{V}}} = \overline{\overline{\text{Q}}} \cdot \overline{\overline{\text{W}}} \cdot \overline{\overline{\lambda}}^{-1}$$

Complete Solution

Combined electric and magnetic solutions

$$\boldsymbol{\Psi}(z') = \begin{bmatrix} E_x(z') \\ E_y(z') \\ \tilde{H}_x(z') \\ \tilde{H}_y(z') \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}} & \overline{\overline{\mathrm{W}}} \\ -\overline{\overline{\mathrm{V}}} & \overline{\overline{\mathrm{V}}} \end{bmatrix} \cdot \begin{bmatrix} e^{-\overline{\overline{\lambda}} z'} & 0 \\ 0 & e^{+\overline{\overline{\lambda}} z'} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \end{bmatrix}$$

Forward/backward mode amplitudes within a layer

1) Compute P and Q from the layer parameters and the incidence angle

$$\overline{\overline{\mathrm{P}}} = \frac{1}{\epsilon_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix}$$

$$\overline{\overline{\mathrm{Q}}} = \frac{1}{\mu_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix}$$

2) Compute Ω

$$\overline{\overline{\Omega}}^2 = \overline{\overline{\mathrm{P}}} \cdot \overline{\overline{\mathrm{Q}}}$$

3) Find the eigen-vectors and eigen-values of Ω

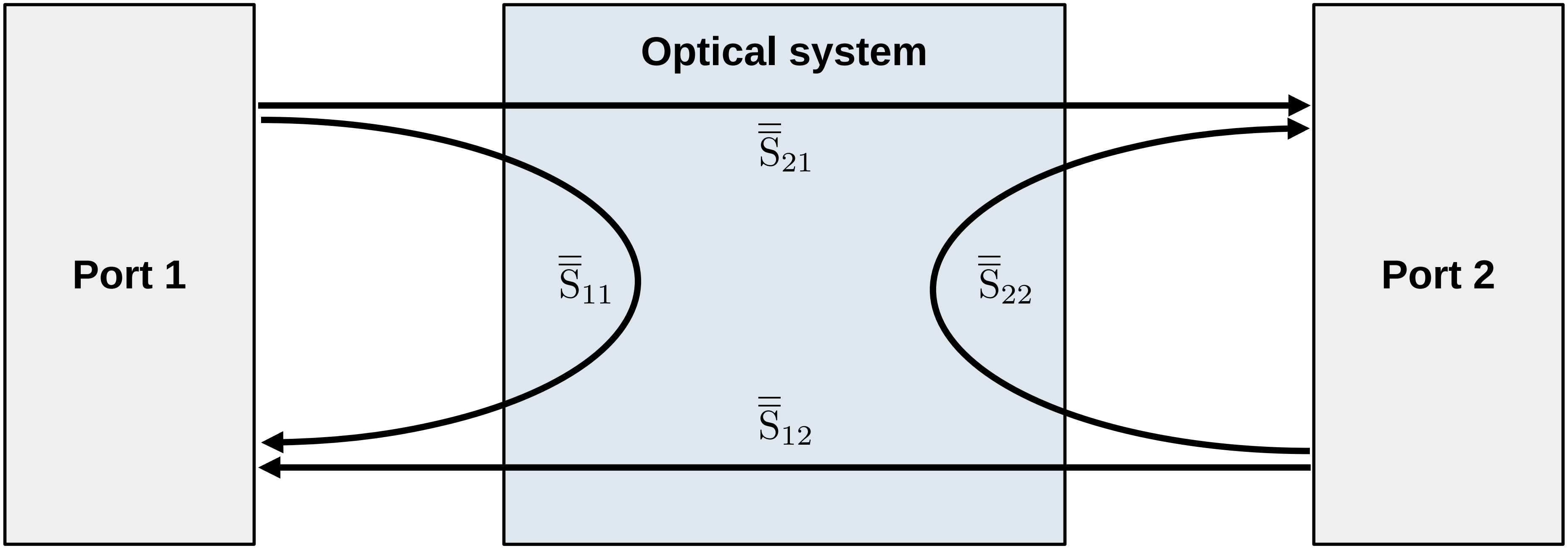
$\overline{\overline{\mathrm{W}}}$: matrix of the eigen-vectors of $\overline{\overline{\Omega}}$

$\overline{\overline{\lambda}}$: matrix of the eigen-values of $\overline{\overline{\Omega}}$

4) Compute V

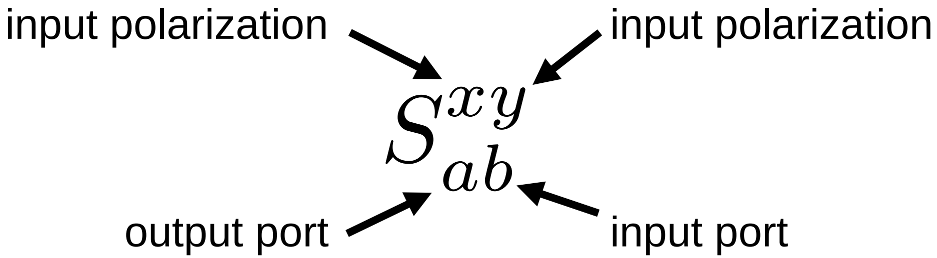
$$\overline{\overline{\mathrm{V}}} = \overline{\overline{\mathrm{Q}}} \cdot \overline{\overline{\mathrm{W}}} \cdot \overline{\overline{\lambda}}^{-1}$$

The Scattering Matrix



Assuming x/y polarizations

$$\overline{\overline{S}}_{ab} = \begin{bmatrix} S_{ab}^{xy} & S_{ab}^{yy} \\ S_{ab}^{yx} & S_{ab}^{yy} \end{bmatrix}$$

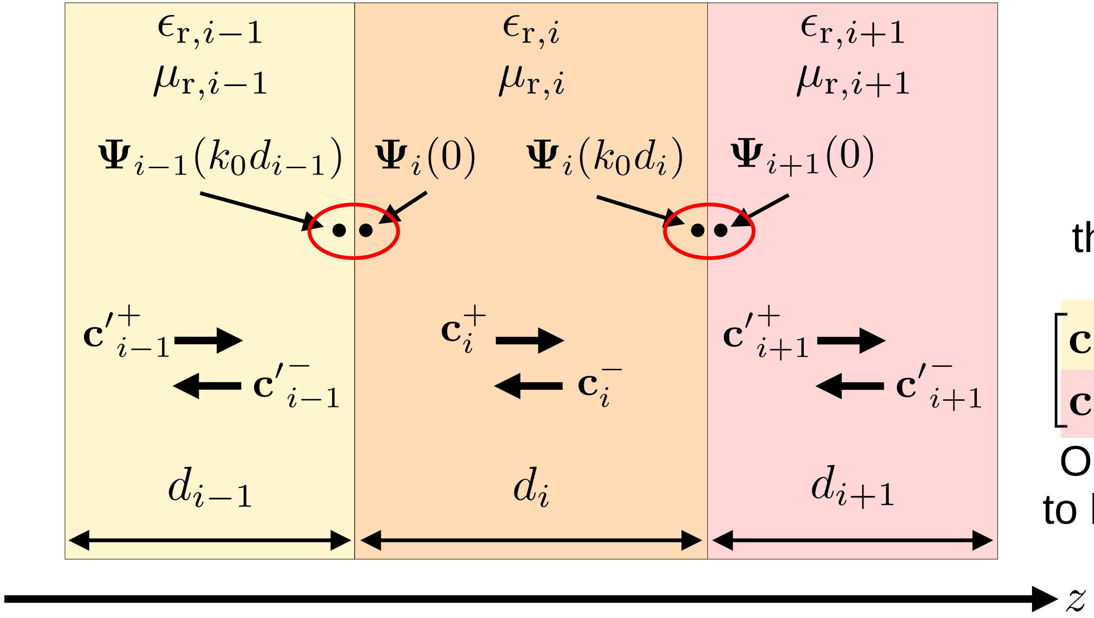


Connecting the Layers Together via Scattering Matrices

$$\mathbf{\Psi}(z') = \begin{bmatrix} E_x(z') \\ E_y(z') \\ \tilde{H}_x(z') \\ \tilde{H}_y(z') \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}} & \overline{\overline{\mathrm{W}}} \\ -\overline{\overline{\mathrm{V}}} & \overline{\overline{\mathrm{V}}} \end{bmatrix} \cdot \begin{bmatrix} e^{-\overline{\overline{\lambda}}z'} & 0 \\ 0 & e^{+\overline{\overline{\lambda}}z'} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \end{bmatrix}$$

Tangential field components are equal at the interfaces

Match boundary conditions at both interfaces



Remember that

$$z' = k_0 z$$

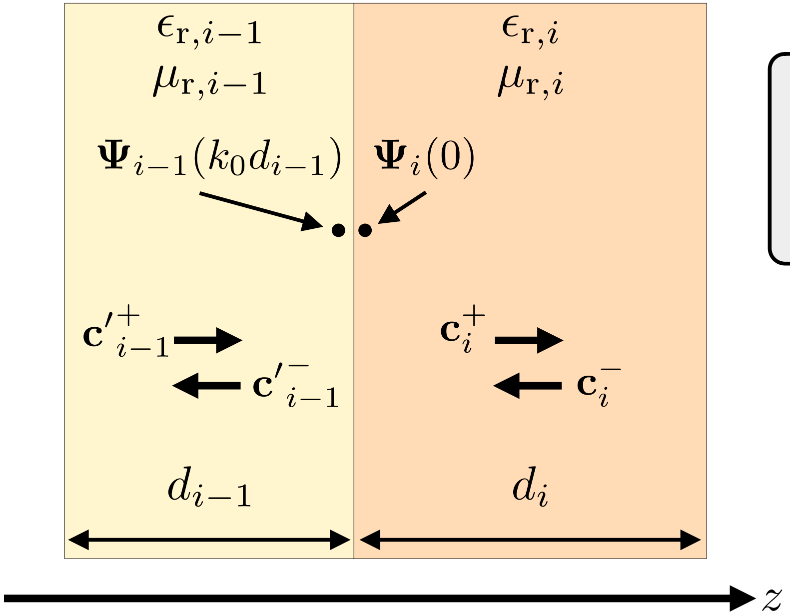
We want to find the scattering matrix of the i^{th}-layer that connects the mode profiles of the *i-1*th and *i+1*th layers together

$$\begin{bmatrix} \mathbf{c'}^{-}_{i-1} \\ \mathbf{c'}^{+}_{i+1} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{S}}}^{(i)}_{11} & \overline{\overline{\mathrm{S}}}^{(i)}_{12} \\ \overline{\overline{\mathrm{S}}}^{(i)}_{21} & \overline{\overline{\mathrm{S}}}^{(i)}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}^{+}_{i-1} \\ \mathbf{c'}^{-}_{i+1} \end{bmatrix}$$

Output to layer *i*

Input to layer *i*

Boundary Conditions at the First Interface



$$\mathbf{\Psi}(z') = \begin{bmatrix} \overline{\overline{\mathrm{W}}} & \overline{\overline{\mathrm{W}}} \\ -\overline{\overline{\mathrm{V}}} & \overline{\overline{\mathrm{V}}} \end{bmatrix} \cdot \begin{bmatrix} e^{-\overline{\overline{\lambda}} z'} & 0 \\ 0 & e^{+\overline{\overline{\lambda}} z'} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}^{+} \\ \mathbf{c}^{-} \end{bmatrix}$$

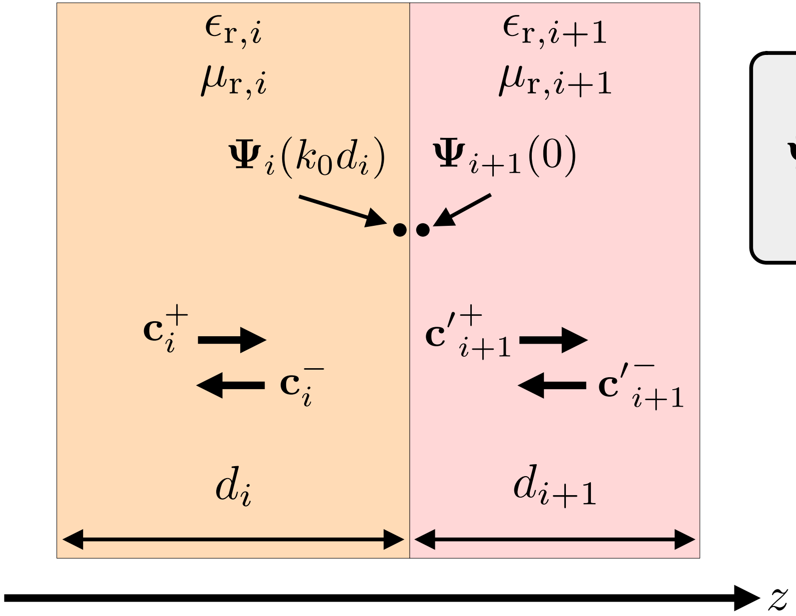
Boundary condition

$$\mathbf{\Psi}_{i-1}(k_0 d_{i-1}) = \mathbf{\Psi}_i(0)$$

$$\begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i-1} & \overline{\overline{\mathrm{W}}}_{i-1} \\ -\overline{\overline{\mathrm{V}}}_{i-1} & \overline{\overline{\mathrm{V}}}_{i-1} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} e^{-\overline{\overline{\lambda}}_{i-1} k_0 d_{i-1}} & 0 \\ 0 & e^{\overline{\overline{\lambda}}_{i-1} k_0 d_{i-1}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}^{+}_{i-1} \\ \mathbf{c}^{-}_{i-1} \end{bmatrix}} = \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i} & \overline{\overline{\mathrm{W}}}_{i} \\ -\overline{\overline{\mathrm{V}}}_{i} & \overline{\overline{\mathrm{V}}}_{i} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}^{+}_{i} \\ \mathbf{c}^{-}_{i} \end{bmatrix}$$

$$\begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i-1} & \overline{\overline{\mathrm{W}}}_{i-1} \\ -\overline{\overline{\mathrm{V}}}_{i-1} & \overline{\overline{\mathrm{V}}}_{i-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}'^{+}_{i-1} \\ \mathbf{c}'^{-}_{i-1} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i} & \overline{\overline{\mathrm{W}}}_{i} \\ -\overline{\overline{\mathrm{V}}}_{i} & \overline{\overline{\mathrm{V}}}_{i} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}^{+}_{i} \\ \mathbf{c}^{-}_{i} \end{bmatrix}$$

Boundary Conditions at the Second Interface



$$\mathbf{\Psi}(z') = \begin{bmatrix} \overline{\overline{\mathrm{W}}} & \overline{\overline{\mathrm{W}}} \\ -\overline{\overline{\mathrm{V}}} & \overline{\overline{\mathrm{V}}} \end{bmatrix} \cdot \begin{bmatrix} e^{-\overline{\overline{\lambda}} z'} & 0 \\ 0 & e^{+\overline{\overline{\lambda}} z'} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \end{bmatrix}$$

Boundary condition

$$\mathbf{\Psi}_i(k_0 d_i) = \mathbf{\Psi}_{i+1}(0)$$

$$\begin{bmatrix} \overline{\overline{\mathrm{W}}}_i & \overline{\overline{\mathrm{W}}}_i \\ -\overline{\overline{\mathrm{V}}}_i & \overline{\overline{\mathrm{V}}}_i \end{bmatrix} \cdot \begin{bmatrix} e^{-\overline{\overline{\lambda}}_i k_0 d_i} & 0 \\ 0 & e^{\overline{\overline{\lambda}}_i k_0 d_i} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i+1} & \overline{\overline{\mathrm{W}}}_{i+1} \\ -\overline{\overline{\mathrm{V}}}_{i+1} & \overline{\overline{\mathrm{V}}}_{i+1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i+1}^+ \\ \mathbf{c'}_{i+1}^- \end{bmatrix}$$

Combining Together the Interface Conditions

From the first interface, we have

$$\begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i-1} & \overline{\overline{\mathrm{W}}}_{i-1} \\ -\overline{\overline{\mathrm{V}}}_{i-1} & \overline{\overline{\mathrm{V}}}_{i-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i-1}^{+} \\ \mathbf{c'}_{i-1}^{-} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i} & \overline{\overline{\mathrm{W}}}_{i} \\ -\overline{\overline{\mathrm{V}}}_{i} & \overline{\overline{\mathrm{V}}}_{i} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}_{i}^{+} \\ \mathbf{c}_{i}^{-} \end{bmatrix}$$

The scattering matrix of the i[th] layer

$$\begin{bmatrix} \mathbf{c'}_{i-1}^{-} \\ \mathbf{c'}_{i+1}^{+} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{S}}}_{11}^{(i)} & \overline{\overline{\mathrm{S}}}_{12}^{(i)} \\ \overline{\overline{\mathrm{S}}}_{21}^{(i)} & \overline{\overline{\mathrm{S}}}_{22}^{(i)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i-1}^{+} \\ \mathbf{c'}_{i+1}^{-} \end{bmatrix}$$

From the second interface, we have

$$\begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i} & \overline{\overline{\mathrm{W}}}_{i} \\ -\overline{\overline{\mathrm{V}}}_{i} & \overline{\overline{\mathrm{V}}}_{i} \end{bmatrix} \cdot \begin{bmatrix} e^{-\overline{\overline{\lambda}}_i k_0 d_i} & 0 \\ 0 & e^{\overline{\overline{\lambda}}_i k_0 d_i} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}_{i}^{+} \\ \mathbf{c}_{i}^{-} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i+1} & \overline{\overline{\mathrm{W}}}_{i+1} \\ -\overline{\overline{\mathrm{V}}}_{i+1} & \overline{\overline{\mathrm{V}}}_{i+1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i+1}^{+} \\ \mathbf{c'}_{i+1}^{-} \end{bmatrix}$$

From these two equations, we express the mode profiles of the *i*[th] layer as

$$\begin{bmatrix} \mathbf{c}_{i}^{+} \\ \mathbf{c}_{i}^{-} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i} & \overline{\overline{\mathrm{W}}}_{i} \\ -\overline{\overline{\mathrm{V}}}_{i} & \overline{\overline{\mathrm{V}}}_{i} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i-1} & \overline{\overline{\mathrm{W}}}_{i-1} \\ -\overline{\overline{\mathrm{V}}}_{i-1} & \overline{\overline{\mathrm{V}}}_{i-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i-1}^{+} \\ \mathbf{c'}_{i-1}^{-} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{c}_{i}^{+} \\ \mathbf{c}_{i}^{-} \end{bmatrix} = \begin{bmatrix} e^{\overline{\overline{\lambda}}_i k_0 d_i} & 0 \\ 0 & e^{-\overline{\overline{\lambda}}_i k_0 d_i} \end{bmatrix} \cdot \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i} & \overline{\overline{\mathrm{W}}}_{i} \\ -\overline{\overline{\mathrm{V}}}_{i} & \overline{\overline{\mathrm{V}}}_{i} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i+1} & \overline{\overline{\mathrm{W}}}_{i+1} \\ -\overline{\overline{\mathrm{V}}}_{i+1} & \overline{\overline{\mathrm{V}}}_{i+1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i+1}^{+} \\ \mathbf{c'}_{i+1}^{-} \end{bmatrix}$$

We now equate these two relations and get rid of the mode profiles of the *i*[th] layer

Combining Together the Interface Conditions

$$\begin{cases} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{W}}}_i & \overline{\overline{\mathrm{W}}}_i \\ -\overline{\overline{\mathrm{V}}}_i & \overline{\overline{\mathrm{V}}}_i \end{bmatrix}^{-1} \cdot \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i-1} & \overline{\overline{\mathrm{W}}}_{i-1} \\ -\overline{\overline{\mathrm{V}}}_{i-1} & \overline{\overline{\mathrm{V}}}_{i-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i-1}^+ \\ \mathbf{c'}_{i-1}^- \end{bmatrix} \\ \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} e^{\overline{\overline{\lambda}}_i k_0 d_i} & 0 \\ 0 & e^{-\overline{\overline{\lambda}}_i k_0 d_i} \end{bmatrix} \cdot \begin{bmatrix} \overline{\overline{\mathrm{W}}}_i & \overline{\overline{\mathrm{W}}}_i \\ -\overline{\overline{\mathrm{V}}}_i & \overline{\overline{\mathrm{V}}}_i \end{bmatrix}^{-1} \cdot \begin{bmatrix} \overline{\overline{\mathrm{W}}}_{i+1} & \overline{\overline{\mathrm{W}}}_{i+1} \\ -\overline{\overline{\mathrm{V}}}_{i+1} & \overline{\overline{\mathrm{V}}}_{i+1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i+1}^+ \\ \mathbf{c'}_{i+1}^- \end{bmatrix} \end{cases}$$

For convenience, we use the following substitution

$$\begin{bmatrix} \overline{\overline{\mathrm{W}}}_i & \overline{\overline{\mathrm{W}}}_i \\ -\overline{\overline{\mathrm{V}}}_i & \overline{\overline{\mathrm{V}}}_i \end{bmatrix}^{-1} \cdot \begin{bmatrix} \overline{\overline{\mathrm{W}}}_j & \overline{\overline{\mathrm{W}}}_j \\ -\overline{\overline{\mathrm{V}}}_j & \overline{\overline{\mathrm{V}}}_j \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \overline{\overline{\mathrm{A}}}_{ij} & \overline{\overline{\mathrm{B}}}_{ij} \\ \overline{\overline{\mathrm{B}}}_{ij} & \overline{\overline{\mathrm{A}}}_{ij} \end{bmatrix} \quad \text{where} \quad \begin{cases} \overline{\overline{\mathrm{A}}}_{ij} = \overline{\overline{\mathrm{W}}}_i^{-1} \cdot \overline{\overline{\mathrm{W}}}_j + \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_j \\ \overline{\overline{\mathrm{B}}}_{ij} = \overline{\overline{\mathrm{W}}}_i^{-1} \cdot \overline{\overline{\mathrm{W}}}_j - \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_j \end{cases}$$

We finally obtain a relation with only the mode profiles of the *i-1*th and *i+1*th layer

$$\begin{bmatrix} \overline{\overline{\mathrm{A}}}_{i,i-1} & \overline{\overline{\mathrm{B}}}_{i,i-1} \\ \overline{\overline{\mathrm{B}}}_{i,i-1} & \overline{\overline{\mathrm{A}}}_{i,i-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i-1}^+ \\ \mathbf{c'}_{i-1}^- \end{bmatrix} = \begin{bmatrix} e^{\overline{\overline{\lambda}}_i k_0 d_i} & 0 \\ 0 & e^{-\overline{\overline{\lambda}}_i k_0 d_i} \end{bmatrix} \cdot \begin{bmatrix} \overline{\overline{\mathrm{A}}}_{i,i+1} & \overline{\overline{\mathrm{B}}}_{i,i+1} \\ \overline{\overline{\mathrm{B}}}_{i,i+1} & \overline{\overline{\mathrm{A}}}_{i,i+1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i+1}^+ \\ \mathbf{c'}_{i+1}^- \end{bmatrix}$$

General Expression of the Scattering Matrix

$$\begin{bmatrix} \overline{\overline{\mathrm{A}}}_{i,i-1} & \overline{\overline{\mathrm{B}}}_{i,i-1} \\ \overline{\overline{\mathrm{B}}}_{i,i-1} & \overline{\overline{\mathrm{A}}}_{i,i-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}^{+}_{i-1} \\ \mathbf{c'}^{-}_{i-1} \end{bmatrix} = \begin{bmatrix} e^{\overline{\overline{\lambda}}_i k_0 d_i} & 0 \\ 0 & e^{-\overline{\overline{\lambda}}_i k_0 d_i} \end{bmatrix} \cdot \begin{bmatrix} \overline{\overline{\mathrm{A}}}_{i,i+1} & \overline{\overline{\mathrm{B}}}_{i,i+1} \\ \overline{\overline{\mathrm{B}}}_{i,i+1} & \overline{\overline{\mathrm{A}}}_{i,i+1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}^{+}_{i+1} \\ \mathbf{c'}^{-}_{i+1} \end{bmatrix}$$

To find the scattering matrix, we need to transform a bit this expression

$$\begin{bmatrix} \mathbf{c'}^{-}_{i-1} \\ \mathbf{c'}^{+}_{i+1} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{S}}}^{(i)}_{11} & \overline{\overline{\mathrm{S}}}^{(i)}_{12} \\ \overline{\overline{\mathrm{S}}}^{(i)}_{21} & \overline{\overline{\mathrm{S}}}^{(i)}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}^{+}_{i-1} \\ \mathbf{c'}^{-}_{i+1} \end{bmatrix}$$

$$\overline{\overline{\mathrm{S}}}^{(i)}_{11} = \left(\overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i+1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1}\right)^{-1} \cdot \left(\overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i+1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{B}}}_{i,i-1}\right)$$

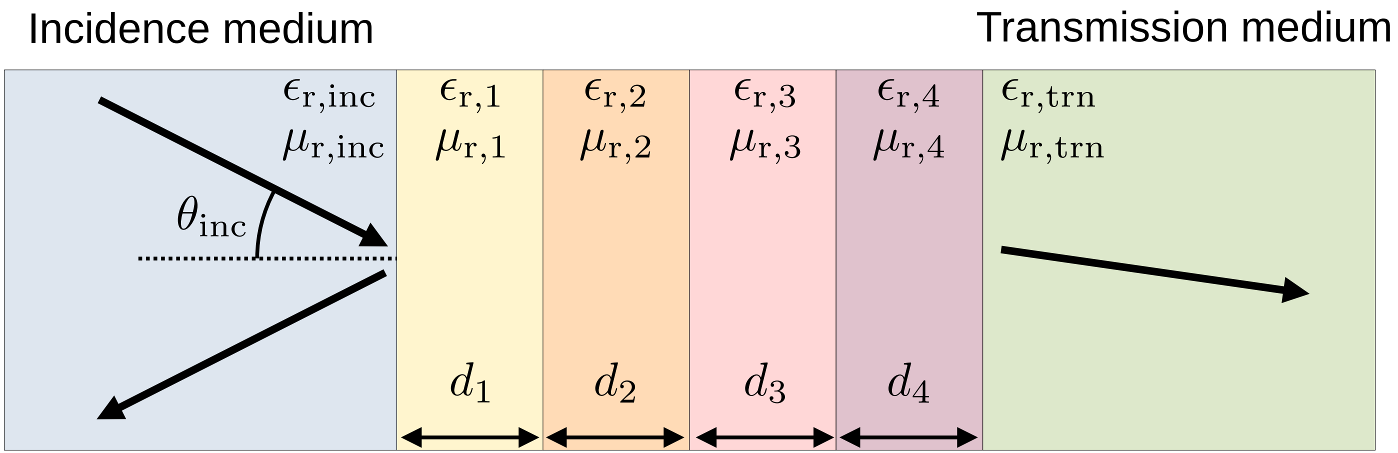
$$\overline{\overline{\mathrm{S}}}^{(i)}_{12} = \left(\overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i+1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1}\right)^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \left(\overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i+1} \cdot \overline{\overline{\mathrm{B}}}_{i,i+1}\right)$$

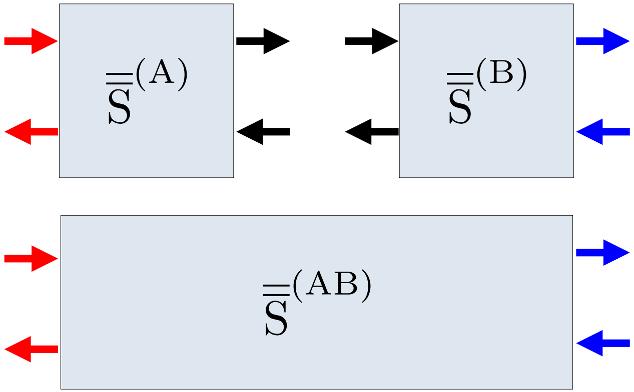
$$\overline{\overline{\mathrm{S}}}^{(i)}_{21} = \left(\overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1}\right)^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \left(\overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i-1} \cdot \overline{\overline{\mathrm{B}}}_{i,i-1}\right)$$

$$\overline{\overline{\mathrm{S}}}^{(i)}_{22} = \left(\overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1}\right)^{-1} \cdot \left(\overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}^{-1}_{i,i-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{B}}}_{i,i+1}\right)$$

where $\overline{\overline{\mathrm{X}}}_i = e^{-\overline{\overline{\lambda}}_i k_0 d_i}$

Global Scattering Matrix





$$\overline{\overline{\mathrm{S}}}^{(\mathrm{global})} = \overline{\overline{\mathrm{S}}}^{(\mathrm{inc})} \otimes \left[\overline{\overline{\mathrm{S}}}^{(1)} \otimes \overline{\overline{\mathrm{S}}}^{(2)} \otimes \cdots \otimes \overline{\overline{\mathrm{S}}}^{(N-1)} \otimes \overline{\overline{\mathrm{S}}}^{(N)} \right] \otimes \overline{\overline{\mathrm{S}}}^{(\mathrm{trn})}$$

$$\begin{bmatrix} \mathbf{c'}_{i-1}^{-} \\ \mathbf{c'}_{i+1}^{+} \end{bmatrix} = \begin{bmatrix} \overline{\overline{\mathrm{S}}}_{11}^{(i)} & \overline{\overline{\mathrm{S}}}_{12}^{(i)} \\ \overline{\overline{\mathrm{S}}}_{21}^{(i)} & \overline{\overline{\mathrm{S}}}_{22}^{(i)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c'}_{i-1}^{+} \\ \mathbf{c'}_{i+1}^{-} \end{bmatrix}$$

Redheffer product $\overline{\overline{\mathrm{S}}}^{(\mathrm{A})} \otimes \overline{\overline{\mathrm{S}}}^{(\mathrm{B})} = \overline{\overline{\mathrm{S}}}^{(\mathrm{AB})}$ is defined from the expression of the scattering matrix and by connecting the input and output of $\mathrm{S}^{(\mathrm{A})}$ and $\mathrm{S}^{(\mathrm{B})}$ together

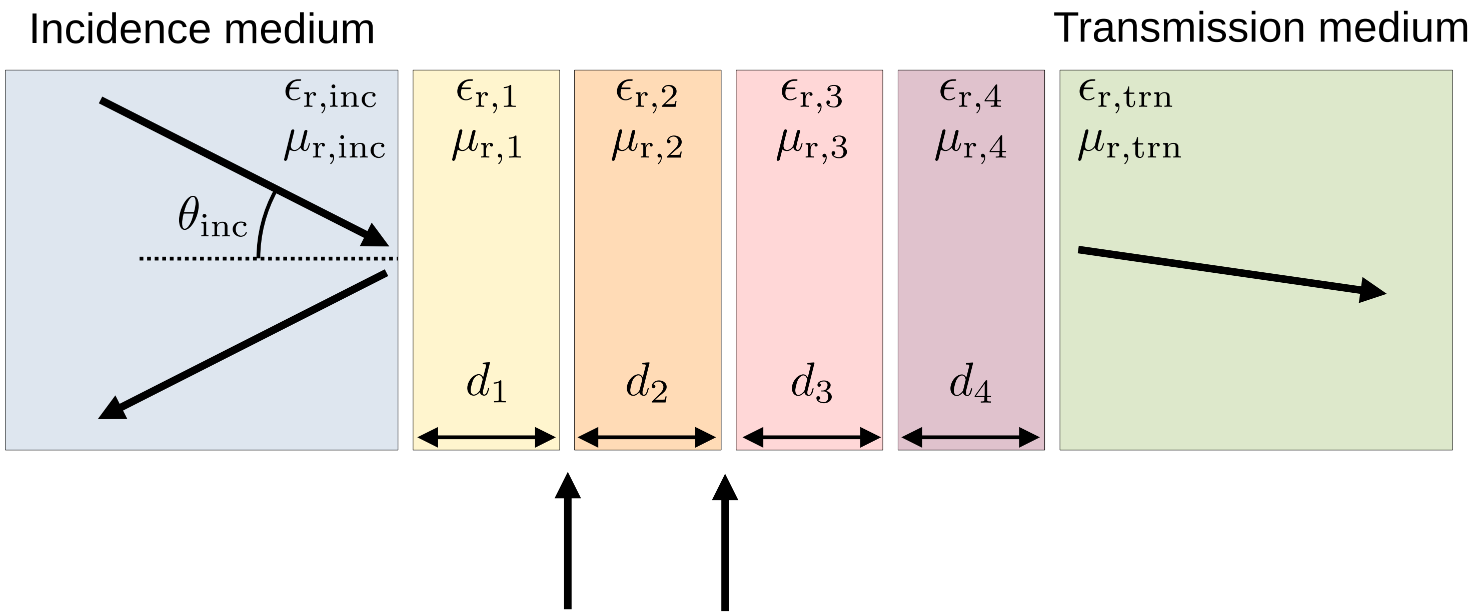
$$\overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{A})} + \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{A})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \cdot \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \cdot \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{A})}$$

$$\overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{A})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \cdot \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{B})}$$

$$\overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{B})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \cdot \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{A})}$$

$$\overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{B})} + \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{B})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \cdot \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \cdot \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{B})}$$

Problem Simplification Using Gap Media



What if each layer was surrounded by zero-thickness gap media?

This makes each layer problem symmetric without affecting the overall scattering response!

Scattering Matrix with Gap Media

Without gaps

$$\begin{cases}
\overline{\overline{\mathrm{S}}}_{11}^{(i)} = \left(\overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}_{i,i+1}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1}\right)^{-1} \cdot \left(\overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}_{i,i+1}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{B}}}_{i,i-1}\right) \\
\overline{\overline{\mathrm{S}}}_{12}^{(i)} = \left(\overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}_{i,i+1}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1}\right)^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \left(\overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{B}}}_{i,i+1} \cdot \overline{\overline{\mathrm{A}}}_{i,i+1}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{i,i+1}\right) \\
\overline{\overline{\mathrm{S}}}_{21}^{(i)} = \left(\overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}_{i,i-1}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1}\right)^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \left(\overline{\overline{\mathrm{A}}}_{i,i-1} - \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}_{i,i-1}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{i,i-1}\right) \\
\overline{\overline{\mathrm{S}}}_{22}^{(i)} = \left(\overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}_{i,i-1}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i+1}\right)^{-1} \cdot \left(\overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i,i-1} \cdot \overline{\overline{\mathrm{A}}}_{i,i-1}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{A}}}_{i,i+1} - \overline{\overline{\mathrm{B}}}_{i,i+1}\right)
\end{cases}$$

With gaps

$$\begin{cases}
\overline{\overline{\mathrm{S}}}_{11}^{(i)} = \left(\overline{\overline{\mathrm{A}}}_{i} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i} \cdot \overline{\overline{\mathrm{A}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i}\right)^{-1} \cdot \left(\overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i} \cdot \overline{\overline{\mathrm{A}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{A}}}_{i} - \overline{\overline{\mathrm{B}}}_{i}\right) \\
\overline{\overline{\mathrm{S}}}_{12}^{(i)} = \left(\overline{\overline{\mathrm{A}}}_{i} - \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i} \cdot \overline{\overline{\mathrm{A}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \overline{\overline{\mathrm{B}}}_{i}\right)^{-1} \cdot \overline{\overline{\mathrm{X}}}_i \cdot \left(\overline{\overline{\mathrm{A}}}_{i} - \overline{\overline{\mathrm{B}}}_{i} \cdot \overline{\overline{\mathrm{A}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{i}\right) \\
\overline{\overline{\mathrm{S}}}_{21}^{(i)} = \overline{\overline{\mathrm{S}}}_{12}^{(i)} \\
\overline{\overline{\mathrm{S}}}_{22}^{(i)} = \overline{\overline{\mathrm{S}}}_{11}^{(i)}
\end{cases}$$

gap parameters

where
$$\begin{cases}
\overline{\overline{\mathrm{A}}}_{i} = \overline{\overline{\mathrm{W}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{W}}}_{0} + \overline{\overline{\mathrm{V}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{V}}}_{0} \\
\overline{\overline{\mathrm{B}}}_{i} = \overline{\overline{\mathrm{W}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{W}}}_{0} - \overline{\overline{\mathrm{V}}}_{i}^{-1} \cdot \overline{\overline{\mathrm{V}}}_{0}
\end{cases}$$

Simplification for Linear Homogeneous Isotropic Media

$$\overline{\overline{\mathrm{P}}} = \frac{1}{\epsilon_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix}$$

$$\overline{\overline{\mathrm{Q}}} = \frac{1}{\mu_\mathrm{r}} \begin{bmatrix} \tilde{k}_x \tilde{k}_y & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_\mathrm{r}\mu_\mathrm{r} & -\tilde{k}_x \tilde{k}_y \end{bmatrix}$$

In a LHI medium, we have

$$\epsilon_\mathrm{r}\mu_\mathrm{r} = \tilde{k}_x^2 + \tilde{k}_y^2 + \tilde{k}_z^2 = \tilde{k}^2$$

$$\overline{\overline{\Omega}}^2 = \overline{\overline{\mathrm{P}}} \cdot \overline{\overline{\mathrm{Q}}} = -\tilde{k}_z^2 \overline{\overline{\mathrm{I}}}$$

$$\overline{\overline{\mathrm{I}}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

..it follows that..

Eigen-vector matrix of Ω

$$\overline{\overline{\mathrm{W}}} = \overline{\overline{\mathrm{I}}}$$

Eigen-value matrix of Ω

$$\overline{\overline{\lambda}} = j\tilde{k}_z \overline{\overline{\mathrm{I}}} \quad \longrightarrow \quad \overline{\overline{\Omega}}^2 = \overline{\overline{\lambda}}^2$$

..and that..

$$\overline{\overline{\mathrm{V}}} = \overline{\overline{\mathrm{Q}}} \cdot \overline{\overline{\mathrm{W}}} \cdot \overline{\overline{\lambda}}^{-1} = -\frac{j}{\tilde{k}_z} \overline{\overline{\mathrm{Q}}}$$

$$\overline{\overline{\mathrm{A}}}_i = \overline{\overline{\mathrm{W}}}_i^{-1} \cdot \overline{\overline{\mathrm{W}}}_0 + \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0 = \overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0$$

$$\overline{\overline{\mathrm{B}}}_i = \overline{\overline{\mathrm{W}}}_i^{-1} \cdot \overline{\overline{\mathrm{W}}}_0 - \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0 = \overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0$$

Parameters of the Gap Medium

We are free to choose the parameters of the gap media as we wish. To prevent k_z from being zero, we may choose

Since $\epsilon_r \mu_r = \tilde{k}_x^2 + \tilde{k}_y^2 + \tilde{k}_z^2$ ⟶ $\begin{cases} \mu_r = 1 \\ \epsilon_r = 1 + \tilde{k}_x^2 + \tilde{k}_y^2 \end{cases}$

Since the problem is invariant by a rotation ϕ, we can restrict our attention to the *xz*-plane

$$\tilde{k}_y = 0$$

$$\overline{\overline{\mathrm{V}}} = -\frac{j}{\tilde{k}_z}\overline{\overline{\mathrm{Q}}}$$

$$\overline{\overline{\mathrm{Q}}} = \frac{1}{\mu_r}\begin{bmatrix} \tilde{k}_x\tilde{k}_y & \epsilon_r\mu_r - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \epsilon_r\mu_r & -\tilde{k}_x\tilde{k}_y \end{bmatrix}$$

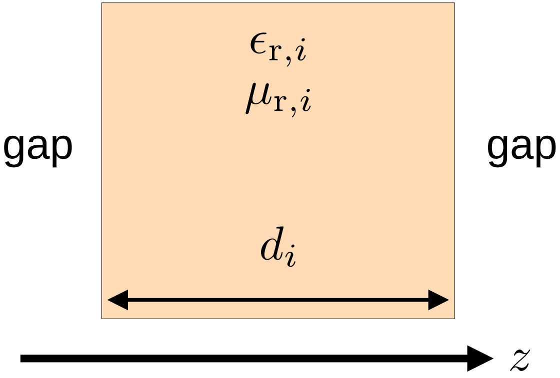
⟶

$$\overline{\overline{\mathrm{V}}}_0 = -j\begin{bmatrix} 0 & 1 \\ -(1+\tilde{k}_x^2) & 0 \end{bmatrix}$$

k_x is given by the incidence angle

$$\tilde{k}_x = n_{\mathrm{inc}} \sin\theta_{\mathrm{inc}}$$

Scattering Matrix of Each Layer



1) define the k-vector

$$\tilde{k}_x = n_{\text{inc}} \sin \theta_{\text{inc}} \qquad \tilde{k}_{z,i} = \sqrt{\epsilon_{r,i}\mu_{r,i} - \tilde{k}_x^2}$$

2) define the V matrix and X parameter

$$\overline{\overline{\mathrm{V}}}_i = -\frac{j}{\tilde{k}_{z,i}\mu_{r,i}} \begin{bmatrix} 0 & \epsilon_{r,i}\mu_{r,i} - \tilde{k}_x^2 \\ -\epsilon_{r,i}\mu_{r,i} & 0 \end{bmatrix}$$

$$X_i = e^{j\tilde{k}_z k_0 d_i}$$

3) define the A, B matrices

$$\overline{\overline{\mathrm{A}}}_i = \overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0$$

$$\overline{\overline{\mathrm{B}}}_i = \overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0$$

$$\overline{\overline{\mathrm{V}}}_0 = -j \begin{bmatrix} 0 & 1 \\ -(1 + \tilde{k}_x^2) & 0 \end{bmatrix}$$

4) define the D matrix

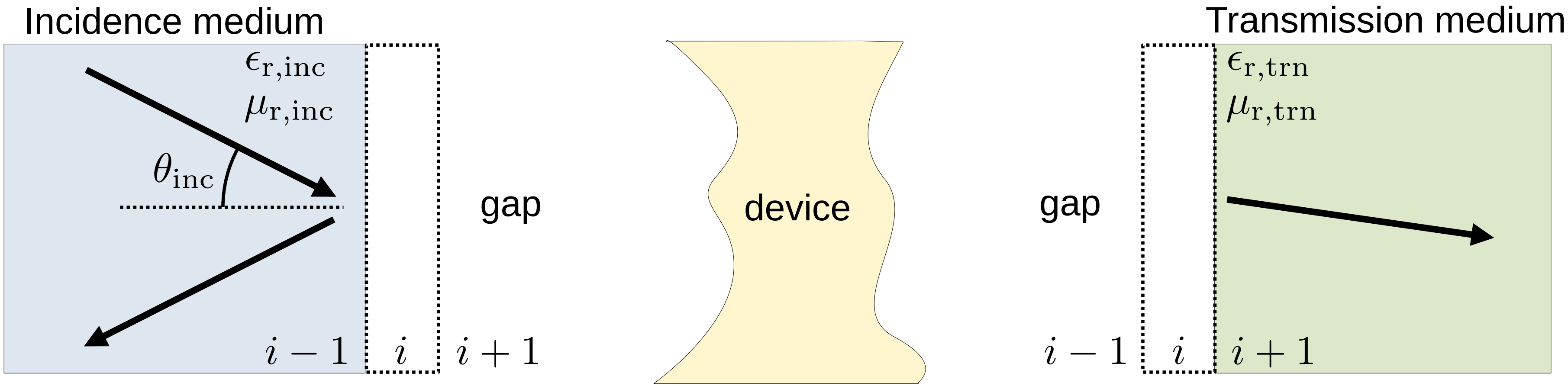
$$\overline{\overline{\mathrm{D}}}_i = \left(\overline{\overline{\mathrm{A}}}_i - X_i^2 \overline{\overline{\mathrm{B}}}_i \cdot \overline{\overline{\mathrm{A}}}_i^{-1} \cdot \overline{\overline{\mathrm{B}}}_i \right)^{-1}$$

5) define the S matrix

$$\overline{\overline{\mathrm{S}}}_{11}^{(i)} = \overline{\overline{\mathrm{S}}}_{22}^{(i)} = \overline{\overline{\mathrm{D}}}_i \cdot \overline{\overline{\mathrm{B}}}_i \left(X_i^2 - 1 \right)$$

$$\overline{\overline{\mathrm{S}}}_{12}^{(i)} = \overline{\overline{\mathrm{S}}}_{21}^{(i)} = \overline{\overline{\mathrm{D}}}_i \cdot \left(\overline{\overline{\mathrm{A}}}_i - \overline{\overline{\mathrm{B}}}_i \cdot \overline{\overline{\mathrm{A}}}_i^{-1} \cdot \overline{\overline{\mathrm{B}}}_i \right) X_i$$

Connection to Incidence and Transmission Media



$$\overline{\overline{\mathrm{A}}}_{i,i+1} = \overline{\overline{\mathrm{A}}}_{i,i} = 2\overline{\overline{\mathrm{I}}}$$

$$\overline{\overline{\mathrm{B}}}_{i,i+1} = \overline{\overline{\mathrm{B}}}_{i,i} = 0$$

$$\overline{\overline{\mathrm{A}}}_{i,i-1} = \overline{\overline{\mathrm{A}}}_{i,i} = 2\overline{\overline{\mathrm{I}}}$$

$$\overline{\overline{\mathrm{B}}}_{i,i-1} = \overline{\overline{\mathrm{B}}}_{i,i} = 0$$

$$\overline{\overline{\mathrm{A}}}_{\mathrm{inc/trn}} = \overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{V}}}_0^{-1} \cdot \overline{\overline{\mathrm{V}}}_{\mathrm{inc/trn}}$$

$$\overline{\overline{\mathrm{B}}}_{\mathrm{inc/trn}} = \overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{V}}}_0^{-1} \cdot \overline{\overline{\mathrm{V}}}_{\mathrm{inc/trn}}$$

$$\overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{inc})} = -\overline{\overline{\mathrm{A}}}_{\mathrm{inc}}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{\mathrm{inc}}$$

$$\overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{inc})} = 2\overline{\overline{\mathrm{A}}}_{\mathrm{inc}}^{-1}$$

$$\overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{inc})} = \frac{1}{2}\left(\overline{\overline{\mathrm{A}}}_{\mathrm{inc}} - \overline{\overline{\mathrm{B}}}_{\mathrm{inc}} \cdot \overline{\overline{\mathrm{A}}}_{\mathrm{inc}}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{\mathrm{inc}}\right)$$

$$\overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{inc})} = \overline{\overline{\mathrm{B}}}_{\mathrm{inc}} \cdot \overline{\overline{\mathrm{A}}}_{\mathrm{inc}}^{-1}$$

where for each medium

$$\overline{\overline{\mathrm{V}}} = -\frac{j}{\tilde{k}_z \mu_{\mathrm{r}}} \begin{bmatrix} 0 & \epsilon_{\mathrm{r}}\mu_{\mathrm{r}} - \tilde{k}_x^2 \\ -\epsilon_{\mathrm{r}}\mu_{\mathrm{r}} & 0 \end{bmatrix}$$

$$\tilde{k}_z = \sqrt{\epsilon_{\mathrm{r}}\mu_{\mathrm{r}} - \tilde{k}_x^2}$$

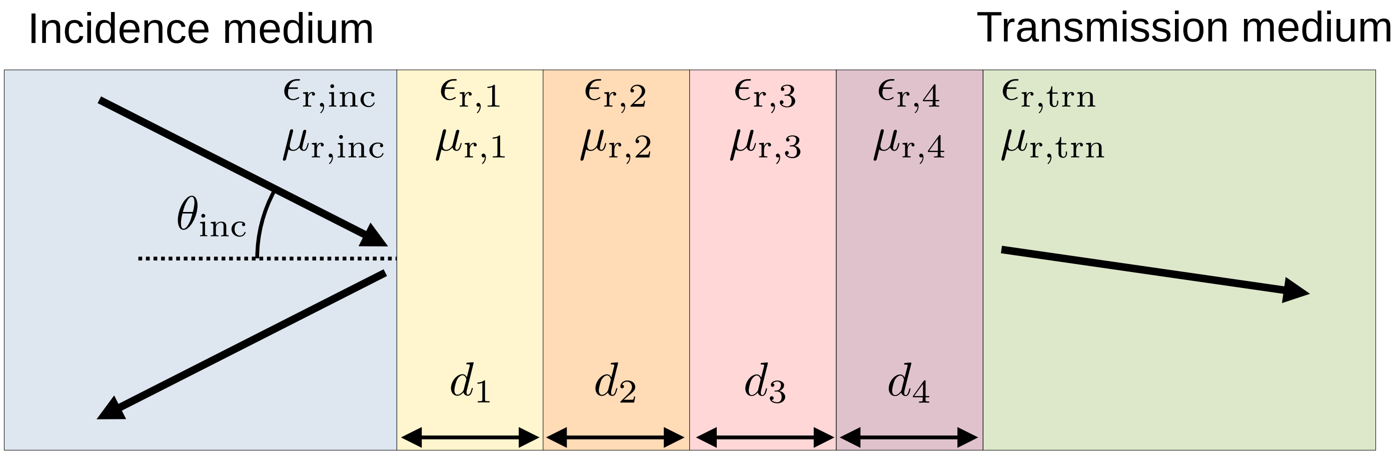
$$\overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{trn})} = \overline{\overline{\mathrm{B}}}_{\mathrm{trn}} \cdot \overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1}$$

$$\overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{trn})} = \frac{1}{2}\left(\overline{\overline{\mathrm{A}}}_{\mathrm{trn}} - \overline{\overline{\mathrm{B}}}_{\mathrm{trn}} \cdot \overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{\mathrm{trn}}\right)$$

$$\overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{trn})} = 2\overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1}$$

$$\overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{trn})} = -\overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{\mathrm{trn}}$$

Global Scattering Matrix



$$\overline{\overline{\mathrm{S}}}^{(\mathrm{global})} = \overline{\overline{\mathrm{S}}}^{(\mathrm{inc})} \otimes \left[\overline{\overline{\mathrm{S}}}^{(1)} \otimes \overline{\overline{\mathrm{S}}}^{(2)} \otimes \cdots \otimes \overline{\overline{\mathrm{S}}}^{(N-1)} \otimes \overline{\overline{\mathrm{S}}}^{(N)} \right] \otimes \overline{\overline{\mathrm{S}}}^{(\mathrm{trn})}$$

Redheffer product

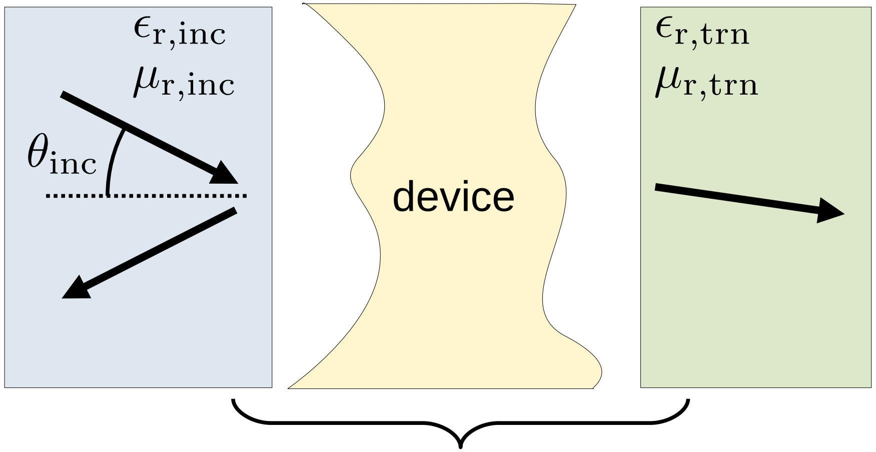
$$\overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{A})} + \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{A})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \cdot \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \cdot \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{A})}$$

$$\overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{A})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \cdot \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{B})}$$

$$\overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{B})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \cdot \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{A})}$$

$$\overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{AB})} = \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{B})} + \overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{B})} \cdot \left(\overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \cdot \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{B})} \right)^{-1} \cdot \overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{A})} \cdot \overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{B})}$$

How to Define the Fields ?



$$\begin{bmatrix} c'^{-}_{inc} \\ c'^{+}_{trn} \end{bmatrix} = \begin{bmatrix} \bar{\bar{S}}^{(global)}_{11} & \bar{\bar{S}}^{(global)}_{12} \\ \bar{\bar{S}}^{(global)}_{21} & \bar{\bar{S}}^{(global)}_{22} \end{bmatrix} \cdot \begin{bmatrix} c'^{+}_{inc} \\ c'^{-}_{trn} \end{bmatrix}$$

We assume the light is incident only from the left → $\mathbf{c}'^{-}_{trn} = 0$

Reflection $\mathbf{c}'^{-}_{inc} = \bar{\bar{S}}^{(global)}_{11} \cdot \mathbf{c}'^{+}_{inc}$

Transmission $\mathbf{c}'^{+}_{trn} = \bar{\bar{S}}^{(global)}_{21} \cdot \mathbf{c}'^{+}_{inc}$

$$\begin{bmatrix} E_x(z') \\ E_y(z') \end{bmatrix} = \bar{\bar{W}} \cdot e^{-\bar{\bar{\lambda}} z'} \cdot \mathbf{c}^{+} + \bar{\bar{W}} \cdot e^{\bar{\bar{\lambda}} z'} \cdot \mathbf{c}^{-}$$

Remember that: $\bar{\bar{W}} = \bar{\bar{I}}$ and we can set $z' = 0$

$$\mathbf{c} = \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$\mathbf{E}_{ref} = \bar{\bar{S}}^{(global)}_{11} \cdot \mathbf{E}_{inc}$$

$$\mathbf{E}_{trn} = \bar{\bar{S}}^{(global)}_{21} \cdot \mathbf{E}_{inc}$$

How to Define the Scattered Power ?

$$\mathbf{E}_{\text{ref}} = \overline{\overline{\text{S}}}_{11}^{(\text{global})} \cdot \mathbf{E}_{\text{inc}}$$

$$\mathbf{E}_{\text{trn}} = \overline{\overline{\text{S}}}_{21}^{(\text{global})} \cdot \mathbf{E}_{\text{inc}}$$

To compute the scattered power, we must find the z-component of the fields

$$E_z = -\frac{k_x E_x + k_y E_y}{k_z} \quad \xrightarrow{\text{since } k_y = 0} \quad E_z = -\frac{k_x}{k_z} E_x$$

Reflectance and transmittance

$$R = |r|^2 \qquad T = |t|^2 \text{Re}\left\{\frac{k_{z,\text{trn}}}{\mu_{\text{r,trn}}}\right\} \text{Re}\left\{\frac{\mu_{\text{r,inc}}}{k_{z,\text{inc}}}\right\}$$

where

$$|r|^2 = \frac{|E_{x,\text{ref}}|^2 + |E_{y,\text{ref}}|^2 + |E_{z,\text{ref}}|^2}{|E_{x,\text{inc}}|^2 + |E_{y,\text{inc}}|^2 + |E_{z,\text{inc}}|^2}$$

$$|t|^2 = \frac{|E_{x,\text{trn}}|^2 + |E_{y,\text{trn}}|^2 + |E_{z,\text{trn}}|^2}{|E_{x,\text{inc}}|^2 + |E_{y,\text{inc}}|^2 + |E_{z,\text{inc}}|^2}$$

What Have We Learned So Far….

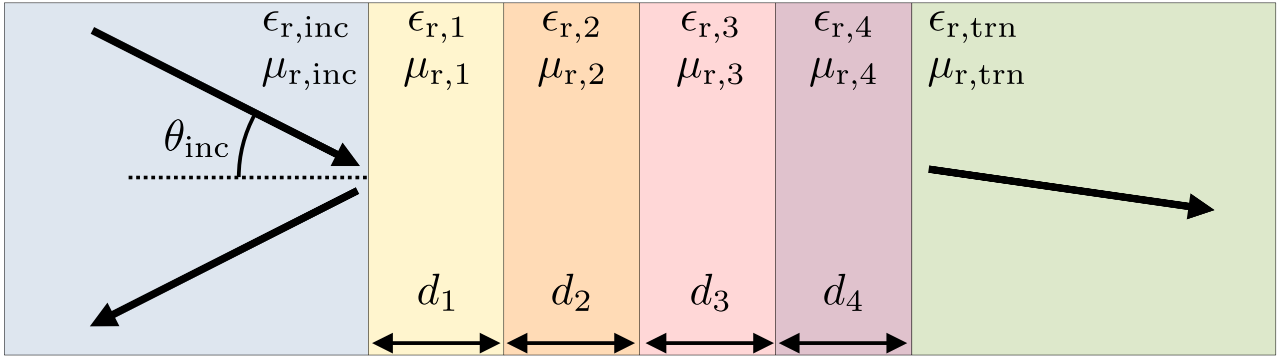
- For numerical techniques, we prefer to normalize Maxwell equations to avoid big and small numbers
- We formulate a differential equation only in terms of tangential field components
- This allows us to find the scattering matrix of each layers
- The global scattering matrix of the entire device is found by “multiplying” each individual scattering matrix using the Redheffer product
- The problem is significantly simplified by introducing zero-thickness gap layers. This symmetrizes the layers making the computation of the scattering matrices much simpler.
- The fact that we are considering spatially uniform and isotropic layers further simplifies the problem since we do not need to compute eigen vectors and eigen values matrices

Python Implementation of the Transfer Matrix Method

In what follows, I am using

```
from numpy import *
```

Defining the Initial Parameters



Define the parameters of the incidence and transmission media as well as each layer

It allows us to define

$$\tilde{k}_x = n_{\mathrm{inc}} \sin\theta_{\mathrm{inc}}$$
$$\tilde{k}_{z,\mathrm{inc}} = n_{\mathrm{inc}} \cos\theta_{\mathrm{inc}}$$

```
tet = 0           # incidence angle in [rad]
lam = 1000        # wavelength in [nm]

eri = 1           # permittivity of incidence medium
mri = 1           # permeability of incidence medium

ert = 1           # permittivity of transmission medium
mrt = 1           # permeability of transmission medium

ER = [2]          # list containing the permittivity of the layers
MR = [1]          # list containing the permeability of the layers
d  = [500]        # list containing the length in [nm] of the layers

# ---------------------------------------------------------
k0  = 2*pi/lam      # free-space wavenumber
kx  = emath.sqrt(eri*mri)*sin(tet)
kzi = emath.sqrt(eri*mri)*cos(tet)
```

Defining the Field Polarization and Gap Parameters

The incident electric field is generally defined via the polarization unit vectors: $\hat{\mathbf{p}}_{\mathrm{TE}},\ \hat{\mathbf{p}}_{\mathrm{TM}}$

Since the wave is incident in the xz-plane, we simply have

$$\mathbf{E}_{\mathrm{inc,TE}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad E_{z,\mathrm{inc,TE}} = 0$$

$$\mathbf{E}_{\mathrm{inc,TM}} = \begin{bmatrix} \cos\theta_{\mathrm{inc}} \\ 0 \end{bmatrix} \qquad E_{z,\mathrm{inc,TM}} = -\sin\theta_{\mathrm{inc}}$$

```
Ei = array([cos(tet),0])
```

Gap media matrix and its inverse

$$\overline{\overline{\mathrm{V}}}_0 = -j \begin{bmatrix} 0 & 1 \\ -(1+\tilde{k}_x^2) & 0 \end{bmatrix}$$

```
V0  = -1j*array([[0, 1], [-(1+kx**2), 0]])
IV0 = linalg.inv(V0)
```

Implementation of the Redheffer Product

Used to combine scattering matrices $\overline{\overline{S}}^{(A)} \otimes \overline{\overline{S}}^{(B)} = \overline{\overline{S}}^{(AB)}$

$$\overline{\overline{S}}_{11}^{(AB)} = \overline{\overline{S}}_{11}^{(A)} + \overline{\overline{S}}_{12}^{(A)} \cdot \left(\overline{\overline{I}} - \overline{\overline{S}}_{11}^{(B)} \cdot \overline{\overline{S}}_{22}^{(A)} \right)^{-1} \cdot \overline{\overline{S}}_{11}^{(B)} \cdot \overline{\overline{S}}_{21}^{(A)}$$

$$\overline{\overline{S}}_{12}^{(AB)} = \overline{\overline{S}}_{12}^{(A)} \cdot \left(\overline{\overline{I}} - \overline{\overline{S}}_{11}^{(B)} \cdot \overline{\overline{S}}_{22}^{(A)} \right)^{-1} \cdot \overline{\overline{S}}_{12}^{(B)}$$

$$\overline{\overline{S}}_{21}^{(AB)} = \overline{\overline{S}}_{21}^{(B)} \cdot \left(\overline{\overline{I}} - \overline{\overline{S}}_{22}^{(A)} \cdot \overline{\overline{S}}_{11}^{(B)} \right)^{-1} \cdot \overline{\overline{S}}_{21}^{(A)}$$

$$\overline{\overline{S}}_{22}^{(AB)} = \overline{\overline{S}}_{22}^{(B)} + \overline{\overline{S}}_{21}^{(B)} \cdot \left(\overline{\overline{I}} - \overline{\overline{S}}_{22}^{(A)} \cdot \overline{\overline{S}}_{11}^{(B)} \right)^{-1} \cdot \overline{\overline{S}}_{22}^{(A)} \cdot \overline{\overline{S}}_{12}^{(B)}$$

```
def redheffer(SA11,SA12,SA21,SA22,SB11,SB12,SB21,SB22):

    I = array([[1, 0],[0, 1]])

    D = SA12.dot(linalg.inv(I - SB11.dot(SA22)))
    F = SB21.dot(linalg.inv(I - SA22.dot(SB11)))

    SAB11 = SA11 + D.dot(SB11.dot(SA21))
    SAB12 = D.dot(SB12)
    SAB21 = F.dot(SA21)
    SAB22 = SB22 + F.dot(SA22.dot(SB12))

    return(SAB11,SAB12,SAB21,SAB22)
```

Defining the Connection to Incidence Medium

We start by initializing the global scattering matrix to that of an empty system

$$\overline{\overline{S}}^{(\text{global})} = \begin{bmatrix} \overline{\overline{S}}_{11}^{(\text{global})} & \overline{\overline{S}}_{12}^{(\text{global})} \\ \overline{\overline{S}}_{21}^{(\text{global})} & \overline{\overline{S}}_{22}^{(\text{global})} \end{bmatrix} = \begin{bmatrix} \overline{\overline{0}} & \overline{\overline{I}} \\ \overline{\overline{I}} & \overline{\overline{0}} \end{bmatrix}$$

$$\tilde{k}_z = \sqrt{\epsilon_\text{r}\mu_\text{r} - \tilde{k}_x^2} \qquad \overline{\overline{V}} = -\frac{j}{\tilde{k}_z\mu_\text{r}} \begin{bmatrix} 0 & \epsilon_\text{r}\mu_\text{r} - \tilde{k}_x^2 \\ -\epsilon_\text{r}\mu_\text{r} & 0 \end{bmatrix}$$

$$\overline{\overline{A}}_{\text{inc/trn}} = \overline{\overline{I}} + \overline{\overline{V}}_0^{-1} \cdot \overline{\overline{V}}_{\text{inc/trn}}$$
$$\overline{\overline{B}}_{\text{inc/trn}} = \overline{\overline{I}} - \overline{\overline{V}}_0^{-1} \cdot \overline{\overline{V}}_{\text{inc/trn}}$$

$$\overline{\overline{S}}_{11}^{(\text{inc})} = -\overline{\overline{A}}_{\text{inc}}^{-1} \cdot \overline{\overline{B}}_{\text{inc}}$$
$$\overline{\overline{S}}_{12}^{(\text{inc})} = 2\overline{\overline{A}}_{\text{inc}}^{-1}$$
$$\overline{\overline{S}}_{21}^{(\text{inc})} = \frac{1}{2}\left(\overline{\overline{A}}_{\text{inc}} - \overline{\overline{B}}_{\text{ref}} \cdot \overline{\overline{A}}_{\text{inc}}^{-1} \cdot \overline{\overline{B}}_{\text{inc}}\right)$$
$$\overline{\overline{S}}_{22}^{(\text{inc})} = \overline{\overline{B}}_{\text{inc}} \cdot \overline{\overline{A}}_{\text{inc}}^{-1}$$

```
I = array([[1, 0],[0, 1]])

# Initialize global scattering matrix
S11G = array([[0, 0],[0, 0]])
S22G = array([[0, 0],[0, 0]])
S21G = I
S12G = I

# Initialize incidence medium scattering matrix
kzi = emath.sqrt(eri*mri-kx**2)
Vi = 1/(1j*kzi*mri)*array([[0,eri*mri-kx**2],
                           [-eri*mri,0]])

A    = I + IV0.dot(Vi)
B    = I - IV0.dot(Vi)
IA   = linalg.inv(A)
S11i = -IA.dot(B)
S12i = 2*IA
S21i = 1/2*(A - B.dot(IA.dot(B)))
S22i = B.dot(IA)

# Update global scattering matrix
S11G, S12G, S21G, S22G =
redheffer(S11G,S12G,S21G,S22G,S11i,S12i,S21i,S22i)
```

Loop Over Each Layer

$$\tilde{k}_{z,i} = \sqrt{\epsilon_{\mathrm{r},i}\mu_{\mathrm{r},i} - \tilde{k}_x^2}$$

$$\overline{\overline{\mathrm{V}}}_i = -\frac{j}{\tilde{k}_{z,i}\mu_{\mathrm{r},i}}\begin{bmatrix} 0 & \epsilon_{\mathrm{r},i}\mu_{\mathrm{r},i} - \tilde{k}_x^2 \\ -\epsilon_{\mathrm{r},i}\mu_{\mathrm{r},i} & 0 \end{bmatrix}$$

$$X_i = e^{j\tilde{k}_z k_0 d_i}$$

$$\overline{\overline{\mathrm{A}}}_i = \overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0$$

$$\overline{\overline{\mathrm{B}}}_i = \overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{V}}}_i^{-1} \cdot \overline{\overline{\mathrm{V}}}_0$$

$$\overline{\overline{\mathrm{D}}}_i = \left(\overline{\overline{\mathrm{A}}}_i - X_i^2 \overline{\overline{\mathrm{B}}}_i \cdot \overline{\overline{\mathrm{A}}}_i^{-1} \cdot \overline{\overline{\mathrm{B}}}_i\right)^{-1}$$

$$\overline{\overline{\mathrm{S}}}_{11}^{(i)} = \overline{\overline{\mathrm{S}}}_{22}^{(i)} = \overline{\overline{\mathrm{D}}}_i \cdot \overline{\overline{\mathrm{B}}}_i \left(X_i^2 - 1\right)$$

$$\overline{\overline{\mathrm{S}}}_{12}^{(i)} = \overline{\overline{\mathrm{S}}}_{21}^{(i)} = \overline{\overline{\mathrm{D}}}_i \cdot \left(\overline{\overline{\mathrm{A}}}_i - \overline{\overline{\mathrm{B}}}_i \cdot \overline{\overline{\mathrm{A}}}_i^{-1} \cdot \overline{\overline{\mathrm{B}}}_i\right) X_i$$

We loop over each layer

```
for i in range(len(ER)):
    # Initialize transmission medium scattering matrix
    kz  = emath.sqrt(ER[i]*MR[i]-kx**2)
    V   = 1/(1j*kz*MR[i])*array([[0,ER[i]*MR[i]-kx**2],
                                  [-ER[i]*MR[i],0]])

    IV  = linalg.inv(V)
    X   = exp(1j*kz*k0*d[i])
    A   = I + IV.dot(V0)
    B   = I - IV.dot(V0)
    IA  = linalg.inv(A)
    D   = linalg.inv(A - X**2*B.dot(IA.dot(B)))

    S11 = D.dot(B)*(X**2 - 1)
    S12 = D.dot(A - B.dot(IA.dot(B)))*X
    S22 = S11
    S21 = S12

    # Update global scattering matrix
    S11G, S12G, S21G, S22G =
redheffer(S11G,S12G,S21G,S22G,S11,S12,S21,S22)
```

Defining the Connection to Transmission Medium

$$\tilde{k}_z = \sqrt{\epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2} \qquad \overline{\overline{\mathrm{V}}} = -\frac{j}{\tilde{k}_z\mu_\mathrm{r}}\begin{bmatrix} 0 & \epsilon_\mathrm{r}\mu_\mathrm{r} - \tilde{k}_x^2 \\ -\epsilon_\mathrm{r}\mu_\mathrm{r} & 0 \end{bmatrix}$$

$$\overline{\overline{\mathrm{A}}}_{\mathrm{inc/trn}} = \overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{V}}}_0^{-1} \cdot \overline{\overline{\mathrm{V}}}_{\mathrm{inc/trn}}$$

$$\overline{\overline{\mathrm{B}}}_{\mathrm{inc/trn}} = \overline{\overline{\mathrm{I}}} - \overline{\overline{\mathrm{V}}}_0^{-1} \cdot \overline{\overline{\mathrm{V}}}_{\mathrm{inc/trn}}$$

$$\overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{trn})} = \overline{\overline{\mathrm{B}}}_{\mathrm{trn}} \cdot \overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1}$$

$$\overline{\overline{\mathrm{S}}}_{12}^{(\mathrm{trn})} = \frac{1}{2}\left(\overline{\overline{\mathrm{A}}}_{\mathrm{trn}} - \overline{\overline{\mathrm{B}}}_{\mathrm{trn}} \cdot \overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{\mathrm{trn}}\right)$$

$$\overline{\overline{\mathrm{S}}}_{21}^{(\mathrm{trn})} = 2\overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1}$$

$$\overline{\overline{\mathrm{S}}}_{22}^{(\mathrm{trn})} = -\overline{\overline{\mathrm{A}}}_{\mathrm{trn}}^{-1} \cdot \overline{\overline{\mathrm{B}}}_{\mathrm{trn}}$$

```
# Initialize transmission medium scattering matrix
kzt = emath.sqrt(ert*mrt-kx**2)
Vt = 1/(1j*kzt*mrt)*array([[0,ert*mrt-kx**2],
                           [-ert*mrt,0]])

A    = I + IV0.dot(Vt)
B    = I - IV0.dot(Vt)
IA   = linalg.inv(A)
S11t = B.dot(IA)
S12t = 1/2*(A - B.dot(IA.dot(B)))
S21t = 2*IA
S22t = -IA.dot(B)

# Update global scattering matrix
S11G, S12G, S21G, S22G =
redheffer(S11G,S12G,S21G,S22G,S11t,S12t,S21t,S22t)
```

Compute the Scattered Power Coefficients

$$\mathbf{E}_{\text{ref}} = \overline{\overline{\text{S}}}_{11}^{(\text{global})} \cdot \mathbf{E}_{\text{inc}}$$

$$\mathbf{E}_{\text{trn}} = \overline{\overline{\text{S}}}_{21}^{(\text{global})} \cdot \mathbf{E}_{\text{inc}}$$

$$E_z = -\frac{k_x}{k_z} E_x$$

$$|r|^2 = \frac{|E_{x,\text{ref}}|^2 + |E_{y,\text{ref}}|^2 + |E_{z,\text{ref}}|^2}{|E_{x,\text{inc}}|^2 + |E_{y,\text{inc}}|^2 + |E_{z,\text{inc}}|^2}$$

$$|t|^2 = \frac{|E_{x,\text{trn}}|^2 + |E_{y,\text{trn}}|^2 + |E_{z,\text{trn}}|^2}{|E_{x,\text{inc}}|^2 + |E_{y,\text{inc}}|^2 + |E_{z,\text{inc}}|^2}$$

$$R = |r|^2$$

$$T = |t|^2 \text{Re}\left\{\frac{k_{z,\text{trn}}}{\mu_{\text{r,trn}}}\right\} \text{Re}\left\{\frac{\mu_{\text{r,inc}}}{k_{z,\text{inc}}}\right\}$$

```
# Calculate scattered fields
Er = S11G.dot(Ei)
Et = S21G.dot(Ei)

# Calculate longitudinal component of the fields
Ezr = -kx*Er[0]/kzi
Ezt = -kx*Et[0]/kzt

# Calculate reflectance and transmittance
R = (abs(Er[0])**2+abs(Er[1])**2+abs(Ezr)**2)
T = (abs(Et[0])**2+abs(Et[1])**2+abs(Ezt)**2)
T *= real(kzt/mrt)*real(mri/kzi)
```

Note: I assumed normalized incident field, i.e., $|E_i| = 1$

Electric Field Computation

Compute the Fields

The fields in the incidence and transmission media are easily computed from the global scattering matrix

$$\mathbf{E}_{\text{ref}}(x,z) = e^{-jk_x x}\left[\bar{\bar{\text{I}}}e^{-jk_{z,\text{i}}z} + \bar{\bar{\text{S}}}_{11}^{(\text{global})}e^{jk_{z,\text{ref}}z}\right]\cdot\mathbf{E}_{\text{inc}}$$

$$\mathbf{E}_{\text{trn}}(x,z) = \bar{\bar{\text{S}}}_{21}^{(\text{global})}\cdot\mathbf{E}_{\text{inc}}e^{-j(k_x x + k_{z,\text{trn}}z)}$$

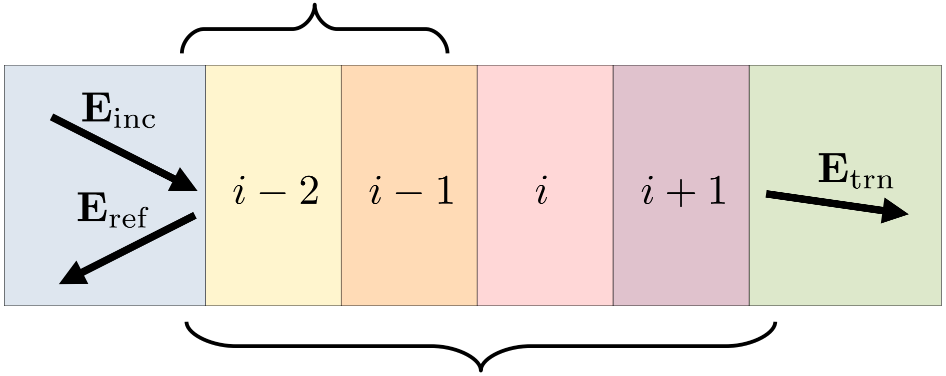
The field within the i^{th} layer is computed from the mode profiles

$$\mathbf{E}_i(x,z_i') = e^{-jk_x x}\left(e^{-j\tilde{k}_{z,i}z_i'}\cdot\mathbf{c}_i^+ + e^{j\tilde{k}_{z,i}z_i'}\cdot\mathbf{c}_i^-\right)$$

The mode profiles of the i^{th} layer are found from those of the $i\text{-}1^{th}$ layer

$$\begin{bmatrix}\mathbf{c}_i^+\\ \mathbf{c}_i^-\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\bar{\bar{\text{A}}}_{i,i-1} & \bar{\bar{\text{B}}}_{i,i-1}\\ \bar{\bar{\text{B}}}_{i,i-1} & \bar{\bar{\text{A}}}_{i,i-1}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{c'}_{i-1}^+\\ \mathbf{c'}_{i-1}^-\end{bmatrix}$$

$\bar{\bar{\text{S}}}^{\text{G}(i-1)}$ "global" scattering matrix up to the $i\text{-}1^{th}$ layer



$\bar{\bar{\text{S}}}^{(\text{global})}$

$$\mathbf{c'}_{i-1}^- = \bar{\bar{\text{S}}}_{12}^{\text{G}(i-1)}\left(\mathbf{c}_{\text{ref}} - \bar{\bar{\text{S}}}_{11}^{\text{G}(i-1)}\cdot\mathbf{c}_{\text{inc}}\right)$$

$$\mathbf{c'}_{i-1}^+ = \bar{\bar{\text{S}}}_{21}^{\text{G}(i-1)}\cdot\mathbf{c}_{\text{inc}} + \bar{\bar{\text{S}}}_{22}^{\text{G}(i-1)}\cdot\mathbf{c'}_{i-1}^-$$

where $\mathbf{c}_{\text{inc}} = \mathbf{E}_{\text{inc}}$ $\quad \mathbf{c}_{\text{ref}} = \bar{\bar{\text{S}}}_{11}^{(\text{global})}\cdot\mathbf{E}_{\text{inc}}$

Process for Computing the Fields

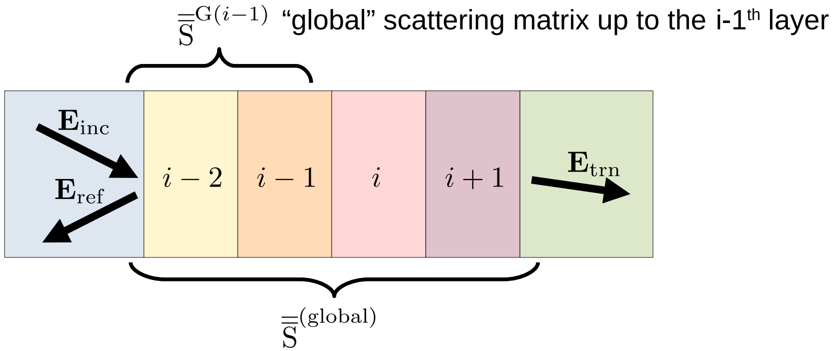
1. Compute the global scattering matrix $\overline{\overline{\mathrm{S}}}^{(\mathrm{global})}$. For each layer, save $\overline{\overline{\mathrm{A}}}$, $\overline{\overline{\mathrm{B}}}$, k_z and $\overline{\overline{\mathrm{S}}}^{\mathrm{G}(i-1)}$
2. Starting from the first layer on the left of the system. We know $\mathbf{c}_{\mathrm{inc}}$ and $\mathbf{c}_{\mathrm{ref}}$ and the value of $\overline{\overline{\mathrm{S}}}^{\mathrm{G}(i-1)}$, which here corresponds to the scattering matrix of the reflection region alone
3. This allows us to compute the $\mathbf{c}'_{i-1}$ coefficients that are just before the first layer using
4. The $\mathbf{c}_i$ coefficients inside the first layer are obtained using
5. Finally, the fields inside the layer are found using
6. Repeat this process for the other layers

$$\begin{cases} \mathbf{c}_{\mathrm{inc}} = \mathbf{E}_{\mathrm{inc}} \\ \mathbf{c}_{\mathrm{ref}} = \overline{\overline{\mathrm{S}}}_{11}^{(\mathrm{global})} \cdot \mathbf{E}_{\mathrm{inc}} \end{cases}$$

$$\begin{cases} \mathbf{c}'^{-}_{i-1} = \overline{\overline{\mathrm{S}}}_{12}^{\mathrm{G}(i-1)} \left(\mathbf{c}_{\mathrm{ref}} - \overline{\overline{\mathrm{S}}}_{11}^{\mathrm{G}(i-1)} \cdot \mathbf{c}_{\mathrm{inc}} \right) \\ \mathbf{c}'^{+}_{i-1} = \overline{\overline{\mathrm{S}}}_{21}^{\mathrm{G}(i-1)} \cdot \mathbf{c}_{\mathrm{inc}} + \overline{\overline{\mathrm{S}}}_{22}^{\mathrm{G}(i-1)} \cdot \mathbf{c}'^{-}_{i-1} \end{cases}$$

$$\begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \overline{\overline{\mathrm{A}}}_{i,i-1} & \overline{\overline{\mathrm{B}}}_{i,i-1} \\ \overline{\overline{\mathrm{B}}}_{i,i-1} & \overline{\overline{\mathrm{A}}}_{i,i-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}'^{+}_{i-1} \\ \mathbf{c}'^{-}_{i-1} \end{bmatrix}$$

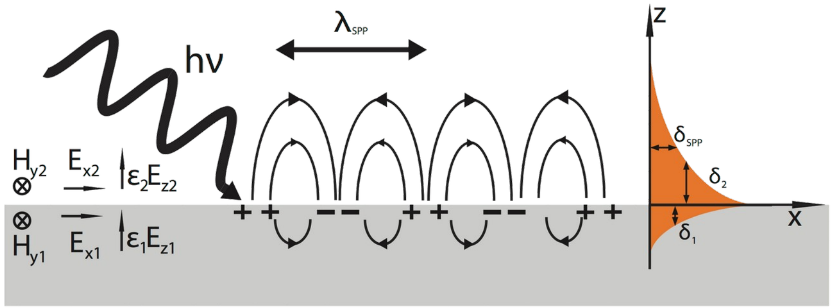
$$\mathbf{E}_i(x, z'_i) = e^{-jk_x x} \left(e^{-j\tilde{k}_{z,i} z'_i} \cdot \mathbf{c}_i^+ + e^{j\tilde{k}_{z,i} z'_i} \cdot \mathbf{c}_i^- \right)$$



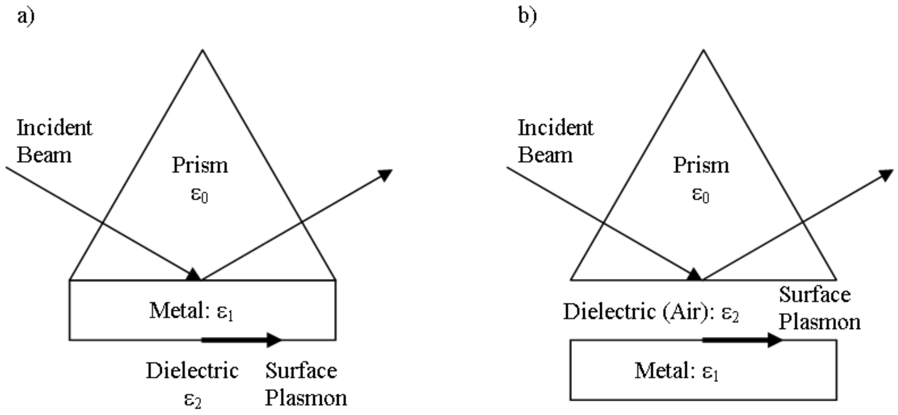
Example of a Multilayer System

How to Excite Surface Plasmon?

Surface plasmon



https://en.wikipedia.org/wiki/Surface_plasmon#/media/File:Sketch_of_surface_plasmon.png

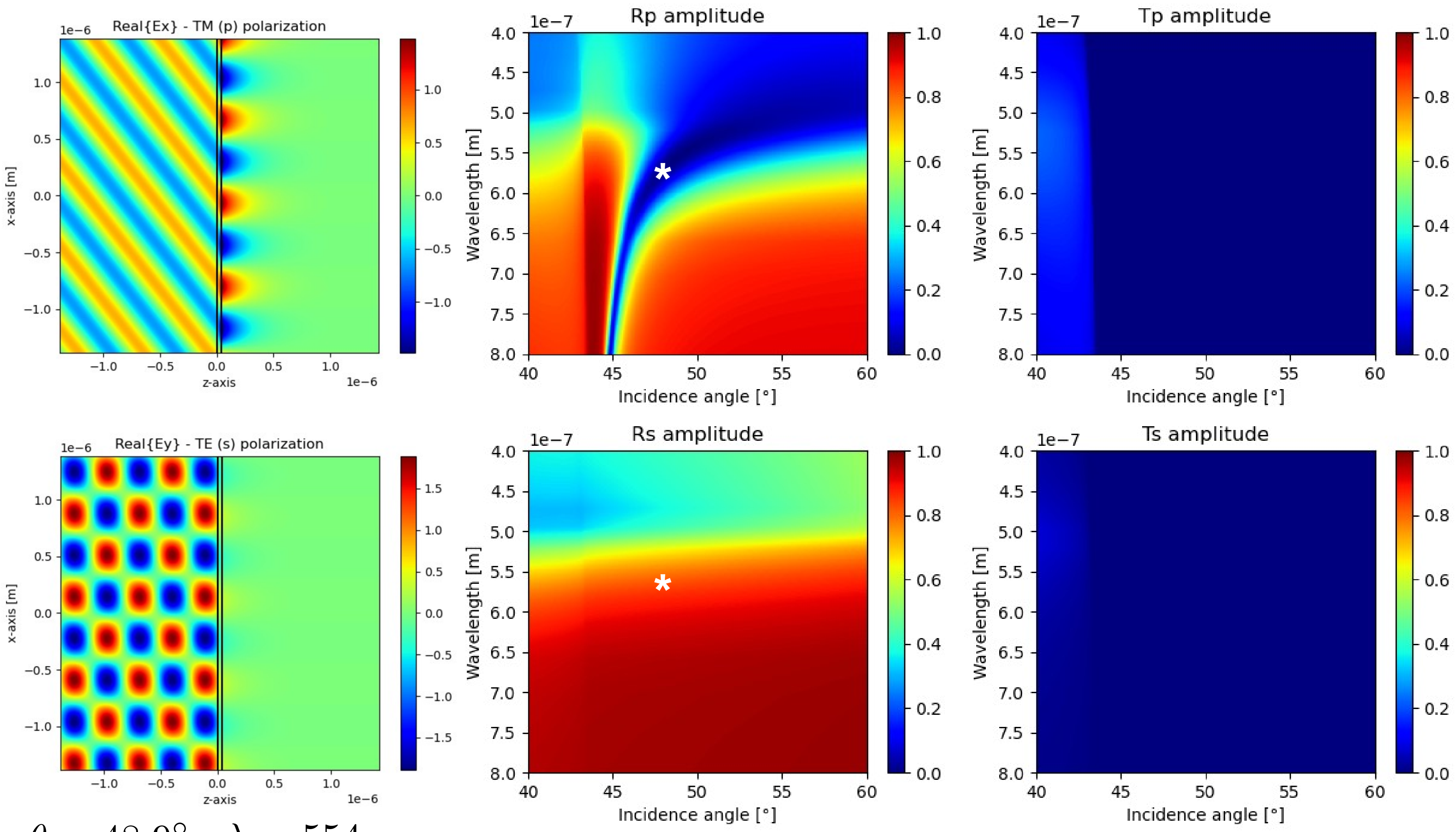


(a) Kretschmann and (b) Otto configuration

https://en.wikipedia.org/wiki/Surface_plasmon_polariton#Propagation_length_and_skin_depth

40 nm of gold

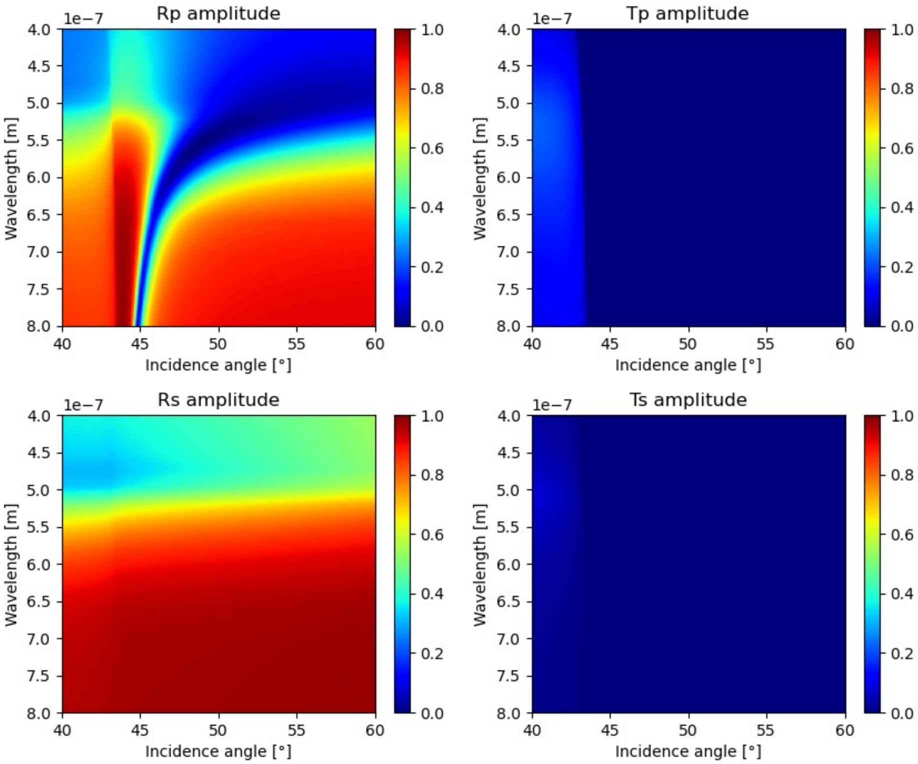
Excitation of Surface Plasmon – Kretschmann Configuration



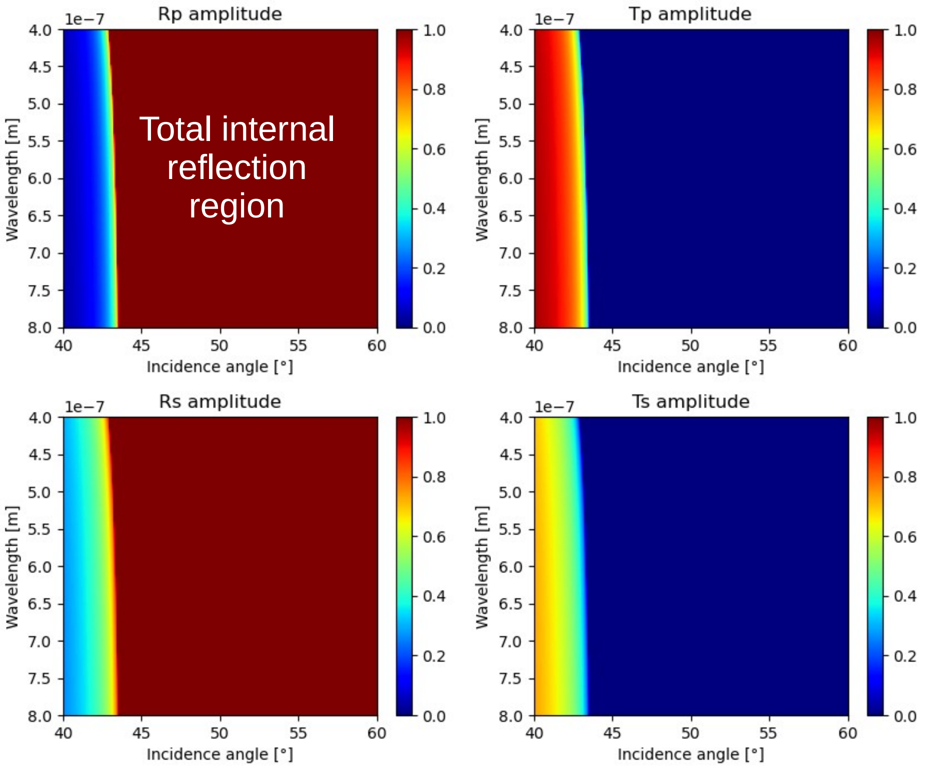
$\theta = 48.9^\circ \quad \lambda = 554 \text{ nm}$

Surface Plasmon Emerges From TIR

With 40 nm of gold



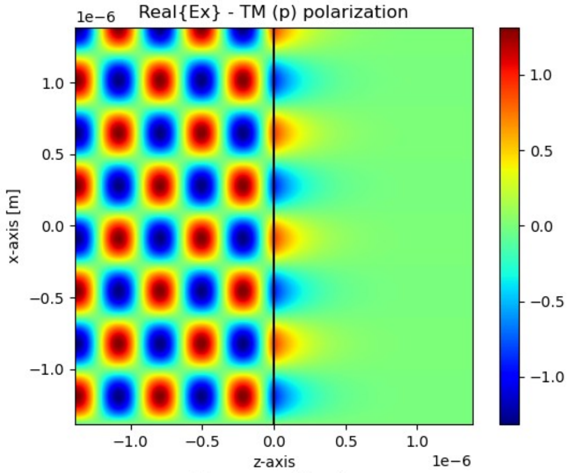
Without gold

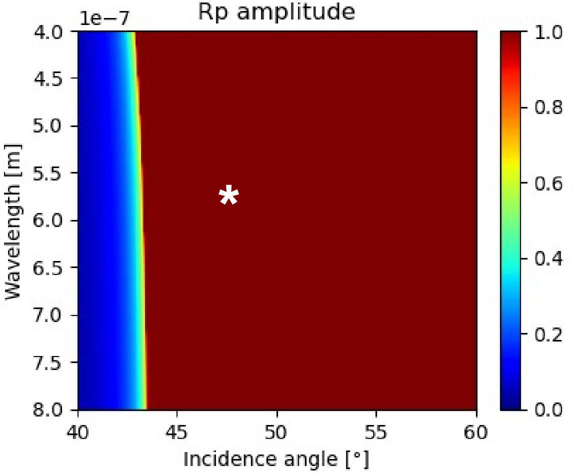


Comparison

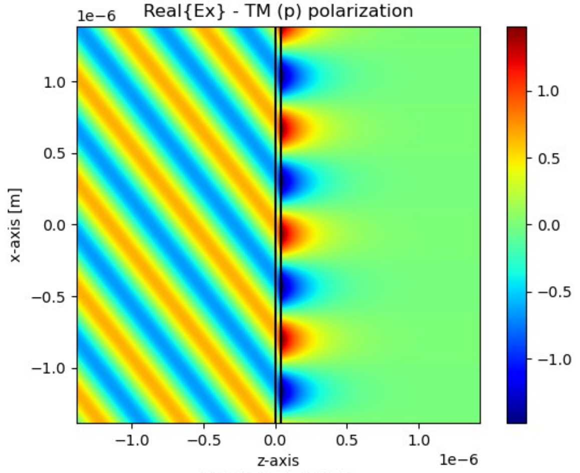
$\theta = 48.9^\circ \quad \lambda = 554 \text{ nm}$

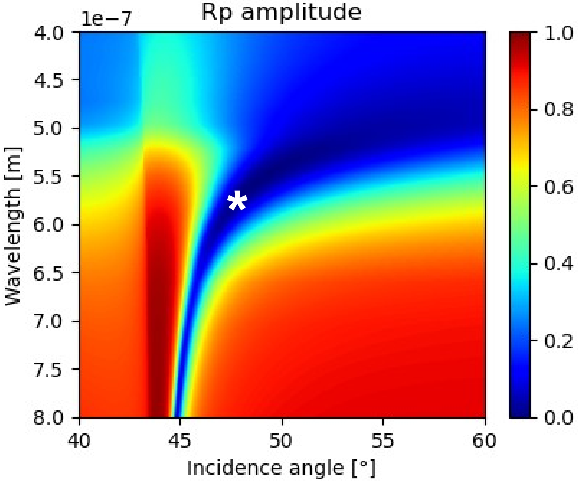
Without gold



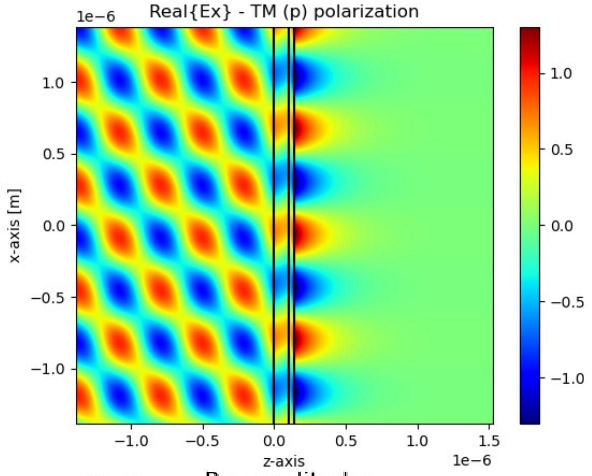


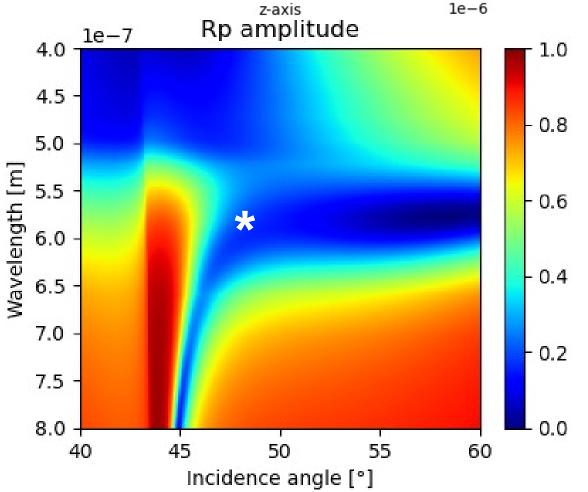
With 40 nm of gold (Kretschmann)





With 40 nm of gold and a 100 nm air gap (Otto)

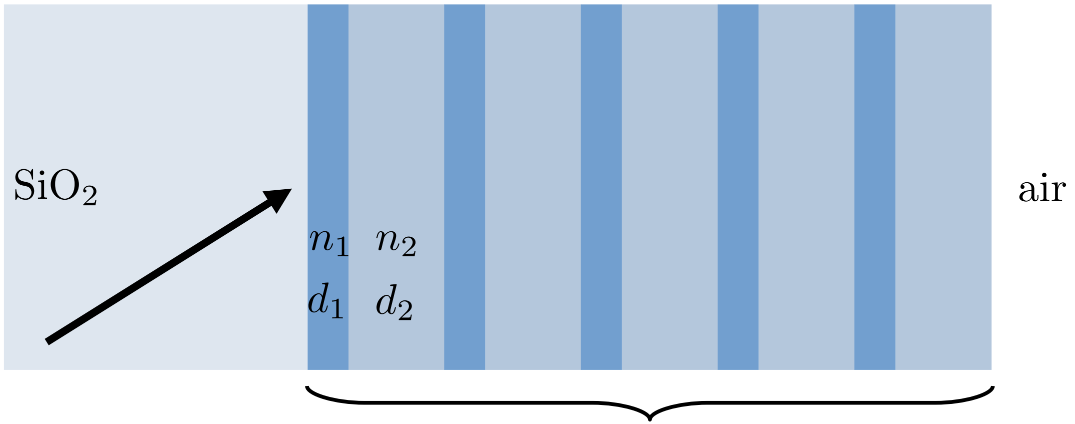




Bloch Surface Wave

Bloch Surface Wave (BSW)

Multilayer periodic structure made of dielectric materials



Five repetitions

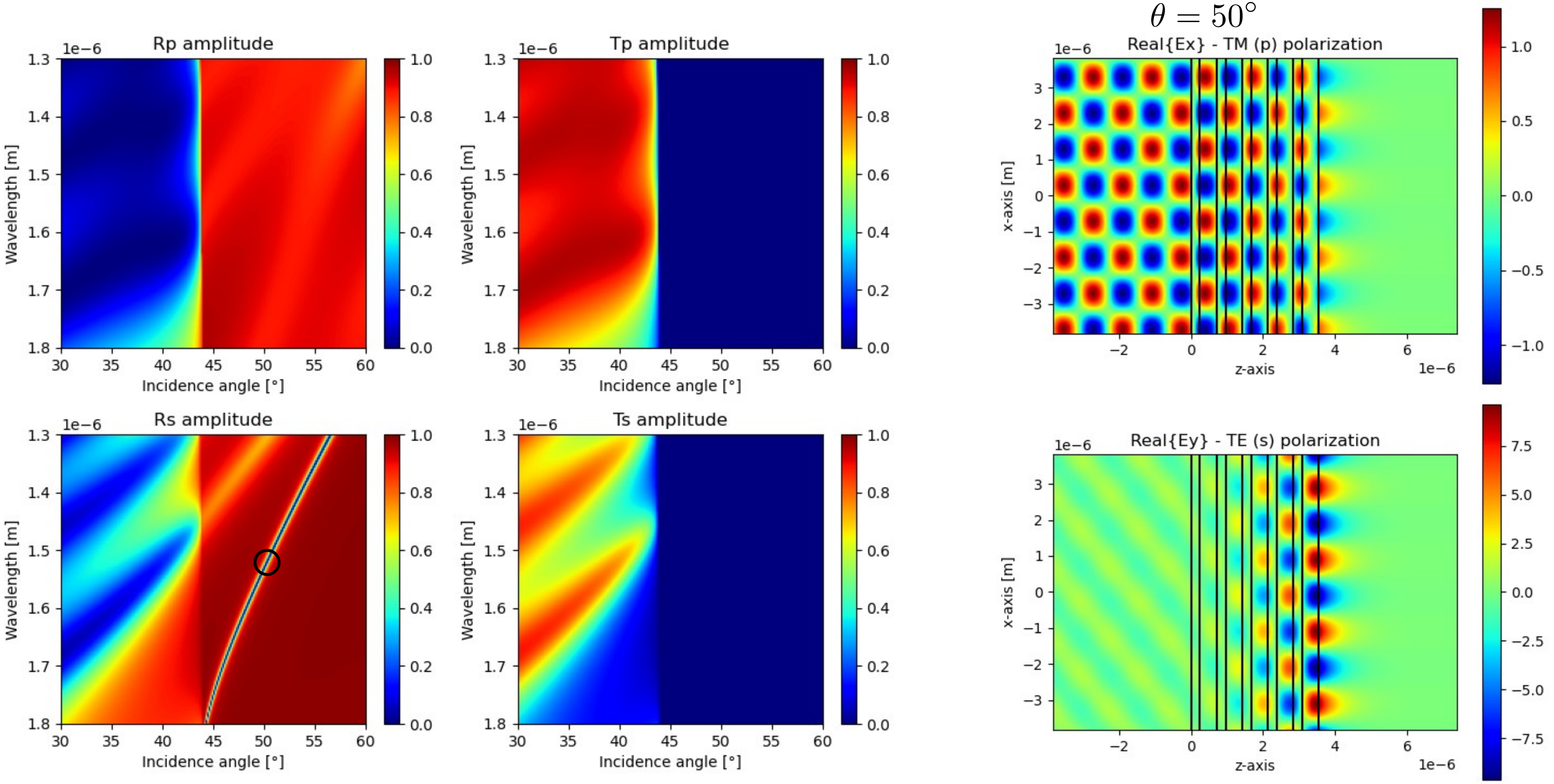
Loss is needed

$$\begin{cases} n_1 = 1.94 + 0.001j \\ n_2 = 1.46 + 0.001j \\ d_1 = 250 \text{ nm} \\ d_2 = 460 \text{ nm} \end{cases}$$

Bloch Surface Wave (BSW)

$\lambda = 1531\ \text{nm}$

$\theta = 50^\circ$



What Have We Learned So Far….

- To couple to surface plasmon, we use evanescent waves generated by the total internal reflection between SiO_2 and air. The gold layer is thin enough to allow this effect to happen
- The evanescent waves generated by the total reflection effect couple to surface plasmons on the metal
- At the plasmon resonance, all the light ends up being absorbed by the metal (no reflection and no transmission)
- Bloch surface waves (BSW) represent an alternative to surface plasmons because the former exist on dielectric (low loss) whereas the latter exist on metal (high loss). BSW therefore have the ability to propagate over very long distance (up to centimeters) due to the low losses, whereas plasmons only propagate for a few microns.