

# Lecture 2

## Waves

# The Wave Equation

# Maxwell Equations for an Anisotropic Medium

Maxwell equations in the absence of impressed sources

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = 0$$



$$\nabla \cdot (\bar{\bar{\mu}} \cdot \mathbf{H}) = 0$$

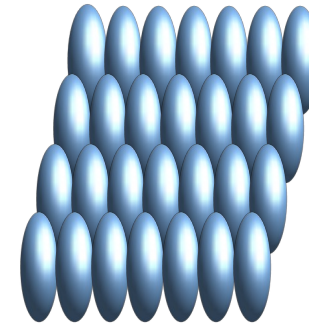
$$\nabla \cdot (\bar{\bar{\epsilon}} \cdot \mathbf{E}) = 0$$

In this case, we cannot conclude that  $\begin{cases} \nabla \cdot \mathbf{H} = 0 \\ \nabla \cdot \mathbf{E} = 0 \end{cases}$

Anisotropic medium response

$$\mathbf{B} = \bar{\bar{\mu}} \cdot \mathbf{H}$$

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E}$$



Medium composed of elongated molecules

also notice that generally  $\begin{cases} \mathbf{E} \nparallel \mathbf{D} \\ \mathbf{H} \nparallel \mathbf{B} \end{cases}$

# Wave Equation in an Anisotropic Medium

Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mathbf{B} \\ \nabla \times \mathbf{H} = j\omega\mathbf{D} \end{cases}$$

Anisotropic constitutive relations

$$\mathbf{B} = \bar{\bar{\mu}} \cdot \mathbf{H}$$

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E}$$



$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\bar{\bar{\mu}} \cdot \mathbf{H} \\ \nabla \times \mathbf{H} = j\omega\bar{\bar{\epsilon}} \cdot \mathbf{E} \end{cases} \longrightarrow \mathbf{H} = -\frac{1}{j\omega}\bar{\bar{\mu}}^{-1} \cdot (\nabla \times \mathbf{E})$$



$$\nabla \times \left[ -\frac{1}{j\omega}\bar{\bar{\mu}}^{-1} \cdot (\nabla \times \mathbf{E}) \right] = j\omega\bar{\bar{\epsilon}} \cdot \mathbf{E}$$



$$\nabla \times \left[ \bar{\bar{\mu}}^{-1} \cdot (\nabla \times \mathbf{E}) \right] = \omega^2\bar{\bar{\epsilon}} \cdot \mathbf{E}$$

Cannot be further simplified because  $\epsilon$  and  $\mu$  are matrices

# Wave Equation in an Isotropic Medium

$$\nabla \times \left[ \bar{\mu}^{-1} \cdot (\nabla \times \mathbf{E}) \right] = \omega^2 \bar{\epsilon} \cdot \mathbf{E}$$

Isotropy  $\bar{\epsilon} = \epsilon \bar{\mathbf{I}}$

$\bar{\mu} = \mu \bar{\mathbf{I}}$

$$\bar{\mathbf{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\nabla \times \left[ \mu^{-1} (\nabla \times \mathbf{E}) \right] = \omega^2 \epsilon \mathbf{E}$$

$$\nabla \times \nabla \times \mathbf{E} = \omega^2 \epsilon \mu \mathbf{E}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \omega^2 \epsilon \mu \mathbf{E}$$

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$

In an isotropic medium

$$\nabla \cdot \mathbf{D} = 0$$

where  $\mathbf{D} = \epsilon \mathbf{E}$

$$\nabla \cdot \mathbf{E} = 0$$

wavenumber

$$k^2 = \omega^2 \epsilon \mu$$

## What Have We Learned So Far....

- An anisotropic medium is characterized by different refractive indices (or permittivities, etc) along the x,y and z directions
- Anisotropy is modeled with material parameters that take the form of matrices instead of scalar values
- In anisotropic media, the vectors **D** and **B** are not necessarily aligned with **E** and **H**
- Remember the form of the wave equation in an isotropic medium

# Cartesian Wave Equation

# Wave Equation in Cartesian Coordinates

$$\boxed{\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0} \longrightarrow \begin{bmatrix} \nabla^2 E_x \\ \nabla^2 E_y \\ \nabla^2 E_z \end{bmatrix} + k^2 \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can be solved for each component individually as a scalar equation

$$(\nabla^2 + k^2) \Psi(x, y, z) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Psi(x, y, z) = 0$$

Can be separated into three functions for x,y,z variables

$$\Psi(x, y, z) = X(x)Y(y)Z(z)$$

The Laplacian operator is defined as

$$\nabla^2 \Psi = \nabla \cdot \nabla \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}$$

# Splitting the Problem

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Psi(x, y, z) = 0 \quad \longleftarrow \quad \Psi(x, y, z) = X(x)Y(y)Z(z)$$

$$X''YZ + XY''Z + XYZ'' + k^2XYZ = 0$$

Divide by XYZ

The spatial variations of the function and its second-order derivative are identical

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + k^2 = 0$$

$$-k_x^2 - k_y^2 - k_z^2 + k^2 = 0$$

$$\frac{d^2}{dx^2} X(x) + k_x^2 X(x) = 0$$

$$\frac{d^2}{dy^2} Y(y) + k_y^2 Y(y) = 0$$

$$\frac{d^2}{dz^2} Z(z) + k_z^2 Z(z) = 0$$

# Solution of the Wave Equation in Cartesian Coordinates

Vector wave equation

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$



Treat as scalar equation

$$(\nabla^2 + k^2) \Psi(x, y, z) = 0$$



Separate variable

$$\Psi(x, y, z) = X(x)Y(y)Z(z)$$

**General solution**

$$\Psi(x, y, z) = \begin{Bmatrix} \cos k_x x \\ \sin k_x x \\ e^{\pm j k_x x} \end{Bmatrix} \begin{Bmatrix} \cos k_y y \\ \sin k_y y \\ e^{\pm j k_y y} \end{Bmatrix} \begin{Bmatrix} \cos k_z z \\ \sin k_z z \\ e^{\pm j k_z z} \end{Bmatrix}$$

Typically used to match wave guide boundary conditions

Used for propagating waves

The final solution is formed by a combination of any of the three functions



**General solution for propagation in a homogeneous medium**

$$\mathbf{E} = \mathbf{E}_0^+ e^{-j\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}_0^- e^{+j\mathbf{k}\cdot\mathbf{r}}$$

Forward wave

Backward wave

# Plane Waves in a Homogeneous Isotropic Medium

$$\mathbf{E} = \mathbf{E}_0^+ e^{-j\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}_0^- e^{+j\mathbf{k}\cdot\mathbf{r}}$$

We will generally consider that

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$$

What about the magnetic field?

From Maxwell eq.  $\nabla \times \mathbf{E} = -j\omega\mathbf{B}$   $\xrightarrow{\mathbf{B} = \mu\mathbf{H}}$   $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

$$\nabla \cdot \mathbf{E} = 0 \rightarrow \mathbf{k} \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0 \rightarrow \mathbf{k} \cdot \mathbf{H} = 0$$

$$\mathbf{H} = \frac{e^{-j\mathbf{k}\cdot\mathbf{r}}}{\omega\mu} \mathbf{k} \times \mathbf{E}_0$$

In general, we have

$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}$$

$$\mathbf{E} = \eta \mathbf{H} \times \hat{\mathbf{k}}$$

where

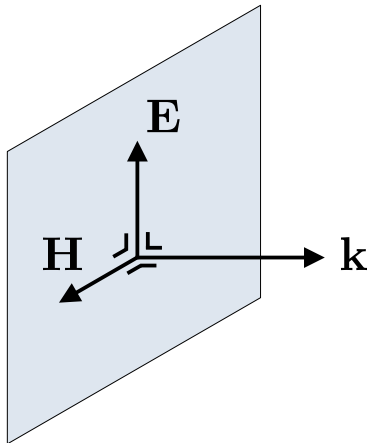
$$k = \omega\sqrt{\epsilon\mu} = \frac{\omega}{c}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|} = \frac{\mathbf{k}}{k}$$

In an isotropic homogeneous medium

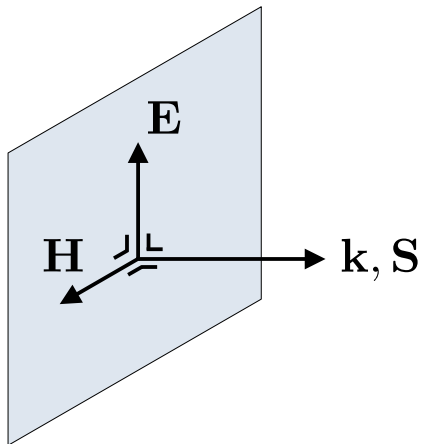
$$\mathbf{E} \perp \mathbf{H} \perp \mathbf{k}$$



# Summary for Plane Waves In Lossless Isotropic Media

General expression of the electric field

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} e^{j\omega t}$$



In a lossless medium,  
 $\mathbf{E}$  and  $\mathbf{H}$  are in phase !

Relations between  $\mathbf{E}$  and  $\mathbf{H}$

$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}$$

$$\mathbf{E} = \eta \mathbf{H} \times \hat{\mathbf{k}}$$

Time-average power flow in [ $\text{W}/\text{m}^2$ ]  
(Poynting vector)

(assuming lossless medium)

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} = \frac{|\mathbf{E}_0|^2}{2\eta} \hat{\mathbf{k}}$$

In an isotropic homogeneous medium

$$\mathbf{E} \perp \mathbf{H} \perp \mathbf{k}$$

Wave vector

$$\mathbf{k} = k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}} + k_z \hat{\mathbf{z}}$$

Wave number

$$k = |\mathbf{k}| = \omega \sqrt{\epsilon \mu} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Wave impedance  
(lossless)

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \in \mathbb{R}$$

Normalized wave vector

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}$$

# What Have We Learned So Far....

- The wave equation may be solved by separation of variables.
- In Cartesian coordinates, this separation leads to solutions given in terms of cosine, sine and complex exponential functions.
- The cosine and sine solutions are typically used in conjunctions with boundary conditions for instance to find the fields inside a rectangular wave guide. For free-space propagation, we use the complex exponentials
- When considering plane waves, we can replace the operator  $\nabla$  by  $-j\mathbf{k}$
- Know how to express the  $\mathbf{H}$  in terms of  $\mathbf{E}$  for a plane wave using  $\mathbf{k}$  and vice versa
- $\mathbf{E}$ ,  $\mathbf{H}$  and  $\mathbf{k}$  form a right-hand triad
- Remember the formula for the time-average Poynting vector (power density). It represents the direction in which the power flows

# Plane Wave Polarization

# Plane Wave Polarization Vector

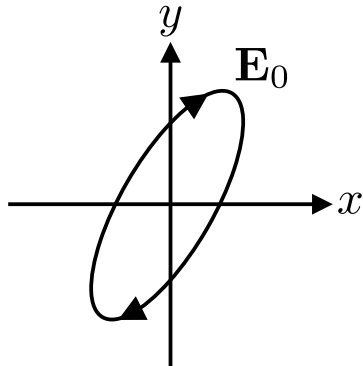
Let's consider a plane wave propagating along z

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} e^{j\omega t} \longrightarrow \mathbf{E} = \mathbf{E}_0 e^{-jkz} e^{j\omega t}$$

The polarization in the xy-plane is  $\mathbf{E}_0 = E_0 \begin{bmatrix} \cos \theta e^{j\alpha_x} \\ \sin \theta e^{j\alpha_y} \end{bmatrix}$

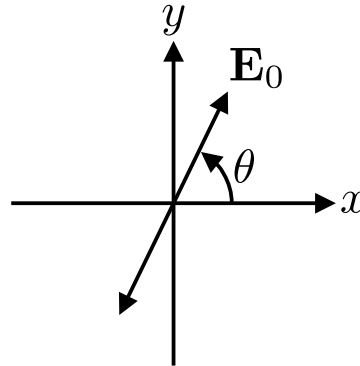
## Elliptical polarization

General case



## Linear polarization

$$\alpha_x = \alpha_y$$

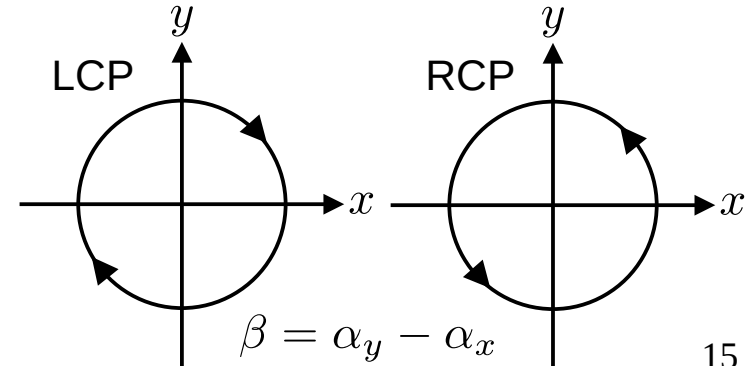


## Circular polarization

$$\theta = \pi/4$$

$$\beta = +\pi/2$$

$$\beta = -\pi/2$$



# Linear Polarization as Sum of LCP and RCP Waves

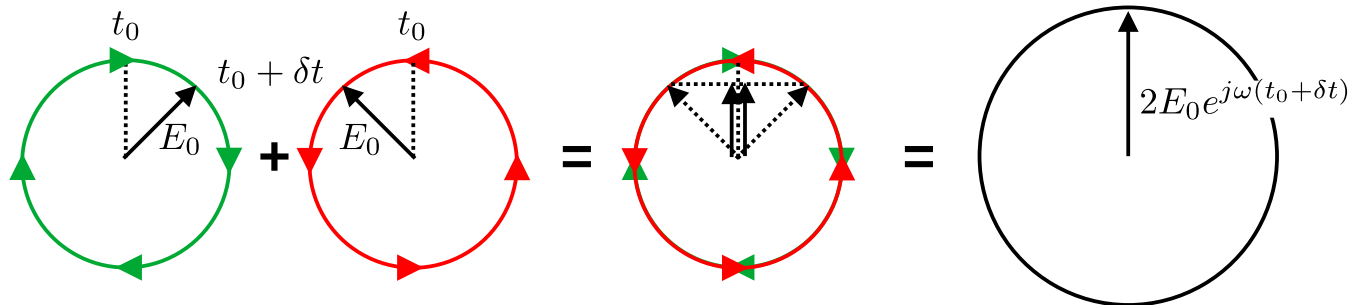
Consider the sum of an LCP and RCP wave

$$\mathbf{E} = E_0 \left( \begin{bmatrix} 1 \\ j \end{bmatrix} + \begin{bmatrix} 1 \\ -j \end{bmatrix} e^{j\phi} \right) e^{-jkz} \longrightarrow \mathbf{E} = E_0 \begin{bmatrix} 1 + e^{j\phi} \\ j(1 - e^{j\phi}) \end{bmatrix} e^{-jkz} = E_0 \begin{bmatrix} e^{-j\frac{\phi}{2}} + e^{j\frac{\phi}{2}} \\ j(e^{-j\frac{\phi}{2}} - e^{j\frac{\phi}{2}}) \end{bmatrix} e^{j\frac{\phi}{2}} e^{-jkz}$$

where  $\phi$  is an arbitrary phase

$$\mathbf{E} = E_0 \left( \begin{bmatrix} 1 \\ j \end{bmatrix} + \begin{bmatrix} 1 \\ -j \end{bmatrix} e^{j\phi} \right) e^{-jkz} = 2E_0 \begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{bmatrix} e^{j\frac{\phi}{2}} e^{-jkz}$$

**LCP**
**RCP**
**Oblique linear polarization**



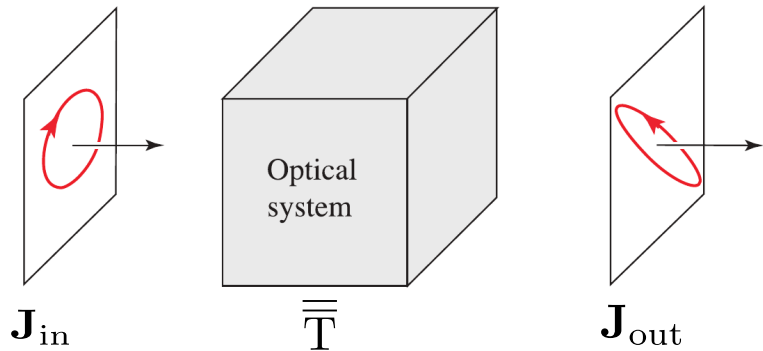
# Jones Calculus

Jones vectors represent the normalized polarization state of light

$$\mathbf{E}_0 = E_0 \begin{bmatrix} \cos \theta e^{i\alpha_x} \\ \sin \theta e^{i\alpha_y} \end{bmatrix} = E_0 \begin{bmatrix} A_x \\ A_y \end{bmatrix} \longrightarrow \mathbf{J} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

Ex.: LP  $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$     LCP  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix}$     RCP  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -j \end{bmatrix}$

The transmission polarization effects of a system may be represented in terms of a Jones matrix



$$\mathbf{J}_{\text{out}} = \overline{\overline{\mathbf{T}}} \cdot \mathbf{J}_{\text{in}}$$

**Examples:**  
Jones matrices representing polarizers

Linear polarizer along  $\theta$   $\overline{\overline{\mathbf{T}}} = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$

RCP polarizer

$$\overline{\overline{\mathbf{T}}} = \frac{1}{2} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}$$

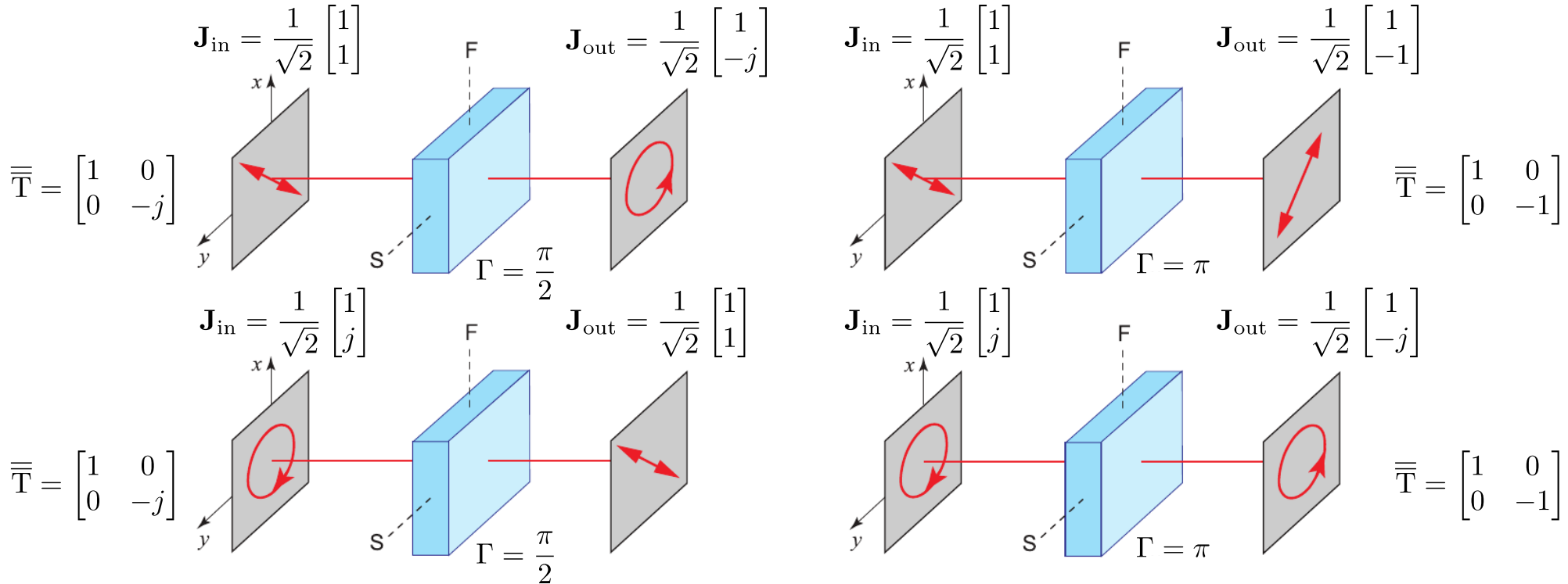
LCP polarizer

$$\overline{\overline{\mathbf{T}}} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}$$

# Wave Retarders/ Wave Plates

A wave plate is a birefringent device that creates a phase shift  $\Gamma$  between the  $x$ - and  $y$ -components of the field

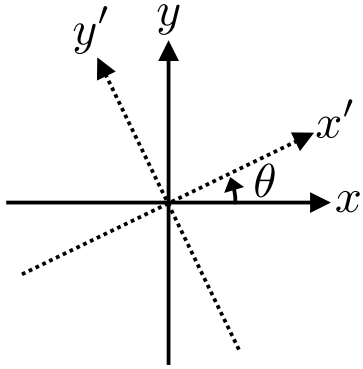
$$\overline{\overline{\mathbf{T}}} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\Gamma} \end{bmatrix}$$



(a) Quarter-wave retarder

(b) Half-wave retarder

# Rotation of Jones Matrices



The Jones vector in the rotated system of coordinates is

$$\mathbf{J}' = \overline{\overline{\mathbf{R}}}(\theta) \cdot \mathbf{J}$$

↓ Expressing  $\mathbf{J}$  in terms of  $\mathbf{J}'$

$$\mathbf{J} = \overline{\overline{\mathbf{R}}}(\theta)^{-1} \cdot \mathbf{J}'$$

**Rotation matrix**

$$\overline{\overline{\mathbf{R}}}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Note that

$$\overline{\overline{\mathbf{R}}}(\theta)^{-1} = \overline{\overline{\mathbf{R}}}(\theta)^T = \overline{\overline{\mathbf{R}}}(-\theta)$$

$$\mathbf{J}_{\text{out}} = \overline{\overline{\mathbf{T}}} \cdot \mathbf{J}_{\text{in}}$$

$$\longrightarrow \overline{\overline{\mathbf{R}}}(\theta)^{-1} \cdot \mathbf{J}'_{\text{out}} = \overline{\overline{\mathbf{T}}} \cdot \overline{\overline{\mathbf{R}}}(\theta)^{-1} \cdot \mathbf{J}'_{\text{in}} \longrightarrow \mathbf{J}'_{\text{out}} = \underbrace{\overline{\overline{\mathbf{R}}}(\theta) \cdot \overline{\overline{\mathbf{T}}} \cdot \overline{\overline{\mathbf{R}}}(\theta)^{-1}}_{\overline{\overline{\mathbf{T}}}' } \cdot \mathbf{J}'_{\text{in}}$$

**Rotated Jones matrix**

$$\overline{\overline{\mathbf{T}}}' = \overline{\overline{\mathbf{R}}}(\theta) \cdot \overline{\overline{\mathbf{T}}} \cdot \overline{\overline{\mathbf{R}}}(-\theta)$$

# Conversion Between Linear and Circular Bases

Transformation from  
linear to circular bases

$$\overline{\overline{\Lambda}}_{\text{LP} \rightarrow \text{CP}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}$$

$$\mathbf{E}_{\text{CP}} = \overline{\overline{\Lambda}}_{\text{LP} \rightarrow \text{CP}} \cdot \mathbf{E}_{\text{LP}}$$

$\begin{bmatrix} E_+ \\ E_- \end{bmatrix}$ 
 $\begin{bmatrix} E_x \\ E_y \end{bmatrix}$

where  $\begin{cases} + : \text{RCP} \\ - : \text{LCP} \end{cases}$

**Example:**  $\mathbf{E}_{\text{LP}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix} \longrightarrow \mathbf{E}_{\text{CP}} = \overline{\overline{\Lambda}}_{\text{LP} \rightarrow \text{CP}} \cdot \mathbf{E}_{\text{LP}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In the linear polarization bases, the Jones matrix is  $\overline{\overline{T}}_{\text{LP}} = \begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix}$

**Mapping relations between Jones matrices in linear and circular bases**

$$\overline{\overline{T}}_{\text{CP}} = \begin{bmatrix} t_{++} & t_{+-} \\ t_{-+} & t_{--} \end{bmatrix} = \overline{\overline{\Lambda}} \cdot \overline{\overline{T}}_{\text{LP}} \cdot \overline{\overline{\Lambda}}^{-1} = \frac{1}{2} \begin{bmatrix} t_{xx} + t_{yy} - j(t_{xy} - t_{yx}) & t_{xx} - t_{yy} + j(t_{xy} + t_{yx}) \\ t_{xx} - t_{yy} - j(t_{xy} + t_{yx}) & t_{xx} + t_{yy} + j(t_{xy} - t_{yx}) \end{bmatrix}$$

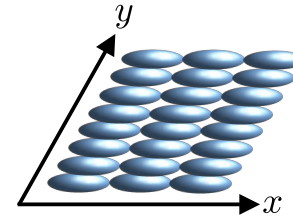
$$\overline{\overline{T}}_{\text{LP}} = \begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix} = \overline{\overline{\Lambda}}^{-1} \cdot \overline{\overline{T}}_{\text{CP}} \cdot \overline{\overline{\Lambda}} = \frac{1}{2} \begin{bmatrix} t_{++} + t_{--} + t_{+-} + t_{-+} & j(t_{++} - t_{--} + t_{-+} - t_{+-}) \\ j(t_{--} - t_{++} + t_{-+} - t_{+-}) & t_{++} + t_{--} - t_{+-} - t_{-+} \end{bmatrix} \quad \text{[20]}$$

# Birefringence and Dichroism

In the linear basis, different refractive indices along x and y leads to different  $t_{xx}$  and  $t_{yy}$

**(Linear) birefringence**

$$n_x \neq n_y \longrightarrow t_{xx} \neq t_{yy}$$



Medium composed of elongated molecules

**Circular birefringence** is the difference of transmission phase between RCP and LCP waves

assuming  $t_{+-} = t_{-+} = 0$  we have

$$\begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} t_{++} + t_{--} & j(t_{++} - t_{--}) \\ j(t_{--} - t_{++}) & t_{++} + t_{--} \end{bmatrix} \xrightarrow{t_{++} = e^{j\phi}, t_{--} = 1} \begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix} = e^{j\frac{\phi}{2}} \begin{bmatrix} \cos\left(\frac{\phi}{2}\right) & -\sin\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) \end{bmatrix}$$

**Inducing a transmission phase shift between LCP and RCP waves leads linear polarization rotation**

**Circular dichroism** is the difference of transmission amplitude between RCP and LCP waves

$$CD = (|t_{++}|^2 + |t_{-+}|^2) - (|t_{--}|^2 + |t_{+-}|^2)$$

# Polarization of Obliquely Propagating Waves

The field is decomposed into TE and TM components

$$\mathbf{E} = A_{\text{TE}} \hat{\mathbf{p}}_{\text{TE}} + A_{\text{TM}} \hat{\mathbf{p}}_{\text{TM}}$$

$$\hat{\mathbf{p}}_{\text{TE}} = \begin{cases} \hat{\mathbf{y}} & \mathbf{k}_i \parallel \hat{\mathbf{n}} \quad (\text{normal incidence}) \\ \frac{\hat{\mathbf{n}} \times \hat{\mathbf{k}}_i}{|\hat{\mathbf{n}} \times \hat{\mathbf{k}}_i|} & \mathbf{k}_i \not\parallel \hat{\mathbf{n}} \end{cases}$$

$$\hat{\mathbf{p}}_{\text{TM}} = \frac{\hat{\mathbf{k}}_i \times \hat{\mathbf{p}}_{\text{TE}}}{|\hat{\mathbf{k}}_i \times \hat{\mathbf{p}}_{\text{TE}}|}$$

In the circular basis, we have

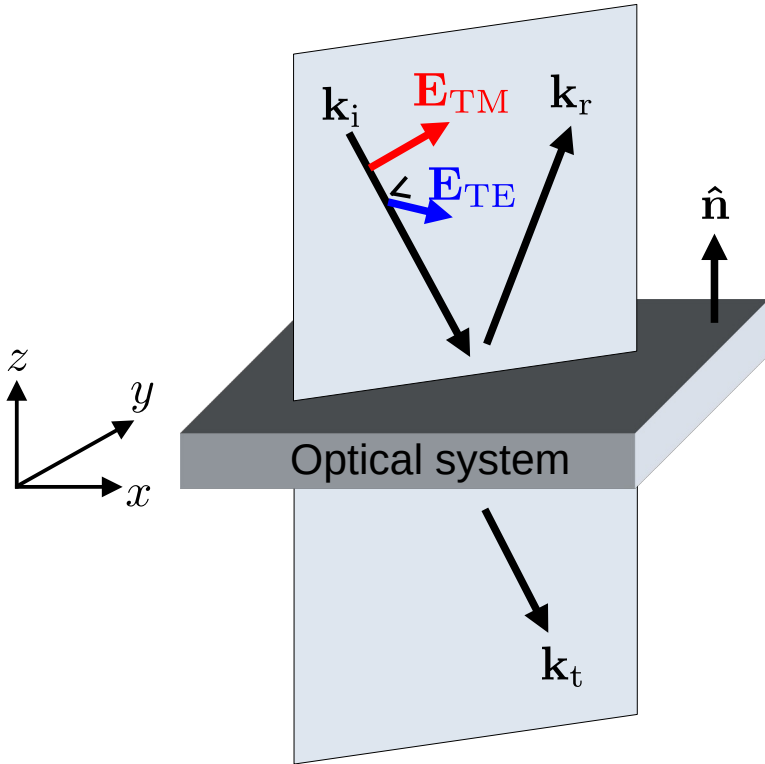
$$\hat{\mathbf{p}}_{\text{RCP}} = \frac{1}{\sqrt{2}} (\hat{\mathbf{p}}_{\text{TM}} - j\hat{\mathbf{p}}_{\text{TE}})$$

$$\hat{\mathbf{p}}_{\text{LCP}} = \frac{1}{\sqrt{2}} (\hat{\mathbf{p}}_{\text{TM}} + j\hat{\mathbf{p}}_{\text{TE}})$$

“TE” = “s” = transverse electric/perpendicular  
= electric field perpendicular to the PI

“TM” = “p” = transverse magnetic/parallel  
= electric field parallel to the PI

Plane of incidence (PI)



For an optical system uniform along  $x$  and  $y$ , the vectors  $\mathbf{k}_i$ ,  $\mathbf{k}_r$  and  $\mathbf{k}_t$  are all in the same plane of incidence

# Polarization Decomposition

Let's assume that the field  $\mathbf{E}$  and its wave vector  $\mathbf{k}$  are known, how to decompose  $\mathbf{E}$  into TE, TM, LCP and RCP components?

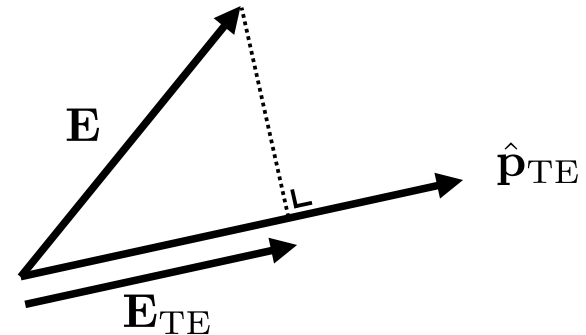
$$\mathbf{E} = A_{\text{TE}} \hat{\mathbf{p}}_{\text{TE}} + A_{\text{TM}} \hat{\mathbf{p}}_{\text{TM}}$$

**Linear basis**

$$\mathbf{E}_{\text{TE}} = (\hat{\mathbf{p}}_{\text{TE}} \cdot \mathbf{E}) \hat{\mathbf{p}}_{\text{TE}}$$

$$\mathbf{E}_{\text{TM}} = (\hat{\mathbf{p}}_{\text{TM}} \cdot \mathbf{E}) \hat{\mathbf{p}}_{\text{TM}}$$

Use vector projection



**Circular basis**

$$\mathbf{E}_{\text{LCP}} = (\hat{\mathbf{p}}_{\text{LCP}}^* \cdot \mathbf{E}) \hat{\mathbf{p}}_{\text{LCP}}$$

$$\mathbf{E}_{\text{RCP}} = (\hat{\mathbf{p}}_{\text{RCP}}^* \cdot \mathbf{E}) \hat{\mathbf{p}}_{\text{RCP}}$$

Note: complex vector projection requires using the complex conjugate (\*)

# What Have We Learned So Far....

- A linearly polarized wave may be expressed by a superposition of an LCP and RCP waves
- Changing the phase shift between these LCP and RCP waves leads to a rotation of the corresponding linear polarization. This is a mechanism used to design polarization rotators
- Jones vectors describe the polarization of input and output waves. Jones matrices describe the polarization response of an optical system
- Remember the principle for rotating a matrix (change of basis)
- Half-wave plates create a  $\pi$  phase shift between x and y polarized waves, which leads to polarization rotation and a change of handedness for circularly polarized waves
- Quarter-wave plates create a  $\pi/2$  phase shift between x and y polarized waves, which leads to linear-to-circular polarization conversion and vice versa
- Linear birefringence refers to different refractive indices for x and y-polarized waves. Circular birefringence refers to different refractive indices (typically phase shifts) for LCP and RCP waves. Circular dichroism is different absorption for LCP and RCP waves.
- Understand the mechanism to define TE, TM, LCP and RCP waves from the orientation of the device and the direction of  $\mathbf{k}$
- Understand how to obtain the TE, TM, LCP and RCP amplitude from a given electric field

# Vector Transmission-Line Equations

# Redundancy of Maxwell Equations

## Maxwell equations in k-space

$$\mathbf{k} \times \mathbf{H} = -\omega\epsilon\mathbf{E}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}$$

$$\mathbf{k} \cdot \mathbf{H} = 0$$

$$\mathbf{k} \cdot \mathbf{E} = 0$$

$$k_x E_x + k_y E_y + k_z E_z = 0$$

$$E_z = -\frac{1}{k_z} (k_x E_x + k_y E_y)$$

The z-component of the field is fully defined from the tangential ones

Assuming plane waves

$$\mathbf{E}, \mathbf{H} \propto e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\nabla \rightarrow -j\mathbf{k}$$

## Maxwell equations

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \cdot \mathbf{H} = 0$$

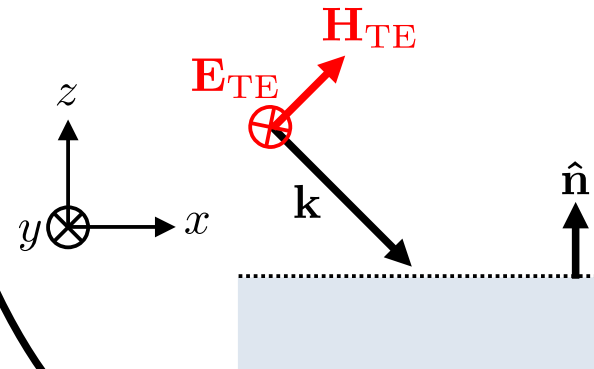
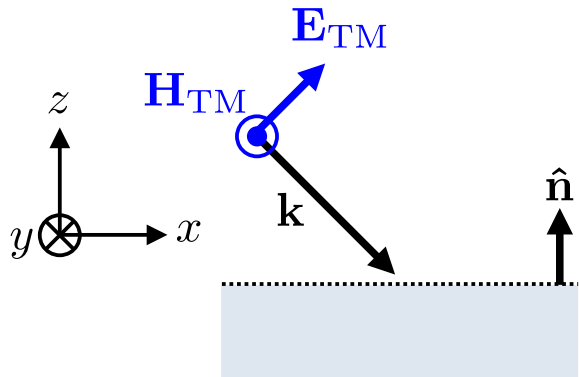
$$\nabla \cdot \mathbf{E} = 0$$

If  $\mathbf{k}$  is known, then it is sufficient to know  $E_x$  and  $E_y$  to define all other field components.

$$\mathbf{H} = \frac{1}{\omega\mu} \mathbf{k} \times \mathbf{E}$$

The magnetic field is fully defined from the electric field

# Impedance of Oblique TE and TM Waves



$$\begin{aligned} \mathbf{k} \times \mathbf{H} &= -\omega\epsilon\mathbf{E} \\ \mathbf{k} \times \mathbf{E} &= \omega\mu\mathbf{H} \end{aligned}$$

The normal field components are

$$E_z = -\frac{1}{k_z} (k_x E_x + k_y E_y)$$

$$H_z = -\frac{1}{k_z} (k_x H_x + k_y H_y)$$

In this example

$$k_y = 0$$

$$k_z E_x - k_x E_z = \omega\mu H_y$$

$$k_z E_x + \frac{k_x^2}{k_z} E_x = \omega\mu H_y$$

$$\frac{k^2}{k_z} E_x = \omega\mu H_y$$

$$\frac{E_x}{H_y} = \frac{k_z}{\omega\epsilon}$$

**TM and TE wave impedances**

$$Z_{\text{TM}} = \frac{k_n}{\omega\epsilon}$$

$$Z_{\text{TE}} = \frac{\omega\mu}{k_n}$$

$$k_z H_x - k_x H_z = -\omega\epsilon E_y$$

$$k_z H_x + \frac{k_x^2}{k_z} H_x = -\omega\epsilon E_y$$

$$\frac{k^2}{k_z} H_x = -\omega\epsilon E_y$$

$$\frac{E_y}{H_x} = -\frac{\omega\mu}{k_z}$$

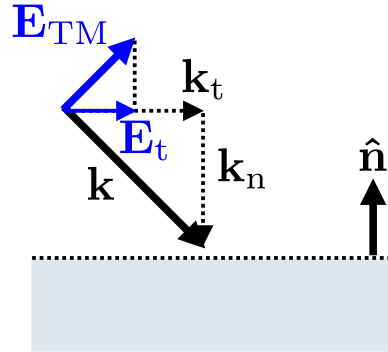
$$k^2 = \omega^2 \epsilon \mu$$

"n" is the component normal to the interface

# Tangential Field Components for TE and TM Waves

The field may be decomposed into tangential and normal components

For a TM wave



$$\begin{cases} \mathbf{E}_t \parallel \mathbf{k}_t \\ \mathbf{E}_t \perp \hat{\mathbf{n}} \times \mathbf{k}_t \end{cases}$$

$$(\hat{\mathbf{n}} \times \mathbf{k}_t \hat{\mathbf{n}} \times \mathbf{k}_t) \cdot \mathbf{E}_t = \hat{\mathbf{n}} \times \mathbf{k}_t [(\hat{\mathbf{n}} \times \mathbf{k}_t) \cdot \mathbf{E}_t] = 0$$

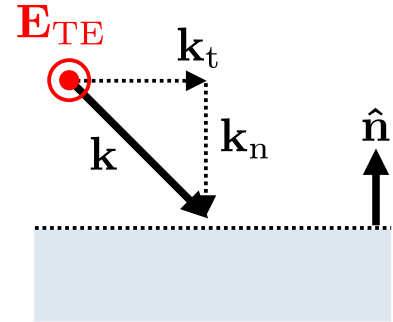
dyad

$$\mathbf{E} = \mathbf{E}_t + \hat{\mathbf{n}} E_n$$

In the xy-plane

$$\mathbf{E} = \begin{bmatrix} E_x \\ E_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} E_z$$

For a TE wave



$$\begin{cases} \mathbf{E}_t \parallel \hat{\mathbf{n}} \times \mathbf{k}_t \\ \mathbf{E}_t \perp \mathbf{k}_t \end{cases}$$

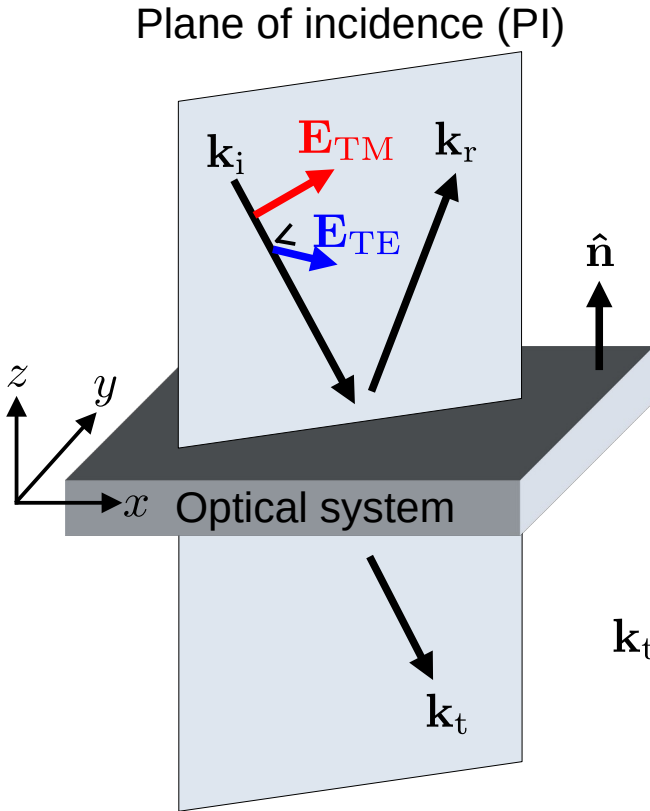
$$\mathbf{k}_t \mathbf{k}_t \cdot \mathbf{E}_t = \mathbf{k}_t (\mathbf{k}_t \cdot \mathbf{E}_t) = 0$$

dyad

Definition of the dyadic or outer product

$$\mathbf{a}\mathbf{b} = \mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

# Definition Example for the xy-plane



For a system in the xy-plane, we have that

$$\mathbf{k}_t = \begin{bmatrix} k_x \\ k_y \\ 0 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{n}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \longrightarrow \quad \hat{\mathbf{n}} \times \mathbf{k}_t = \begin{bmatrix} -k_y \\ k_x \\ 0 \end{bmatrix}$$

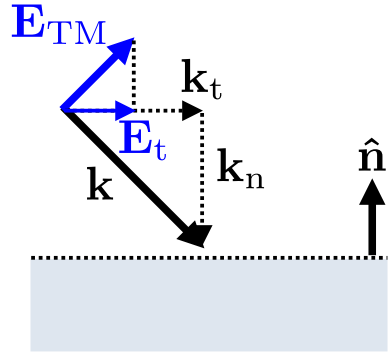
The two dyadic products are then defined as

$$\mathbf{k}_t \mathbf{k}_t = \begin{bmatrix} k_x^2 & k_x k_y & 0 \\ k_y k_x & k_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \hat{\mathbf{n}} \times \mathbf{k}_t \hat{\mathbf{n}} \times \mathbf{k}_t = \begin{bmatrix} k_y^2 & -k_x k_y & 0 \\ -k_y k_x & k_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## Definition of the dyadic or outer product

$$\mathbf{ab} = \mathbf{a} \otimes \mathbf{b} = \mathbf{ab}^T = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

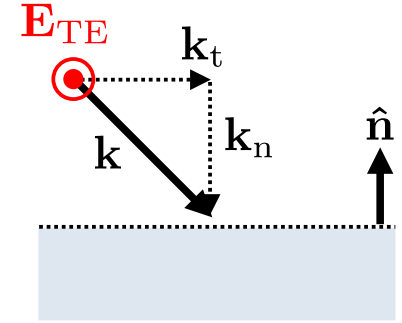
# Tangential Field Components for TE and TM Waves



## TM and TE wave impedances

$$Z_{\text{TM}} = \frac{k_n}{\omega\epsilon}$$

$$Z_{\text{TE}} = \frac{\omega\mu}{k_n}$$



## Generalized impedance dyadic

$$\bar{\bar{Z}} = Z_{\text{TM}} \frac{\mathbf{k}_t \mathbf{k}_t}{k_t^2} + Z_{\text{TE}} \frac{\hat{\mathbf{n}} \times \mathbf{k}_t \hat{\mathbf{n}} \times \mathbf{k}_t}{k_t^2}$$

## Generalized admittance dyadic

$$\bar{\bar{Y}} = \bar{\bar{Z}}^{-1} = \frac{1}{Z_{\text{TM}}} \frac{\mathbf{k}_t \mathbf{k}_t}{k_t^2} + \frac{1}{Z_{\text{TE}}} \frac{\hat{\mathbf{n}} \times \mathbf{k}_t \hat{\mathbf{n}} \times \mathbf{k}_t}{k_t^2}$$

$$\hat{\mathbf{n}} \times \mathbf{k}_t \hat{\mathbf{n}} \times \mathbf{k}_t \cdot \mathbf{E}_t = 0$$

$$\frac{\mathbf{k}_t \mathbf{k}_t}{k_t^2} \cdot \mathbf{E}_t = \mathbf{E}_t$$

$$\mathbf{k}_t \mathbf{k}_t \cdot \mathbf{E}_t = 0$$

$$\frac{\hat{\mathbf{n}} \times \mathbf{k}_t \hat{\mathbf{n}} \times \mathbf{k}_t}{k_t^2} \cdot \mathbf{E}_t = \mathbf{E}_t$$

## All field components in terms of tangential electric field

$$\mathbf{H}_t = \pm \hat{\mathbf{n}} \times \left( \bar{\bar{Y}} \cdot \mathbf{E}_t \right)$$

$$\hat{\mathbf{n}} H_n = \frac{1}{\omega\mu} \mathbf{k}_t \times \mathbf{E}_t$$

$$\hat{\mathbf{n}} E_n = \mp \frac{1}{\omega\epsilon} \left( \mathbf{k}_t \cdot \bar{\bar{Y}} \cdot \mathbf{E}_t \right) \hat{\mathbf{n}}$$

Top and bottom signs correspond to propagation in the positive and negative direction of  $\mathbf{n}$

## What Have We Learned So Far....

- Maxwell equations in the absence of sources (especially charges) are highly redundant
- This redundancy allows us to find the  $\mathbf{H}$  field and the normal component of the  $\mathbf{E}$  field from the knowledge of the tangential components of  $\mathbf{E}$  and  $\mathbf{k}$
- For instance, the divergence of  $\mathbf{E}$  allows us to find  $E_z$  in terms of  $E_x$  and  $E_y$  and  $k_x$  and  $k_y$
- For generality, we can define an impedance/admittance dyadic that allows us to find these field components for arbitrary polarization and propagation angle

# **Conventional Boundary Conditions**

# Unconventional Derivation

## Maxwell equations

$$\nabla \times \mathbf{H} = j\omega\mathbf{D}$$

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$

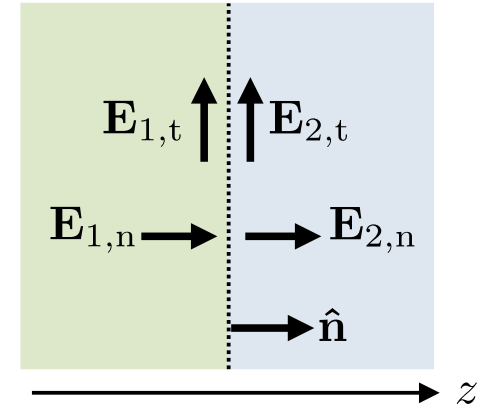
$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Splitting into tangential and normal components

$$\mathbf{E} = \mathbf{E}_t + \hat{\mathbf{n}}E_n$$

$$\nabla = \nabla_t + \hat{\mathbf{n}}\frac{\partial}{\partial z}$$



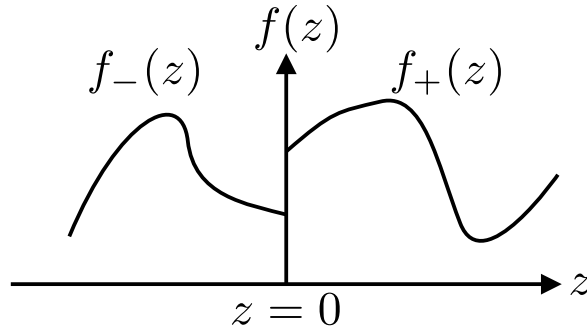
What is the z-derivative of a discontinuous function ?

Unit step function

$$U(z) = \begin{cases} 1, & z > 0 \\ 0, & z < 0 \end{cases}$$

$$\frac{\partial}{\partial z}U(z) = \delta(z)$$

Dirac delta



$$f(z) = f_-(z)[1 - U(z)] + f_+(z)U(z)$$

Taking the derivative along z and applying the chain rule method

$$\frac{\partial}{\partial z}f(z) = f'_-(z)[1 - U(z)] + f'_+(z)U(z) + [f_+(0) - f_-(0)]\delta(z)$$

The derivative of a discontinuous function blows up to infinity!

# Splitting Maxwell Equations into Normal and Tangential Parts

## Maxwell equations

$$\nabla \times \mathbf{H} = j\omega\mathbf{D}$$

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

example

$$\left( \nabla_t + \hat{\mathbf{n}} \frac{\partial}{\partial z} \right) \cdot (\mathbf{B}_t + \hat{\mathbf{n}} B_n) = 0$$

$$\begin{aligned} & \underbrace{\left( \nabla_t \cdot \mathbf{B}_t \right)}_{=0} + \underbrace{\left( \hat{\mathbf{n}} \cdot \nabla_t B_n \right)}_{=0} \\ & \underbrace{\left( \nabla_t \cdot \mathbf{B}_t \right)}_{=0} + \underbrace{\left( \frac{\partial B_n}{\partial z} \right)}_{=0} \end{aligned}$$

$$\nabla_t \cdot \mathbf{B}_t + \frac{\partial B_n}{\partial z} = 0$$

Splitting into tangential and normal components

$$\mathbf{E} = \mathbf{E}_t + \hat{\mathbf{n}} E_n$$

$$\nabla = \nabla_t + \hat{\mathbf{n}} \frac{\partial}{\partial z}$$

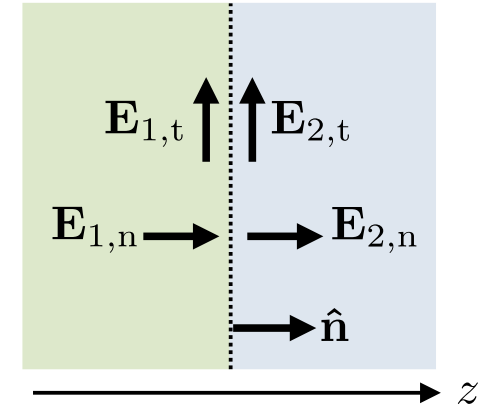
After substitution and simplification

$$\nabla_t \times \hat{\mathbf{n}} H_n + \frac{\partial}{\partial z} (\hat{\mathbf{n}} \times \mathbf{H}_t) = j\omega \mathbf{D}_t$$

$$\nabla_t \times \hat{\mathbf{n}} E_n + \frac{\partial}{\partial z} (\hat{\mathbf{n}} \times \mathbf{E}_t) = -j\omega \mathbf{B}_t$$

$$\nabla_t \cdot \mathbf{D}_t + \frac{\partial D_n}{\partial z} = 0$$

$$\nabla_t \cdot \mathbf{B}_t + \frac{\partial B_n}{\partial z} = 0$$



**These quantities must be continuous at the interface to avoid Dirac deltas and fields blowing up**

# Sourceless Boundary Conditions

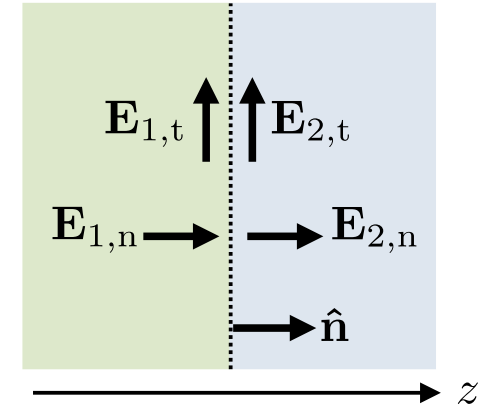
$$\nabla_t \times \hat{\mathbf{n}} H_n + \frac{\partial}{\partial z} (\hat{\mathbf{n}} \times \mathbf{H}_t) = j\omega \mathbf{D}_t$$

$$\nabla_t \times \hat{\mathbf{n}} E_n + \frac{\partial}{\partial z} (\hat{\mathbf{n}} \times \mathbf{E}_t) = -j\omega \mathbf{B}_t$$

$$\nabla_t \cdot \mathbf{D}_t + \frac{\partial}{\partial z} D_n = 0$$

$$\nabla_t \cdot \mathbf{B}_t + \frac{\partial}{\partial z} B_n = 0$$

These quantities must be continuous at the interface to avoid Dirac deltas and fields blowing up



Boundary conditions

$$\Delta \mathbf{H}_t = \mathbf{H}_{2,t} - \mathbf{H}_{1,t} = 0$$

$$\Delta \mathbf{E}_t = \mathbf{E}_{2,t} - \mathbf{E}_{1,t} = 0$$

$$\Delta D_n = D_{2,n} - D_{1,n} = 0$$

$$\Delta B_n = B_{2,n} - B_{1,n} = 0$$

$$\mathbf{H}_{1,t} = \mathbf{H}_{2,t}$$

$$\mathbf{E}_{1,t} = \mathbf{E}_{2,t}$$

$$D_{1,n} = D_{2,n}$$

$$B_{1,n} = B_{2,n}$$

# Boundary Conditions with Sources

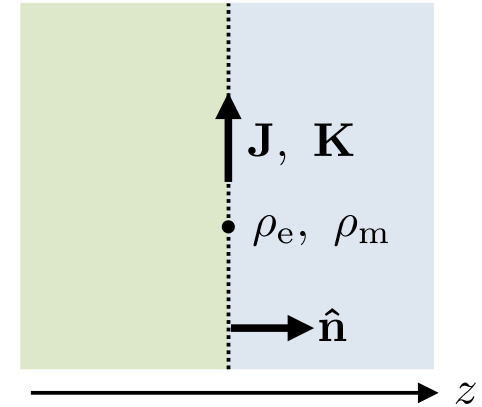
$$\nabla_t \times \hat{\mathbf{n}} H_n + \frac{\partial}{\partial z} (\hat{\mathbf{n}} \times \mathbf{H}_t) = +\mathbf{J}_t \delta(z) + j\omega \mathbf{D}_t$$

$$\nabla_t \times \hat{\mathbf{n}} E_n + \frac{\partial}{\partial z} (\hat{\mathbf{n}} \times \mathbf{E}_t) = -\mathbf{K}_t \delta(z) - j\omega \mathbf{B}_t$$

$$\nabla_t \cdot \mathbf{D}_t + \frac{\partial}{\partial z} D_n = \rho_e \delta(z)$$

$$\nabla_t \cdot \mathbf{B}_t + \frac{\partial}{\partial z} B_n = \rho_m \delta(z)$$

We now consider the presence of surface tangential currents and charge densities



**Boundary conditions**

$$\hat{\mathbf{n}} \times \Delta \mathbf{H}_t = +\mathbf{J}_t$$

$$\hat{\mathbf{n}} \times \Delta \mathbf{E}_t = -\mathbf{K}_t$$

$$\Delta D_n = \rho_e$$

$$\Delta B_n = \rho_m$$

In the presence of impressed sources, these quantities must diverge. Their divergence is equal to the Dirac sources

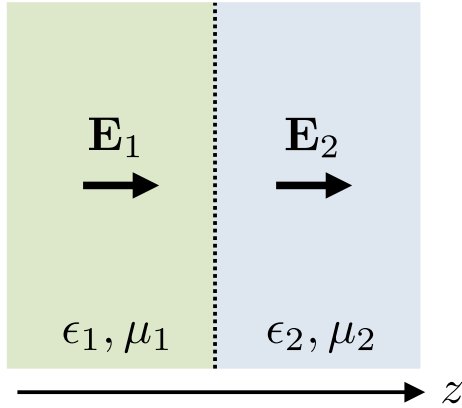
## What Have We Learned So Far....

- We can heuristically derive boundary conditions by considering field discontinuities
- In the absence of sources at the interface, the spatial derivatives across the interface must lead to non-diverging values to prevent fields from blowing up. This allows us to find that the tangential parts of  $\mathbf{E}$  and  $\mathbf{H}$  are continuous, whereas the normal parts of  $\mathbf{D}$  and  $\mathbf{B}$  are continuous across the interface.
- In the presence of sources, the fields are discontinuous such that their divergence (when taking the derivative along  $z$ ) is equal to these sources.

# **Scattering at an Interface Between two Media**

# Why Waves Reflect?

Can a wave fully transmit from medium 1 to medium 2?



Can all the power be transmitted?  
 =  
 Are the Poynting vectors equal?

$$\mathbf{S}_1 = \mathbf{S}_2$$

$$\frac{|\mathbf{E}_1|^2}{2\eta_1} \hat{\mathbf{z}} = \frac{|\mathbf{E}_2|^2}{2\eta_2} \hat{\mathbf{z}}$$



$$\eta_1 = \eta_2$$

From the boundary conditions, we know that

$$\mathbf{E}_{1,t} = \mathbf{E}_{2,t}$$

At normal, incidence

$$\mathbf{E}_1 = \mathbf{E}_{1,t} = \mathbf{E}_{2,t} = \mathbf{E}_2$$



$$\sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_2}{\epsilon_2}}$$

We either have

$$\epsilon_1 = \epsilon_2$$

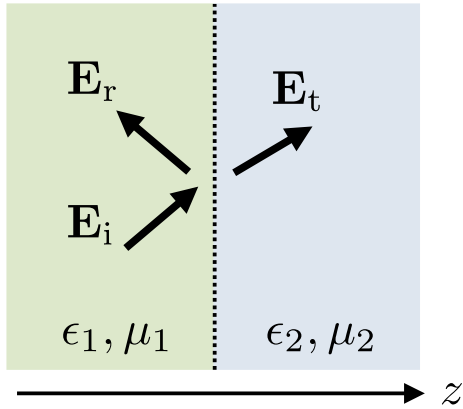
$$\mu_1 = \mu_2$$

or

$$\frac{\mu_1}{\epsilon_1} = \frac{\mu_2}{\epsilon_2}$$

Full transmission only if the two media are (impedance) matched

# Angular Scattering at an Interface



Boundary conditions

$$\begin{aligned} \mathbf{H}_{1,\parallel} &= \mathbf{H}_{2,\parallel} \\ \mathbf{E}_{1,\parallel} &= \mathbf{E}_{2,\parallel} \end{aligned}$$

Incident, reflected and transmitted fields

$$\mathbf{E}_i = (\mathbf{E}_{i,\parallel} + \hat{\mathbf{n}}E_{i,n}) e^{-j\mathbf{k}_i \cdot \mathbf{r}}$$

$$\mathbf{E}_r = (\mathbf{E}_{r,\parallel} + \hat{\mathbf{n}}E_{r,n}) e^{-j\mathbf{k}_r \cdot \mathbf{r}}$$

$$\mathbf{E}_t = (\mathbf{E}_{t,\parallel} + \hat{\mathbf{n}}E_{t,n}) e^{-j\mathbf{k}_t \cdot \mathbf{r}}$$

$$\mathbf{H}_{i,\parallel} e^{-j\mathbf{k}_i \cdot \mathbf{r}} + \mathbf{H}_{r,\parallel} e^{-j\mathbf{k}_r \cdot \mathbf{r}} = \mathbf{H}_{t,\parallel} e^{-j\mathbf{k}_t \cdot \mathbf{r}}$$

$$\mathbf{E}_{i,\parallel} e^{-j\mathbf{k}_i \cdot \mathbf{r}} + \mathbf{E}_{r,\parallel} e^{-j\mathbf{k}_r \cdot \mathbf{r}} = \mathbf{E}_{t,\parallel} e^{-j\mathbf{k}_t \cdot \mathbf{r}}$$

If the interface is at  $z = 0$

$$\mathbf{H}_{i,\parallel} e^{-j\mathbf{k}_{i,\parallel} \cdot \mathbf{r}_{\parallel}} + \mathbf{H}_{r,\parallel} e^{-j\mathbf{k}_{r,\parallel} \cdot \mathbf{r}_{\parallel}} = \mathbf{H}_{t,\parallel} e^{-j\mathbf{k}_{t,\parallel} \cdot \mathbf{r}_{\parallel}}$$

$$\mathbf{E}_{i,\parallel} e^{-j\mathbf{k}_{i,\parallel} \cdot \mathbf{r}_{\parallel}} + \mathbf{E}_{r,\parallel} e^{-j\mathbf{k}_{r,\parallel} \cdot \mathbf{r}_{\parallel}} = \mathbf{E}_{t,\parallel} e^{-j\mathbf{k}_{t,\parallel} \cdot \mathbf{r}_{\parallel}}$$

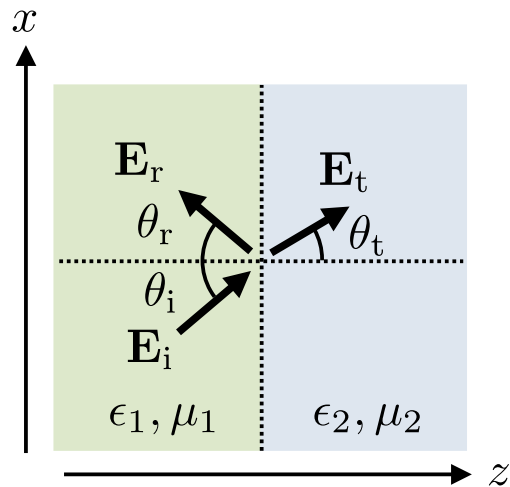
Snell law

Reflection law

$$\mathbf{k}_{t,\parallel} = \mathbf{k}_{i,\parallel} = \mathbf{k}_{r,\parallel}$$

Can only be satisfied if

# Law of Reflection and Refraction



In the  $xz$ -plane

$$\mathbf{k}_i = k_{x,i} \hat{\mathbf{x}} + k_{z,i} \hat{\mathbf{z}}$$

$$\mathbf{k}_{i,\parallel} = k_{x,i} \hat{\mathbf{x}}$$

$$k_{x,i} = k_1 \sin \theta_i$$

Snell law      Reflection law

$$\mathbf{k}_{t,\parallel} = \mathbf{k}_{i,\parallel} = \mathbf{k}_{r,\parallel}$$

$$k_2 \sin \theta_t = k_1 \sin \theta_i$$

$$k_1 \sin \theta_i = k_1 \sin \theta_r$$

$$k_1 = k_0 n_1$$

$$k_2 = k_0 n_2$$

$$n_2 \sin \theta_t = n_1 \sin \theta_i$$

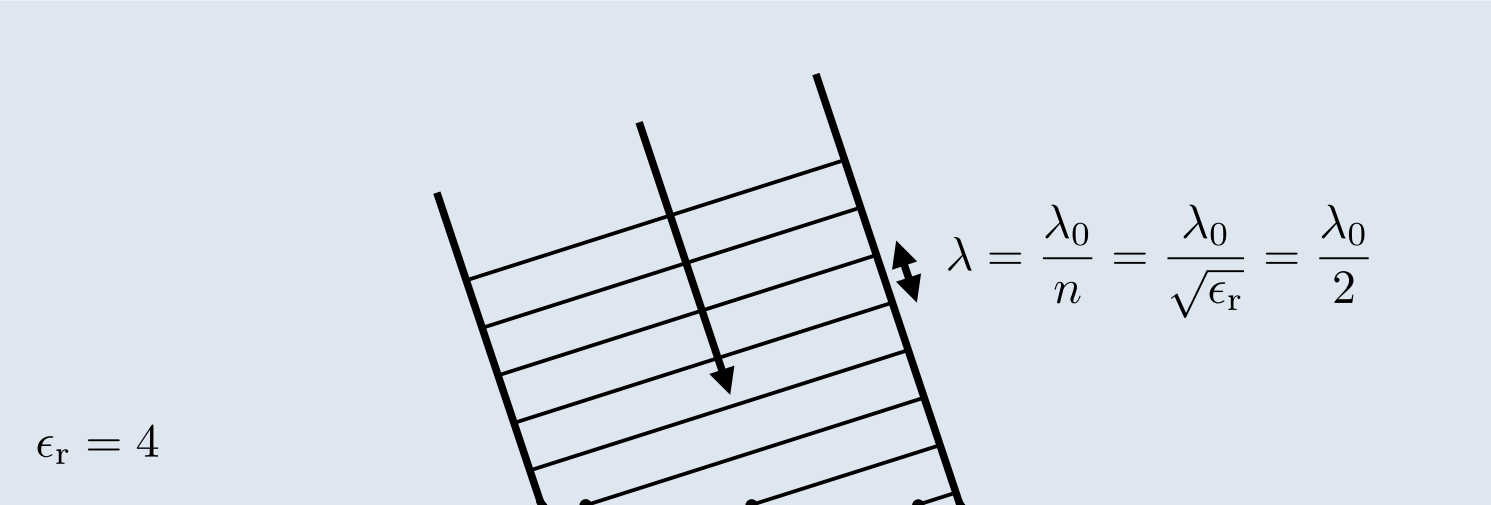
$$\theta_i = \theta_r$$

**Snell law**

**Reflection law**

# Understanding Refraction

Refraction between dielectric materials



$$\lambda = \frac{\lambda_0}{n} = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \frac{\lambda_0}{2}$$

$\epsilon_r = 4$

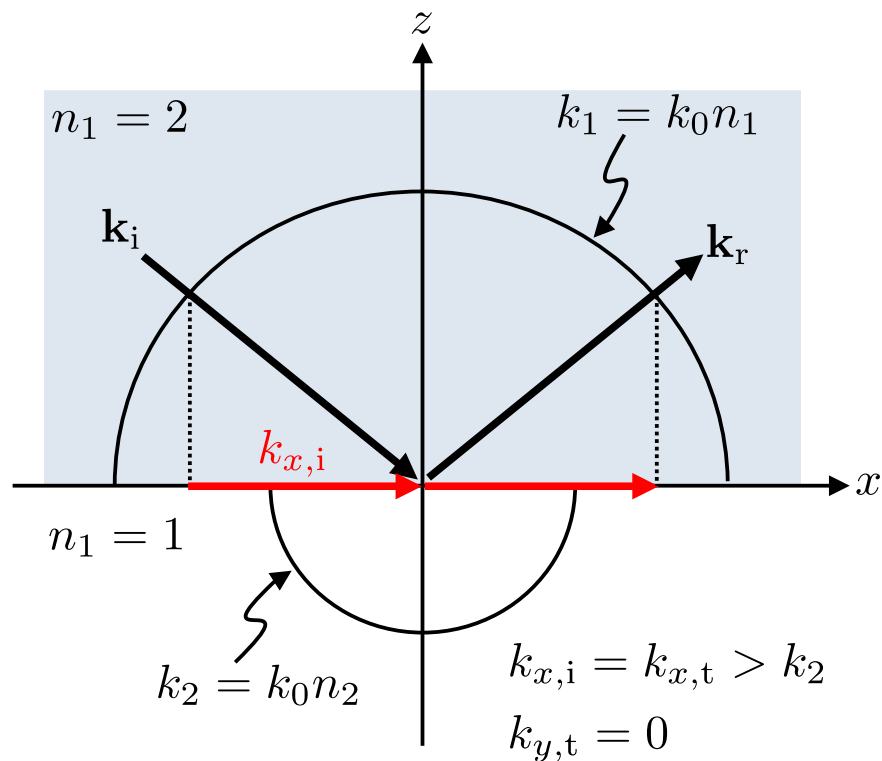
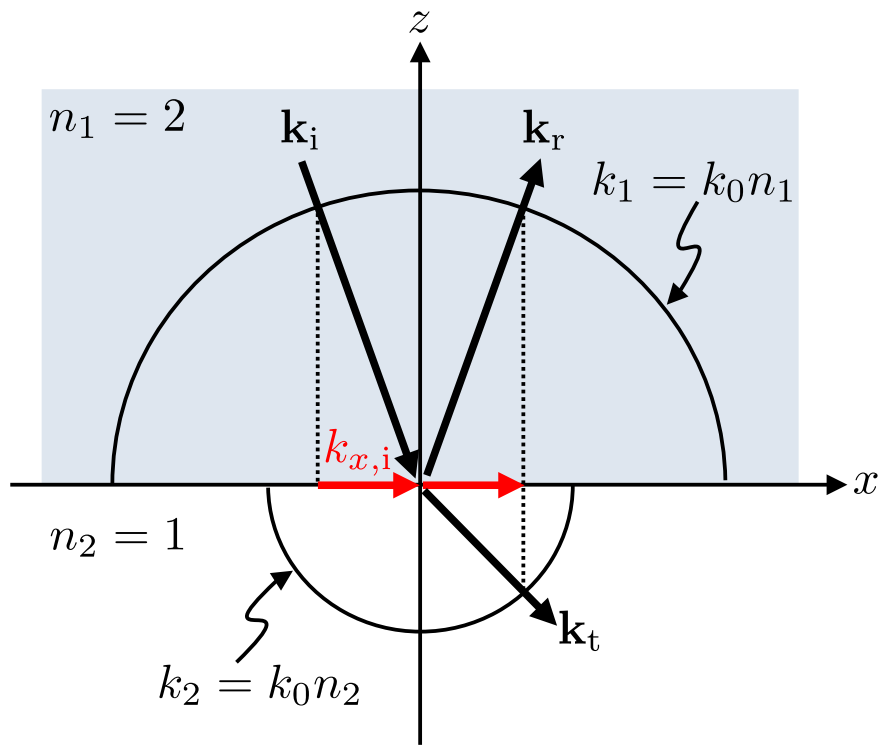
$\epsilon_r = 1$

$$\lambda = \frac{\lambda_0}{n} = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \lambda_0$$

Phase matching condition forces the wave to bend

$$\mathbf{k}_{t,\parallel} = \mathbf{k}_{i,\parallel} = \mathbf{k}_{r,\parallel}$$

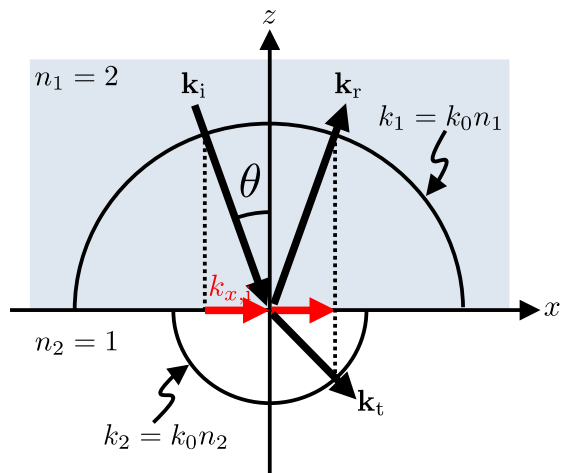
# Understanding Refraction



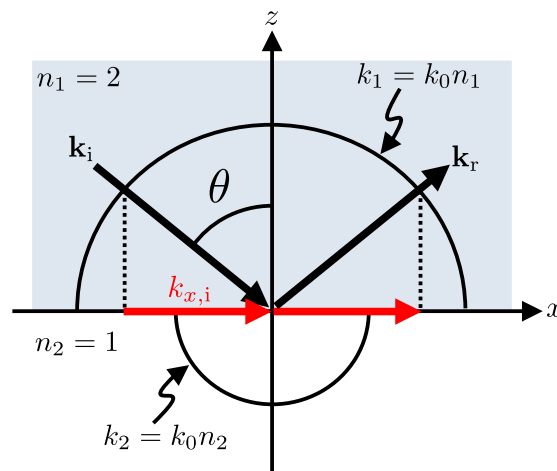
$$\mathbf{k}_{t,\parallel} = \mathbf{k}_{i,\parallel} = \mathbf{k}_{r,\parallel}$$

$$k_{z,t} = \sqrt{k_2^2 - (k_{x,t}^2 + k_{y,t}^2)} \in \mathbb{I}$$

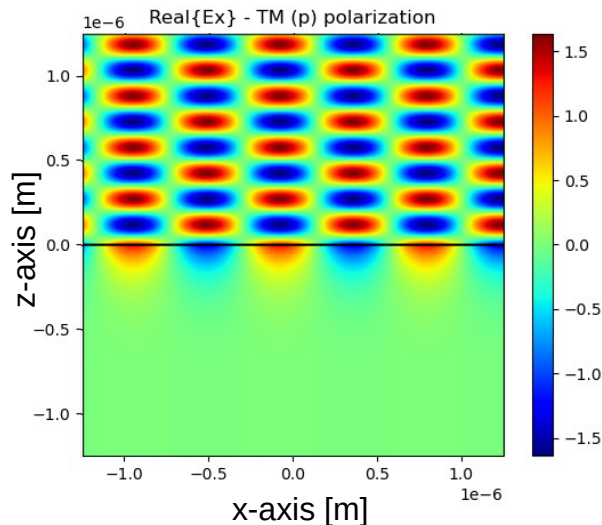
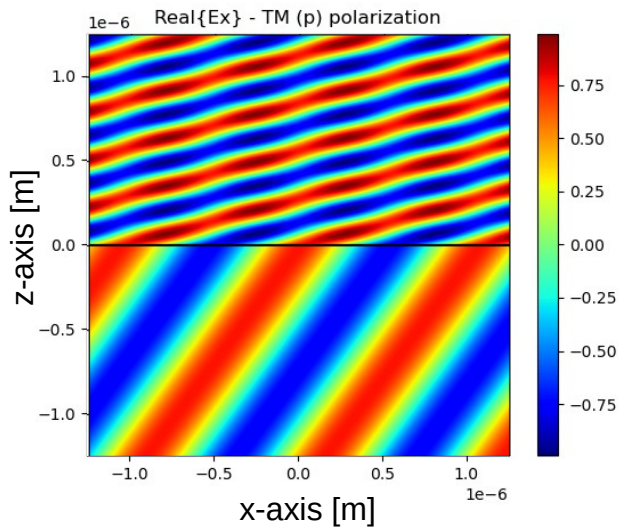
# Refraction and Total Internal Reflection



$\theta = 28^\circ$



$\theta = 35^\circ$



$$\mathbf{E}_t = \mathbf{E}_{0,t} e^{-jk_{x,t}x} e^{-jk_{z,t}z}$$

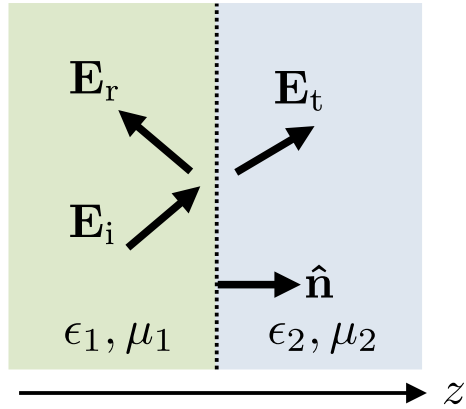
$$k_{z,t} = \sqrt{k_2^2 - (k_{x,t}^2 + k_{y,t}^2)} \in \mathbb{I}$$

$$k_{z,t} = j\alpha$$

$$\mathbf{E}_t = \mathbf{E}_{0,t} e^{-jk_{x,t}x} e^{\alpha z}$$

**Evanescent wave**

# Sourceless Boundary Conditions



Vector transmission line

$$\mathbf{H}_{\parallel} = \pm \hat{\mathbf{n}} \times (\overline{\overline{\mathbf{Y}}} \cdot \mathbf{E}_{\parallel})$$

where

$$\overline{\overline{\mathbf{Y}}} = \frac{\omega \epsilon}{k_n} \frac{\mathbf{k}_{\parallel} \mathbf{k}_{\parallel}}{k_{\parallel}^2} + \frac{k_n}{\omega \mu} \frac{\hat{\mathbf{n}} \times \mathbf{k}_{\parallel} \hat{\mathbf{n}} \times \mathbf{k}_{\parallel}}{k_{\parallel}^2}$$

Boundary conditions

$$\begin{aligned} \mathbf{H}_{1,\parallel} &= \mathbf{H}_{2,\parallel} \\ \mathbf{E}_{1,\parallel} &= \mathbf{E}_{2,\parallel} \end{aligned}$$

Phase matching condition

$$\mathbf{k}_{t,\parallel} = \mathbf{k}_{i,\parallel} = \mathbf{k}_{r,\parallel}$$

$$\mathbf{H}_{i,\parallel} + \mathbf{H}_{r,\parallel} = \mathbf{H}_{t,\parallel}$$

$$\mathbf{E}_{i,\parallel} + \mathbf{E}_{r,\parallel} = \mathbf{E}_{t,\parallel}$$

$$\overline{\overline{\mathbf{Y}}}_1 \cdot \mathbf{E}_{i,\parallel} - \overline{\overline{\mathbf{Y}}}_1 \cdot \mathbf{E}_{r,\parallel} = \overline{\overline{\mathbf{Y}}}_2 \cdot \mathbf{E}_{t,\parallel}$$

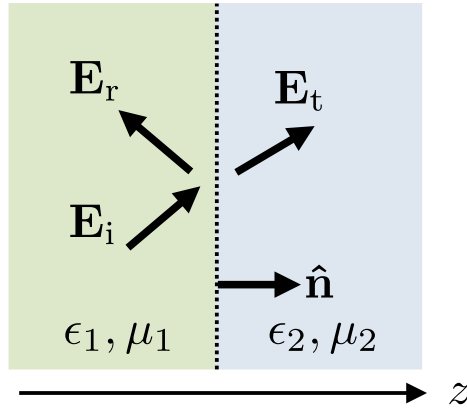
$$\mathbf{E}_{i,\parallel} + \mathbf{E}_{r,\parallel} = \mathbf{E}_{t,\parallel}$$

$$\mathbf{E}_{i,\parallel} - \mathbf{E}_{r,\parallel} = \overline{\overline{\mathbf{Z}}}_1 \cdot \overline{\overline{\mathbf{Z}}}_2^{-1} \cdot \mathbf{E}_{t,\parallel}$$

$$\mathbf{E}_{i,\parallel} + \mathbf{E}_{r,\parallel} = \mathbf{E}_{t,\parallel}$$

$$\overline{\overline{\mathbf{Z}}} = \overline{\overline{\mathbf{Y}}}^{-1}$$

# Solving for the Scattering Dyadics



## Medium impedance dyadic

$$\bar{\bar{Z}} = \frac{k_n}{\omega\epsilon} \frac{\mathbf{k}_{\parallel}\mathbf{k}_{\parallel}}{k_{\parallel}^2} + \frac{\omega\mu}{k_n} \frac{\hat{\mathbf{n}} \times \mathbf{k}_{\parallel} \hat{\mathbf{n}} \times \mathbf{k}_{\parallel}}{k_{\parallel}^2}$$

$$\mathbf{E}_{i,\parallel} - \mathbf{E}_{r,\parallel} = \bar{\bar{Z}}_1 \cdot \bar{\bar{Z}}_2^{-1} \cdot \mathbf{E}_{t,\parallel}$$

$$\mathbf{E}_{i,\parallel} + \mathbf{E}_{r,\parallel} = \mathbf{E}_{t,\parallel}$$

↓ Solving for  $\mathbf{E}_{r,\parallel}$ ,  $\mathbf{E}_{t,\parallel}$

$$\mathbf{E}_{t,\parallel} = 2\bar{\bar{Z}}_2 \cdot \left(\bar{\bar{Z}}_1 + \bar{\bar{Z}}_2\right)^{-1} \cdot \mathbf{E}_{i,\parallel}$$

$$\mathbf{E}_{r,\parallel} = \left(\bar{\bar{Z}}_2 - \bar{\bar{Z}}_1\right) \cdot \left(\bar{\bar{Z}}_1 + \bar{\bar{Z}}_2\right)^{-1} \cdot \mathbf{E}_{i,\parallel}$$

↓

## Scattering dyadics

$$\bar{\bar{t}} = 2\bar{\bar{Z}}_2 \cdot \left(\bar{\bar{Z}}_1 + \bar{\bar{Z}}_2\right)^{-1}$$

$$\bar{\bar{r}} = \left(\bar{\bar{Z}}_2 - \bar{\bar{Z}}_1\right) \cdot \left(\bar{\bar{Z}}_1 + \bar{\bar{Z}}_2\right)^{-1}$$

No reflection if

$$\bar{\bar{Z}}_1 = \bar{\bar{Z}}_2$$

Impedance matching!

# Field Scattering Parameters

## Medium impedance dyadic

$$\bar{\bar{Z}} = \frac{k_n}{\omega\epsilon} \frac{\mathbf{k}_{\parallel}\mathbf{k}_{\parallel}}{k_{\parallel}^2} + \frac{\omega\mu}{k_n} \frac{\hat{\mathbf{n}} \times \mathbf{k}_{\parallel} \hat{\mathbf{n}} \times \mathbf{k}_{\parallel}}{k_{\parallel}^2}$$

## Scattering dyadics

$$\bar{\bar{t}} = 2\bar{\bar{Z}}_2 \cdot (\bar{\bar{Z}}_1 + \bar{\bar{Z}}_2)^{-1}$$

$$\bar{\bar{r}} = (\bar{\bar{Z}}_2 - \bar{\bar{Z}}_1) \cdot (\bar{\bar{Z}}_1 + \bar{\bar{Z}}_2)^{-1}$$



For the TM case, the electric field has both normal and tangential components. We need to project the field to get the tangential component.

$$k_{z,1} = \sqrt{k_1^2 - (k_x^2 + k_y^2)}$$

$$k_{z,2} = \sqrt{k_2^2 - (k_x^2 + k_y^2)}$$

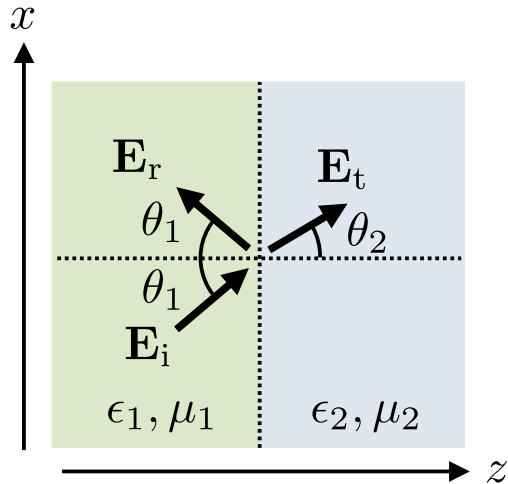
$$t_{\text{TM}}(k_x, k_y) = \frac{k_2}{k_{z,2}} \hat{\mathbf{p}}_{\text{TM}} \cdot \bar{\bar{t}} \cdot \hat{\mathbf{p}}_{\text{TM}} \frac{k_{z,1}}{k_1} = \frac{2\epsilon_2 k_{z,1}}{\epsilon_1 k_{z,2} + \epsilon_2 k_{z,1}} \frac{\eta_2}{\eta_1}$$

$$r_{\text{TM}}(k_x, k_y) = \frac{k_1}{k_{z,1}} \hat{\mathbf{p}}_{\text{TM}} \cdot \bar{\bar{r}} \cdot \hat{\mathbf{p}}_{\text{TM}} \frac{k_{z,1}}{k_1} = \frac{\epsilon_1 k_{z,2} - \epsilon_2 k_{z,1}}{\epsilon_1 k_{z,2} + \epsilon_2 k_{z,1}}$$

$$t_{\text{TE}}(k_x, k_y) = \hat{\mathbf{p}}_{\text{TE}} \cdot \bar{\bar{t}} \cdot \hat{\mathbf{p}}_{\text{TE}} = \frac{2\mu_2 k_{z,1}}{\mu_2 k_{z,1} + \mu_1 k_{z,2}}$$

$$r_{\text{TE}}(k_x, k_y) = \hat{\mathbf{p}}_{\text{TE}} \cdot \bar{\bar{r}} \cdot \hat{\mathbf{p}}_{\text{TE}} = \frac{\mu_2 k_{z,1} - \mu_1 k_{z,2}}{\mu_2 k_{z,1} + \mu_1 k_{z,2}}$$

# Field Scattering Parameters in One Plane of Incidence



Since the media are uniform, we can restrict our analysis to a single plane of incidence. Let's use the  $xz$ -plane implying that

$$k_{z,1} = k_1 \cos \theta_1 \quad k_{z,2} = k_2 \cos \theta_2$$

## TM and TE Fresnel coefficients

$$t_{\text{TM}} = \frac{2\eta_2 \cos \theta_1}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1}$$

$$r_{\text{TM}} = \frac{\eta_2 \cos \theta_2 - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1}$$

$$t_{\text{TE}} = \frac{2\eta_2 \cos \theta_1}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_2}$$

$$r_{\text{TE}} = \frac{\eta_2 \cos \theta_1 - \eta_1 \cos \theta_2}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_2}$$

These are field scattering parameters not power !!

# Power Coefficients

Power coefficients are obtained by considering the Poynting vector

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} = \frac{1}{2} \text{Re} \left\{ (\mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}}) \times \left( \frac{\mathbf{k} \times \mathbf{E}_0}{\omega \mu} e^{-j\mathbf{k} \cdot \mathbf{r}} \right)^* \right\} = \frac{|\mathbf{E}_0|^2}{2k_0 \eta_0} e^{2\text{Im}\{\mathbf{k}\} \cdot \mathbf{r}} \text{Re} \left\{ \frac{\mathbf{k}}{\mu_r} \right\}$$

Power coefficients are the ratios of the Poynting vectors normal to the interface

$$R = \frac{\langle S_{z,r}(\mathbf{r} = 0) \rangle}{\langle S_{z,i}(\mathbf{r} = 0) \rangle} = \frac{\frac{|\mathbf{E}_r|^2}{2k_0 \eta_0} \text{Re} \left\{ \frac{k_{z,r}}{\mu_{r,1}} \right\}}{\frac{|\mathbf{E}_i|^2}{2k_0 \eta_0} \text{Re} \left\{ \frac{k_{z,i}}{\mu_{r,1}} \right\}} = \frac{|\mathbf{E}_r|^2}{|\mathbf{E}_i|^2}$$

$$T = \frac{\langle S_{z,t}(\mathbf{r} = 0) \rangle}{\langle S_{z,i}(\mathbf{r} = 0) \rangle} = \frac{\frac{|\mathbf{E}_t|^2}{2k_0 \eta_0} \text{Re} \left\{ \frac{k_{z,t}}{\mu_{r,2}} \right\}}{\frac{|\mathbf{E}_i|^2}{2k_0 \eta_0} \text{Re} \left\{ \frac{k_{z,i}}{\mu_{r,1}} \right\}} = \frac{|\mathbf{E}_t|^2}{|\mathbf{E}_i|^2} \text{Re} \left\{ \frac{k_{z,t}}{\mu_{r,2}} \right\} \text{Re} \left\{ \frac{\mu_{r,1}}{k_{z,i}} \right\}$$

$$\mathbf{E}_r = r \mathbf{E}_i$$

$$\mathbf{E}_t = t \mathbf{E}_i$$

Either TE and  
TM parameters

## Power coefficients for lossy media

$$R = |r|^2 \quad T = |t|^2 \text{Re} \left\{ \frac{k_{z,t}}{\mu_{r,2}} \right\} \text{Re} \left\{ \frac{\mu_{r,1}}{k_{z,i}} \right\}$$

## Power coefficients for lossless media

$$R = |r|^2 \quad T = |t|^2 \frac{\eta_1 \cos \theta_2}{\eta_2 \cos \theta_1}$$

# What Have We Learned So Far....

- To remove reflection between two media, their impedance must be the same. This is the case if the two media are identical or if the ratio of their permittivity and permeability are equal
- The application of the boundary conditions leads to the phase matching conditions from which we obtain Snell law and the reflection law
- Understand how to graphically represent and explain refraction/reflection
- Total internal reflection leads to evanescent wave in the outside medium
- Fresnel reflection and transmission coefficients are only for the fields not the power
- To obtain the reflectance and transmittance (power), we must consider the ratio of the components of the Poynting vectors normal to the interface.
- The reflectance is expressed as the intensity of the reflection coefficient
- The transmittance is expressed as the intensity of the transmission coefficient and also includes a correcting factor to account for power conservation

# Nonuniform Plane Waves

# Wave Number and Wave Vector

Maxwell equations in k-space

$$\mathbf{k} \times \mathbf{H} = -\omega\epsilon\mathbf{E} \quad \longrightarrow \quad \mathbf{k} \times \mathbf{k} \times \mathbf{E} = -\omega^2\mu\epsilon\mathbf{E}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}$$

$$\mathbf{k} \cdot \mathbf{H} = 0$$

$$\mathbf{k} \cdot \mathbf{E} = 0$$

$$(\mathbf{k} \cdot \mathbf{E})\mathbf{k} - (\mathbf{k} \cdot \mathbf{k})\mathbf{E} = -k^2\mathbf{E}$$

$$(\mathbf{k} \cdot \mathbf{k} - k^2)\mathbf{E} = 0$$

In general, the k-vector may be complex

$$\mathbf{k} = \mathbf{k}_r + j\mathbf{k}_i$$

$$\mathbf{k} \cdot \mathbf{k} = k^2$$

The wavenumber is

$$k = \omega\sqrt{\epsilon\mu}$$

$$k_r^2 - k_i^2 + 2jk_r \cdot k_i = \omega^2\epsilon\mu$$

if  $\mathbf{k}_r \cdot \mathbf{k}_i = 0$  then this relation is real even though the wave vector is complex ( $\mathbf{k}_i \neq 0$ )!

A plane wave is uniform if  $\mathbf{k}_r \parallel \mathbf{k}_i$

# Uniform vs Nonuniform Waves

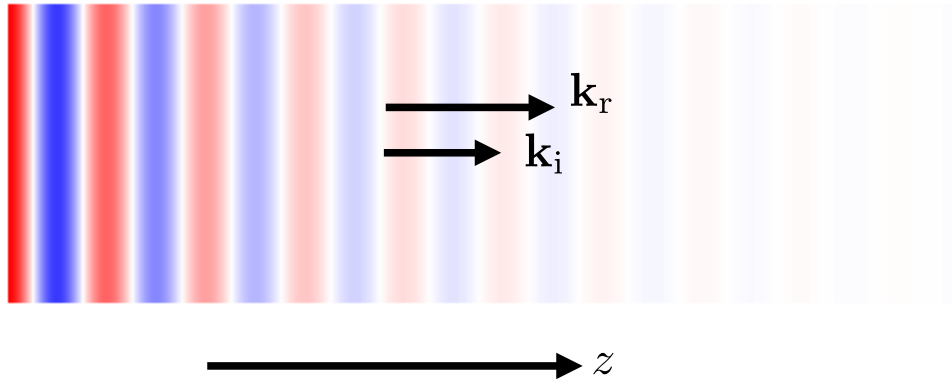
A plane wave is generally expressed as

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_0 e^{-j\mathbf{k}_r\cdot\mathbf{r}} e^{j\mathbf{k}_i\cdot\mathbf{r}} \longrightarrow$$

The corresponding Poynting vector is

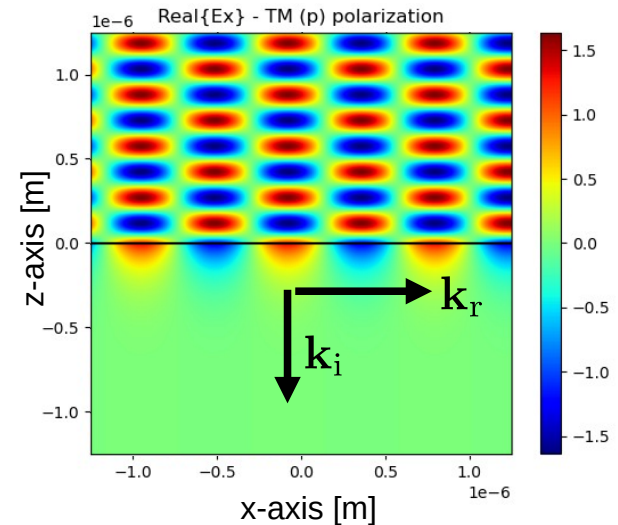
$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} = \frac{|\mathbf{E}_0|^2}{2\omega|\mu|^2} e^{2\mathbf{k}_i\cdot\mathbf{r}} (\mathbf{k}_r\mu_r + \mathbf{k}_i\mu_i)$$

Uniform plane wave in a lossy medium



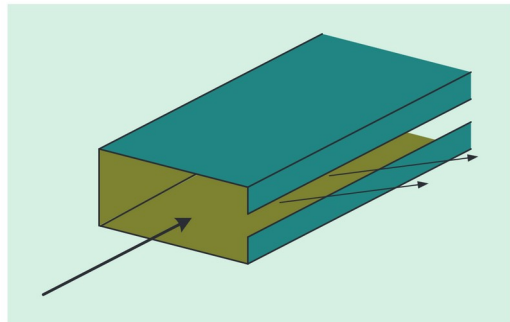
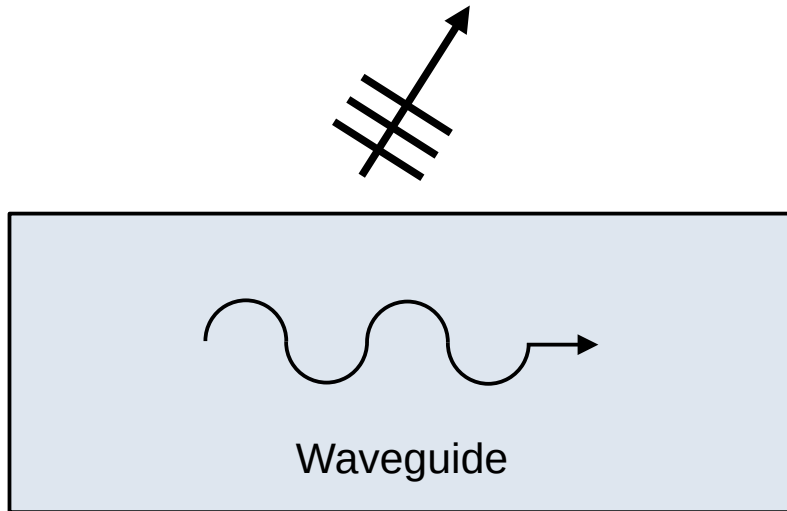
Total internal reflection leads to a nonuniform evanescent wave

$$\theta = 35^\circ$$



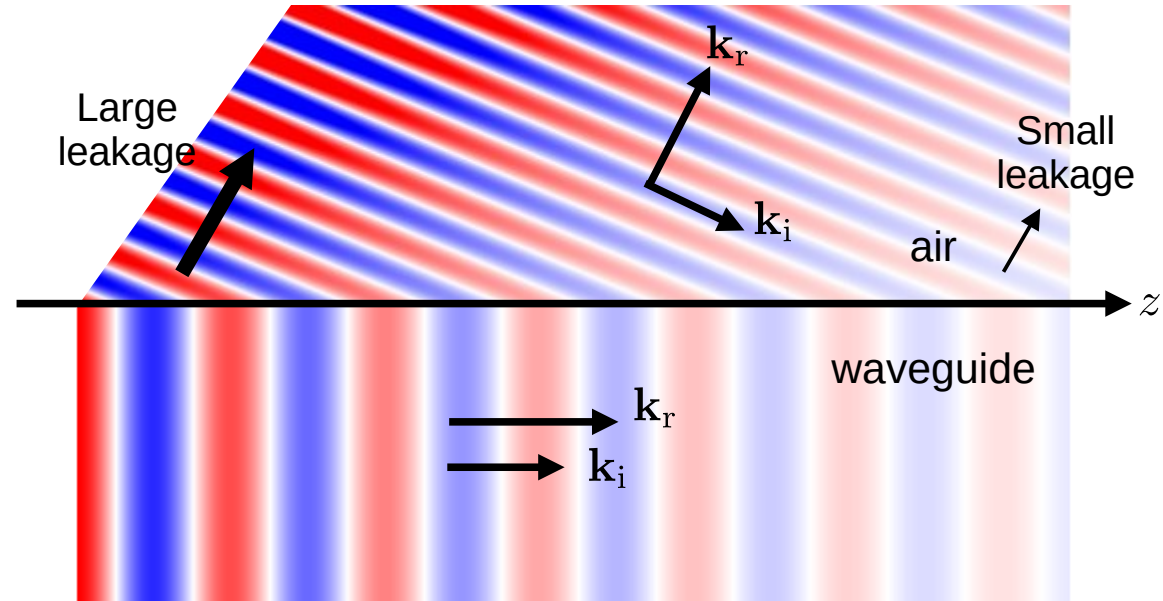
# Leaky Waves

By “opening” the waveguide, we can make the guided wave leak out of it.



**Figure 1.** A rectangular waveguide with a slit cut along the length.

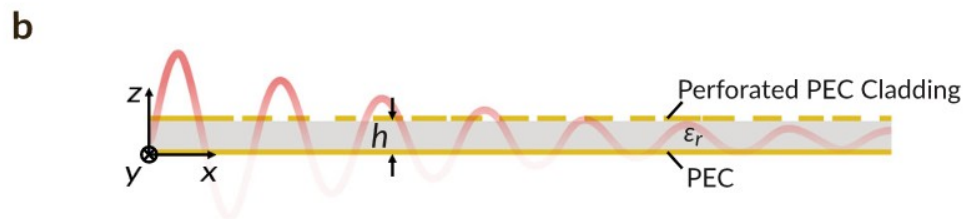
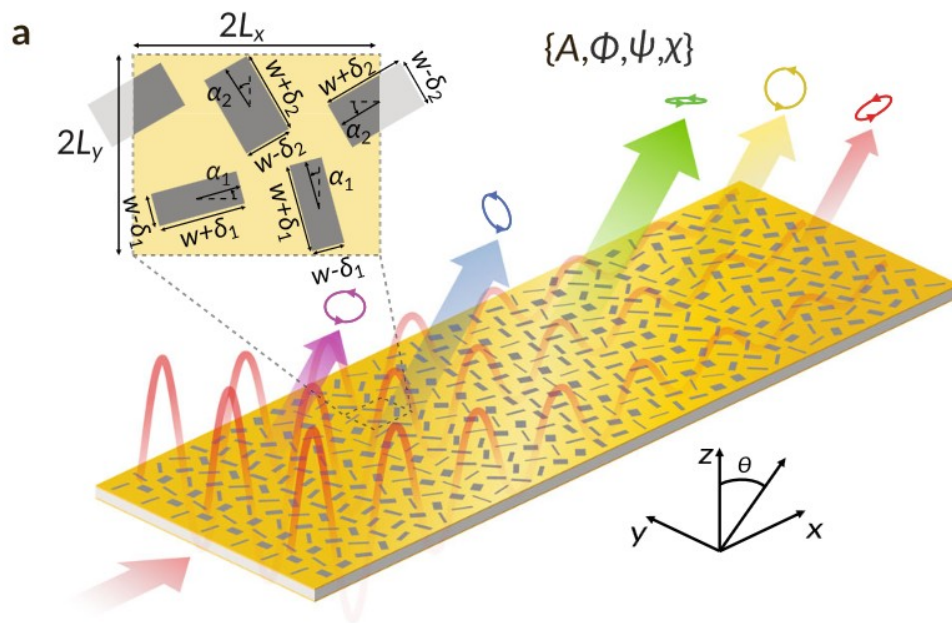
<https://doi.org/10.1109/MMM.2013.2259400>



There is no absorption loss in this system only scattering loss.

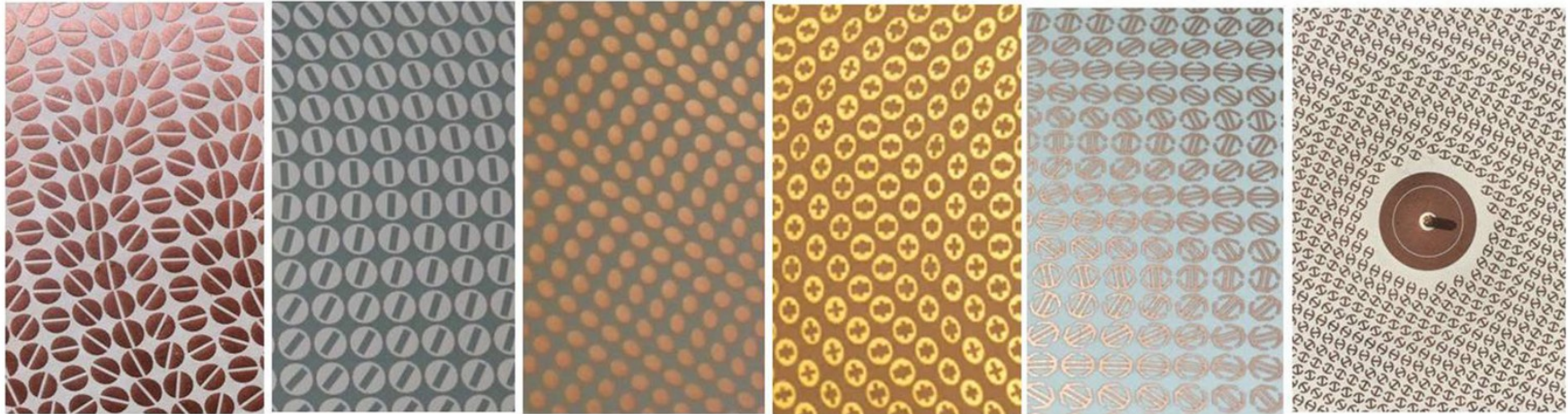
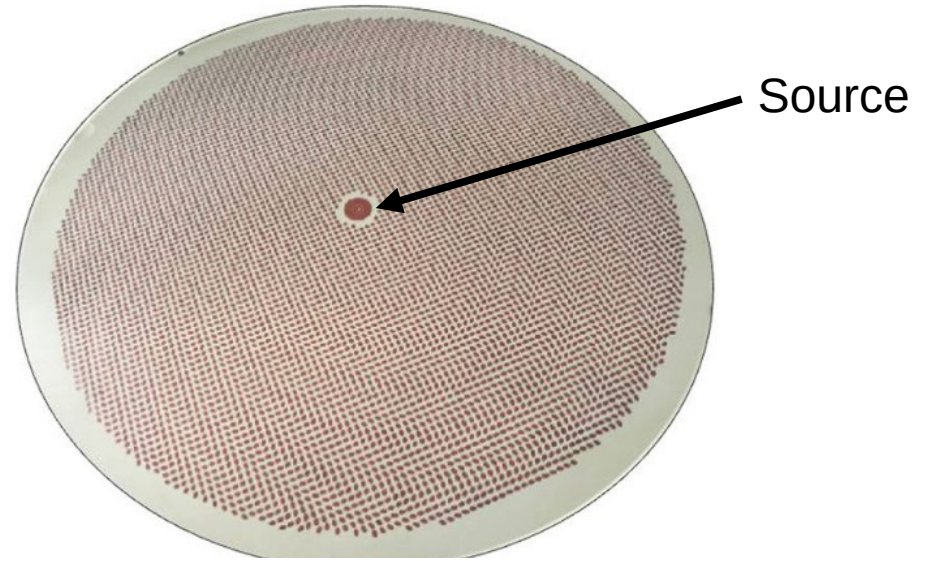
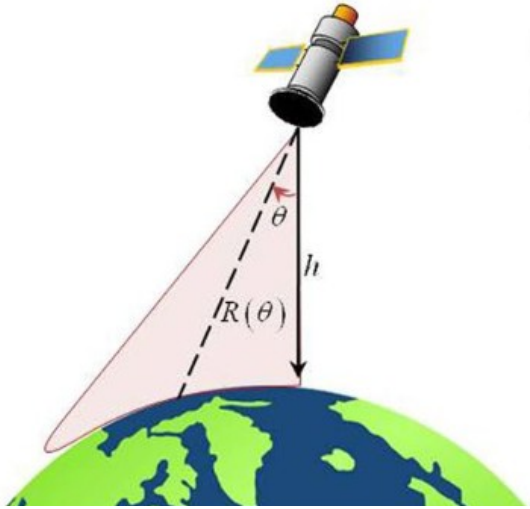
In the waveguide, the wave is uniform but the wave that leaks out of it is nonuniform.

# Examples of Leaky Wave Metamaterials



<https://doi.org/10.1038/s41467-023-39818-2>

# Examples of Leaky Wave Metamaterials

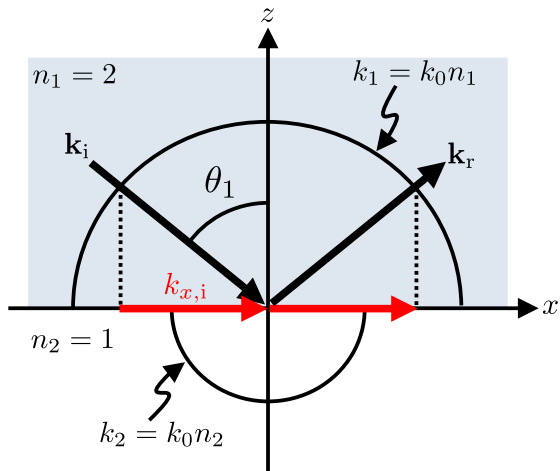


## What Have We Learned So Far....

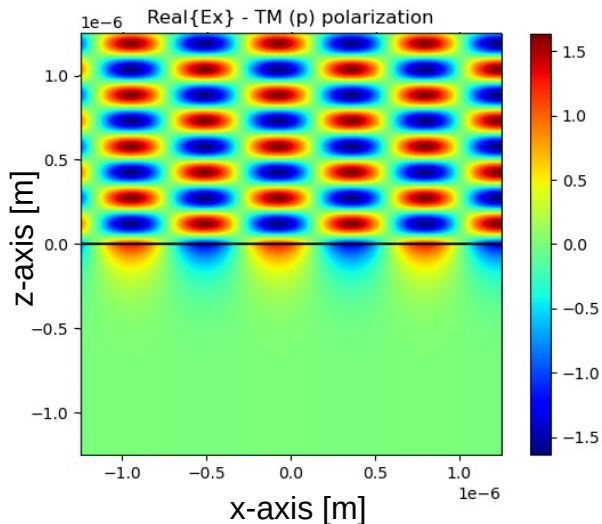
- The wave vector  $\mathbf{k}$  may be decomposed in a real and imaginary vectors  $\mathbf{k}_r$  and  $\mathbf{k}_i$
- If  $\mathbf{k}_r$  and  $\mathbf{k}_i$  are collinear, the wave is uniform. Otherwise, the wave is non-uniform
- Example of uniform waves are plane waves propagating in lossy media
- Example of non-uniform waves are evanescent waves and leaky waves

# Special Angles

# Total Internal Reflection (TIR)



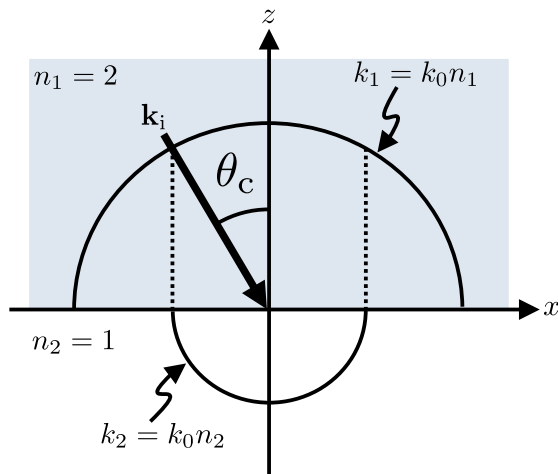
$$\theta_1 = 35^\circ$$



## Snell law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

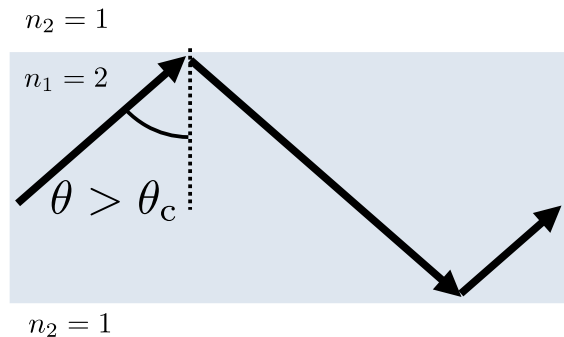
The critical angle  $\theta_1 = \theta_c$  is when  $\theta_2 = 90^\circ$



The critical angle defines an "escape cone" in which the light can leave the high-index material

## Critical angle for TIR

$$\theta_c = \arcsin\left(\frac{n_2}{n_1}\right)$$



Basic principle of waveguiding (optical fibers)

# Brewster Angles

## Fresnel TE reflection coefficient

$$r_{\text{TE}} = \frac{\eta_2 \cos \theta_1 - \eta_1 \cos \theta_2}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_2}$$



$$\eta_2 \cos \theta_1 = \eta_1 \cos \theta_2$$

$$\frac{\mu_2}{\epsilon_2} \cos^2 \theta_1 = \frac{\mu_1}{\epsilon_1} \cos^2 \theta_2$$

$$\frac{\mu_2}{\epsilon_2} (1 - \sin^2 \theta_1) = \frac{\mu_1}{\epsilon_1} (1 - \sin^2 \theta_2)$$

$$\frac{\mu_2}{\epsilon_2} (1 - \sin^2 \theta_1) = \frac{\mu_1}{\epsilon_1} \left( 1 - \frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2} \sin^2 \theta_1 \right)$$

$$\sin^2 \theta_1 = \frac{\mu_2}{\epsilon_1} \left( \frac{\epsilon_2 \mu_1 - \epsilon_1 \mu_2}{\mu_1^2 - \mu_2^2} \right)$$

Assuming:  $\epsilon_1 = \epsilon_2$

$$\theta_{\text{B,TE}} = \arcsin \sqrt{\frac{\mu_2}{\mu_1 + \mu_2}} = \arctan \sqrt{\frac{\mu_2}{\mu_1}}$$

The Brewster angles are when  
 $r_{\text{TE}} = 0$   
 and  
 $r_{\text{TM}} = 0$

## Fresnel TM reflection coefficient

$$r_{\text{TM}} = \frac{\eta_2 \cos \theta_2 - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1}$$



$$\eta_2 \cos \theta_2 = \eta_1 \cos \theta_1$$

$$\frac{\mu_2}{\epsilon_2} \cos^2 \theta_2 = \frac{\mu_1}{\epsilon_1} \cos^2 \theta_1$$

$$\frac{\mu_2}{\epsilon_2} (1 - \sin^2 \theta_2) = \frac{\mu_1}{\epsilon_1} (1 - \sin^2 \theta_1)$$

$$\frac{\mu_2}{\epsilon_2} \left( 1 - \frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2} \sin^2 \theta_1 \right) = \frac{\mu_1}{\epsilon_1} (1 - \sin^2 \theta_1)$$

$$\sin^2 \theta_1 = \frac{\epsilon_2}{\mu_1} \left( \frac{\epsilon_1 \mu_2 - \epsilon_2 \mu_1}{\epsilon_1^2 - \epsilon_2^2} \right)$$

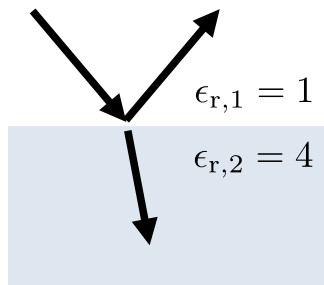
Assuming:  $\mu_1 = \mu_2$

$$\theta_{\text{B,TM}} = \arcsin \sqrt{\frac{\epsilon_2}{\epsilon_1 + \epsilon_2}} = \arctan \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

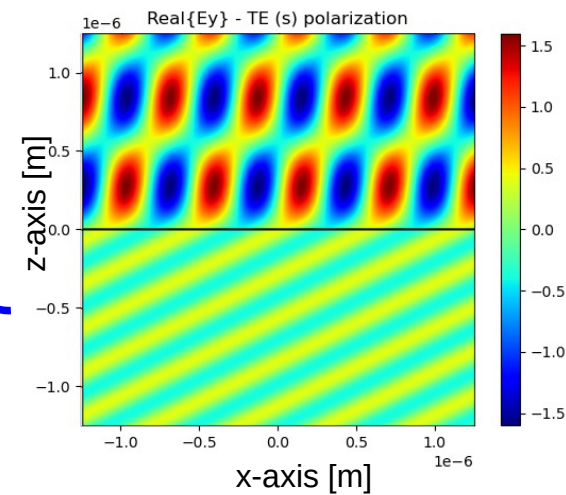
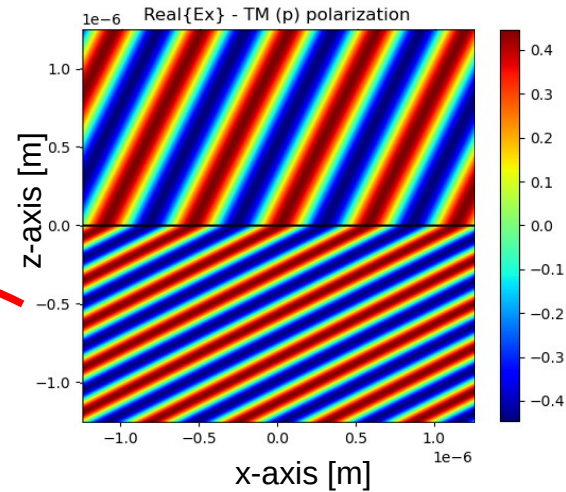
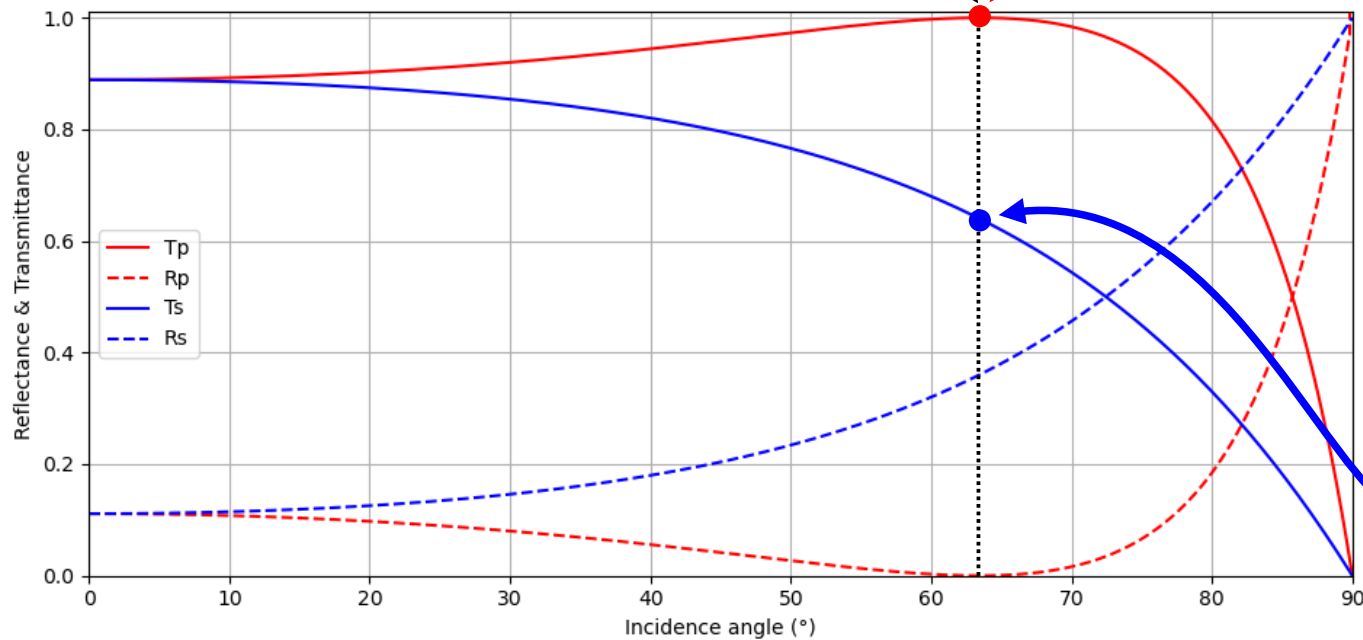
← Using Snell law →

← General solution →

# Brewster Angle Example



$$\theta_{B, TM} = \arctan \sqrt{\frac{\epsilon_2}{\epsilon_1}} \approx 63.43^\circ$$

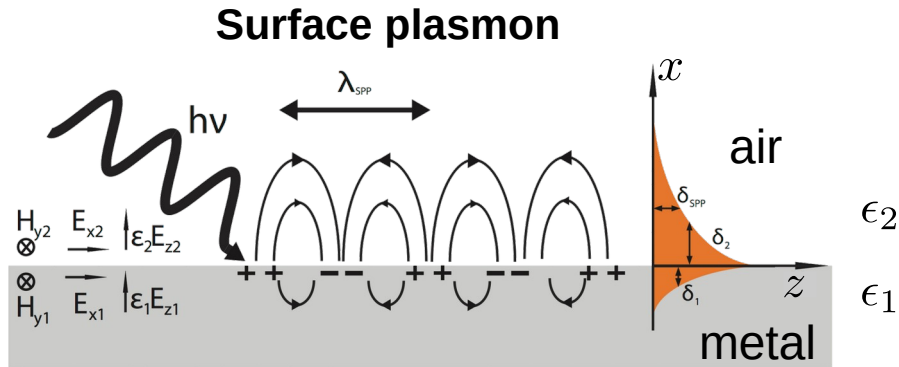


## What Have We Learned So Far....

- The Brewster angle is when the reflection coefficient between two media is zero
- At the Brewster angle, we have separation of polarization since TM becomes reflection-less and only TE is reflected. This can be used as a polarization splitter
- The Brewster angle is also useful for creating reflection-less systems and angular filters

# Surface Plasmons

# What are Surface Plasmons ?



[https://en.wikipedia.org/wiki/Surface\\_plasmon#/media/File:Sketch\\_of\\_surface\\_plasmon.png](https://en.wikipedia.org/wiki/Surface_plasmon#/media/File:Sketch_of_surface_plasmon.png)

In the two media, the fields may be expressed using non-uniform waves

$$\mathbf{E}_i(z) = (E_{x,i}\hat{\mathbf{x}} + E_{z,i}\hat{\mathbf{z}})e^{-\kappa_i|x|}e^{-j\beta z}$$

$$\mathbf{H}_i(z) = \hat{\mathbf{y}}H_{y,i}e^{-\kappa_i|x|}e^{-j\beta z}$$

The wave vector is  $\mathbf{k} = \beta\hat{\mathbf{z}} - j\kappa_i\hat{\mathbf{x}}$

Maxwell equations

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

We restrict our attention to TM waves and consider

$$\frac{\partial}{\partial z} \rightarrow -j\beta \quad \frac{\partial}{\partial y} \rightarrow 0$$

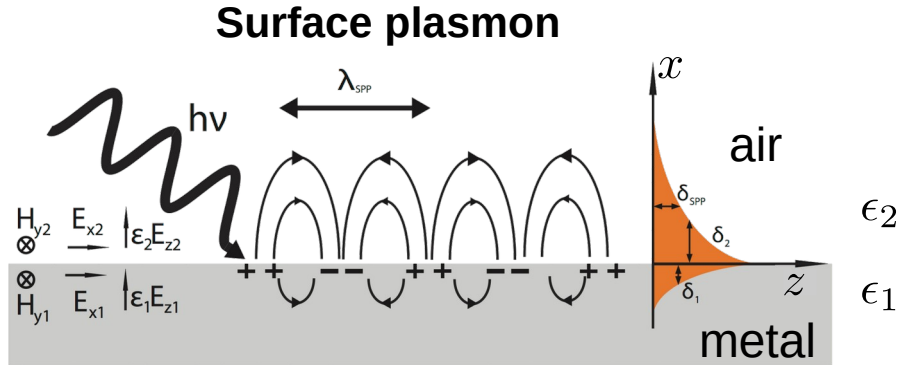


$$j\beta H_y = j\omega\epsilon E_x$$

$$\frac{\partial H_y}{\partial x} = j\omega\epsilon E_z$$

$$j\beta E_x + \frac{\partial E_z}{\partial x} = j\omega\mu H_y$$

# Solving Maxwell Equations



[https://en.wikipedia.org/wiki/Surface\\_plasmon#/media/File:Sketch\\_of\\_surface\\_plasmon.png](https://en.wikipedia.org/wiki/Surface_plasmon#/media/File:Sketch_of_surface_plasmon.png)

In the two media, the fields may be expressed using non-uniform waves

$$\mathbf{E}_i(z) = (E_{x,i}\hat{\mathbf{x}} + E_{z,i}\hat{\mathbf{z}})e^{-\kappa_i|x|}e^{-j\beta z}$$

$$\mathbf{H}_i(z) = \hat{\mathbf{y}}H_{y,i}e^{-\kappa_i|x|}e^{-j\beta z}$$

The wave vector is  $\mathbf{k} = \beta\hat{\mathbf{z}} - j\kappa_i\hat{\mathbf{x}}$

**In the bottom medium**

$$j\beta H_{y,1} = j\omega\epsilon_1 E_{x,1}$$

$$\kappa_1 H_{y,1} = j\omega\epsilon_1 E_{z,1}$$

$$j\beta E_{x,1} + \kappa_1 E_{z,1} = j\omega\mu_1 H_{y,1}$$

$$\downarrow$$

$$k_0^2 \mu_{r,1} \epsilon_{r,1} = \beta^2 - \kappa_1^2$$

$$j\beta H_y = j\omega\epsilon E_x$$

$$\frac{\partial H_y}{\partial x} = j\omega\epsilon E_z$$

$$j\beta E_x + \frac{\partial E_z}{\partial x} = j\omega\mu H_y$$

**In the top medium**

$$j\beta H_{y,2} = j\omega\epsilon_2 E_{x,2}$$

$$-\kappa_2 H_{y,2} = j\omega\epsilon_2 E_{z,2}$$

$$j\beta E_{x,2} - \kappa_2 E_{z,2} = j\omega\mu_2 H_{y,2}$$

$$\downarrow$$

$$k_0^2 \mu_{r,2} \epsilon_{r,2} = \beta^2 - \kappa_2^2$$

# Applying Boundary Conditions

In the bottom medium

$$j\beta H_{y,1} = j\omega\epsilon_1 E_{x,1}$$

$$\kappa_1 H_{y,1} = j\omega\epsilon_1 E_{z,1}$$

$$j\beta E_{x,1} + \kappa_1 E_{z,1} = j\omega\mu_1 H_{y,1}$$

Boundary conditions

$$H_{y,1} = H_{y,2}$$

$$E_{z,1} = E_{z,2}$$

In the top medium

$$j\beta H_{y,2} = j\omega\epsilon_2 E_{x,2}$$

$$-\kappa_2 H_{y,2} = j\omega\epsilon_2 E_{z,2}$$

$$j\beta E_{x,2} - \kappa_2 E_{z,2} = j\omega\mu_2 H_{y,2}$$

$$H_{y,1} = j\omega \frac{\epsilon_1}{\kappa_1} E_{z,1} = -j\omega \frac{\epsilon_2}{\kappa_2} E_{z,2} = H_{y,2}$$

Existence condition

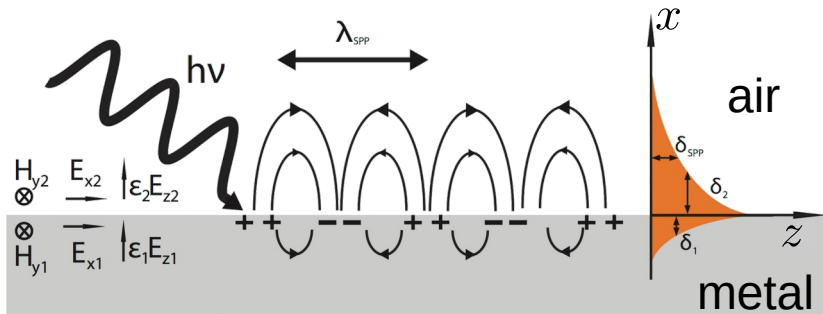
$$\frac{\epsilon_{r,1}}{\kappa_1} + \frac{\epsilon_{r,2}}{\kappa_2} = 0$$

$$k_0^2 \mu_{r,1} \epsilon_{r,1} = \beta^2 - \kappa_1^2$$

$$k_0^2 \mu_{r,2} \epsilon_{r,2} = \beta^2 - \kappa_2^2$$

$$\beta^2 = k_0^2 \left( \frac{\epsilon_{r,1} \epsilon_{r,2}}{\epsilon_{r,2}^2 - \epsilon_{r,1}^2} \right) (\mu_{r,1} \epsilon_{r,2} - \mu_{r,2} \epsilon_{r,1})$$

Surface plasmon



# Applying Boundary Conditions

## Existence condition

$$\frac{\epsilon_{r,1}}{\kappa_1} + \frac{\epsilon_{r,2}}{\kappa_2} = 0$$



$$\epsilon_{r,1} = -\epsilon_{r,2} \frac{\kappa_1}{\kappa_2}$$

since  $\kappa_1 > 0, \kappa_2 > 0$

One of the two media must have a negative permittivity! => metal

## Dispersion relation

$$\beta^2 = k_0^2 \left( \frac{\epsilon_{r,1}\epsilon_{r,2}}{\epsilon_{r,2}^2 - \epsilon_{r,1}^2} \right) (\mu_{r,1}\epsilon_{r,2} - \mu_{r,2}\epsilon_{r,1})$$

assuming no magnetic response

$$\mu_{r,1} = \mu_{r,2} = 1$$



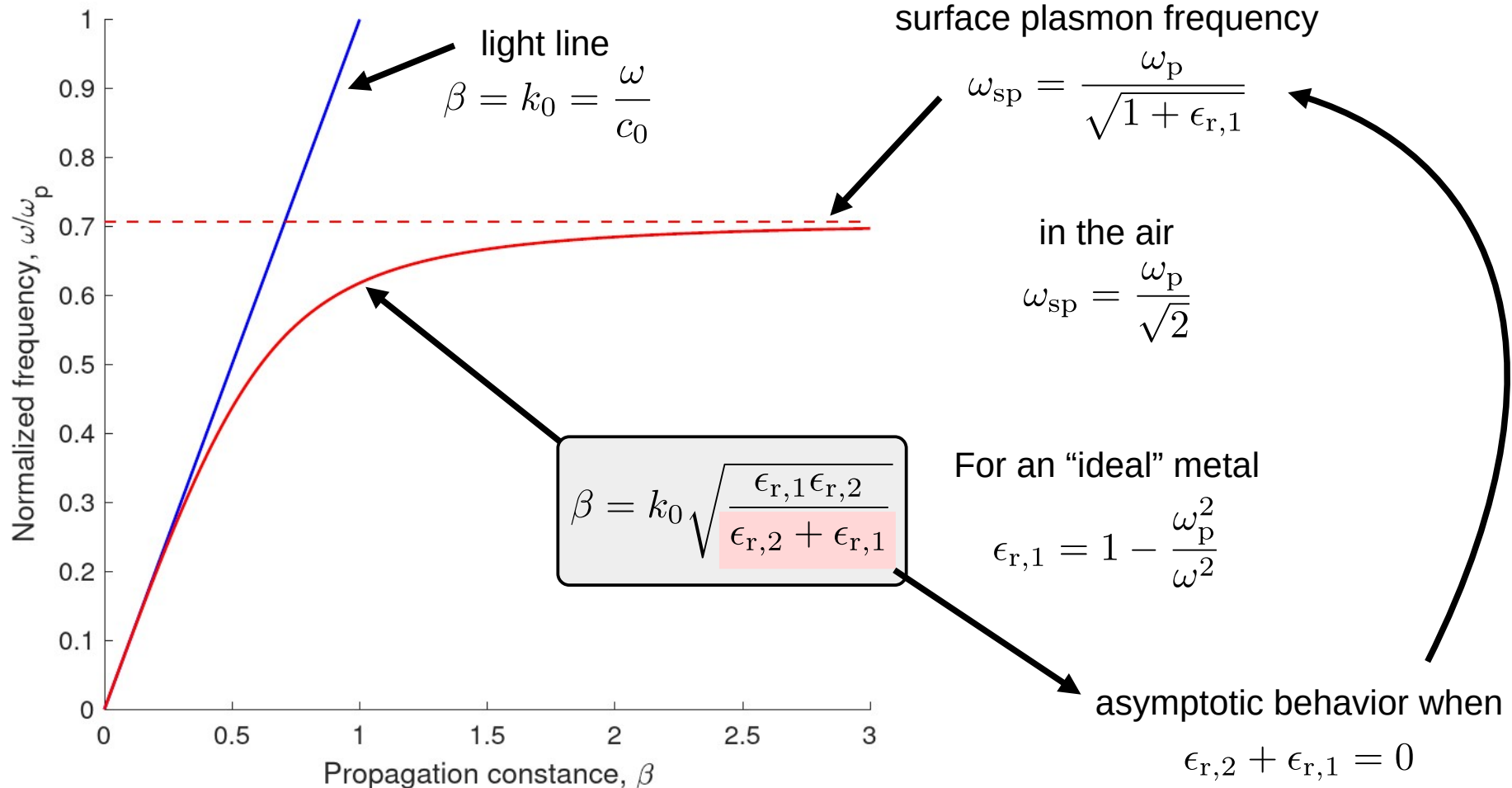
$$\beta = k_0 \sqrt{\frac{\epsilon_{r,1}\epsilon_{r,2}}{\epsilon_{r,2} + \epsilon_{r,1}}}$$

For an “ideal” metal

$$\epsilon_{r,1} = 1 - \frac{\omega_p^2}{\omega^2}$$

$\omega_p$  is the plasma frequency that depends on the properties of the metal

# Dispersion of a Surface Plasmon



# What Have We Learned So Far....

- A surface plasmon is a surface wave propagating at the interface between two media, one having a positive permittivity and the other having a negative permittivity
- A negative permittivity is typical for metals, so surface plasmon are generated on metals
- Since tangential electric fields must be zero on the surface of a metal, the only polarization that allows surface plasmon propagation is TM, whose electric field has a component normal to the metal
- By solving Maxwell equations, we can find the dispersion relation of the surface plasmon
- In a dispersion diagram ( $\omega$ - $k$ ), a plasmon is always below the light line. This means that a plasmon is stuck on the metal and cannot leak out from it. As a consequence, this means that the plasmon cannot be excited by a plane wave.
- The only way to excite a plasmon is by generating an evanescent field (e.g., using a prism and total internal reflection)
- The dispersion of a plasmon is extremely sensitive to a variation of the refractive index making it great for bio-sensing applications