Lecture 1

The Basics

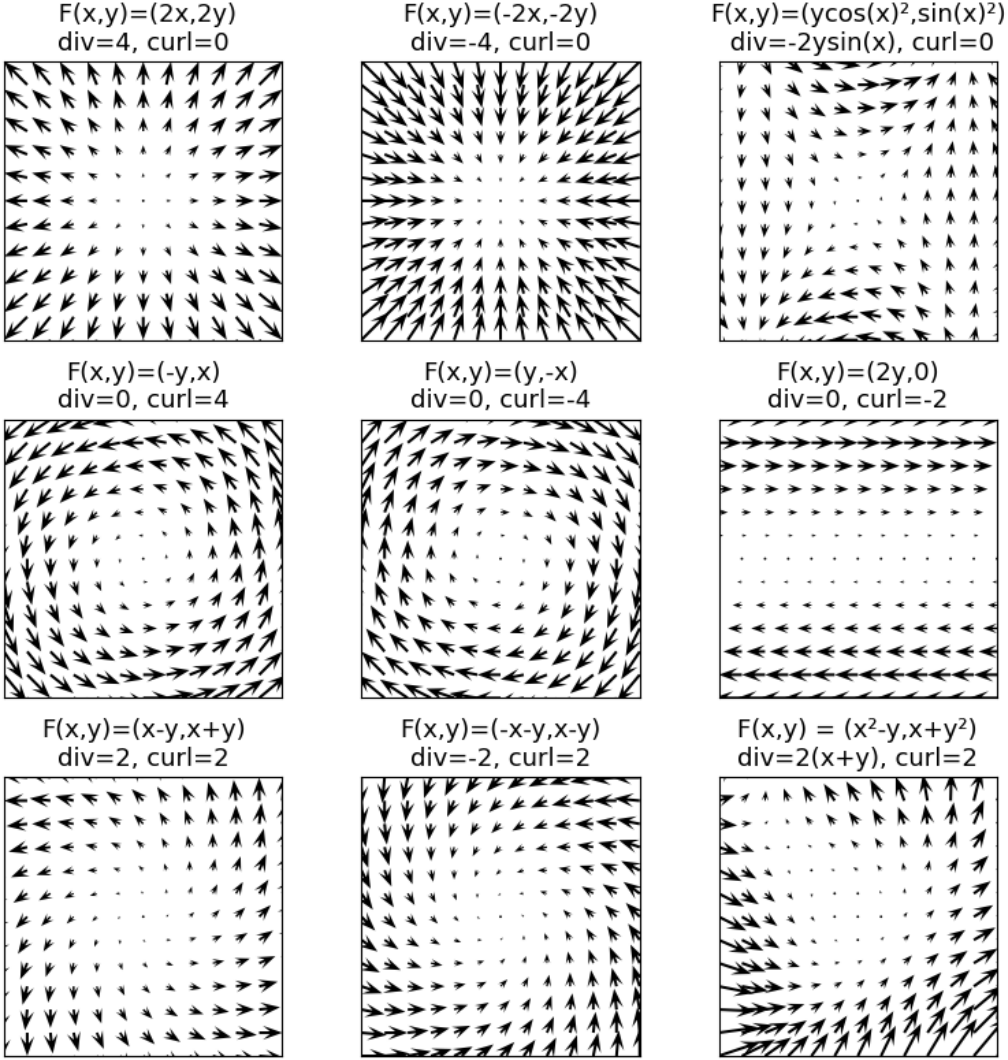
Divergence and Curl Operations

In Cartesian coordinates, we have

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \qquad \nabla = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix}$$

$$\nabla \times \mathbf{F} = \begin{bmatrix} \partial_y F_z - \partial_z F_y \\ \partial_x F_z - \partial_z F_x \\ \partial_x F_y - \partial_y F_x \end{bmatrix}$$

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$$



F(x,y)=(2x,2y)
div=4, curl=0

F(x,y)=(-2x,-2y)
div=-4, curl=0

F(x,y)=(ycos(x)^2,sin(x)^2)
div=-2ysin(x), curl=0

F(x,y)=(-y,x)
div=0, curl=4

F(x,y)=(y,-x)
div=0, curl=-4

F(x,y)=(2y,0)
div=0, curl=-2

F(x,y)=(x-y,x+y)
div=2, curl=2

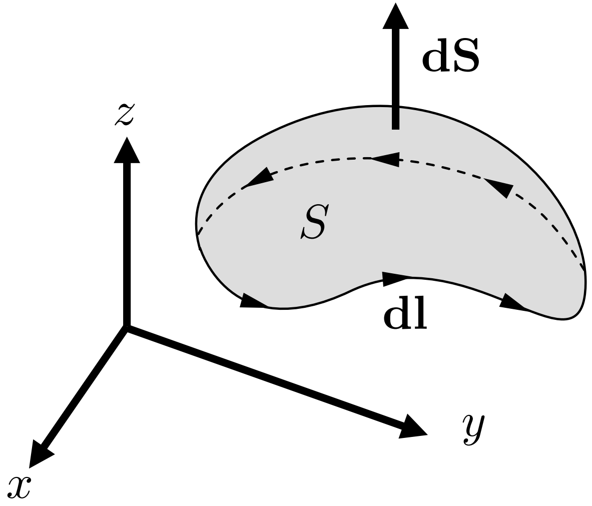
F(x,y)=(-x-y,x-y)
div=-2, curl=2

F(x,y) = (x^2-y,x+y^2)
div=2(x+y), curl=2

The Stokes and Divergence Theorems

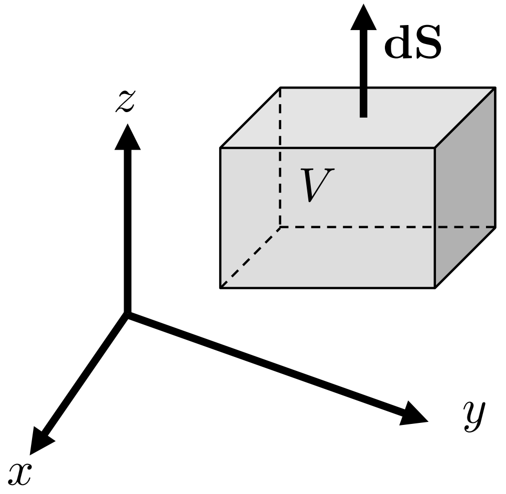
Stokes theorem

$$\iint (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = \oint \mathbf{F} \cdot \mathbf{dl}$$



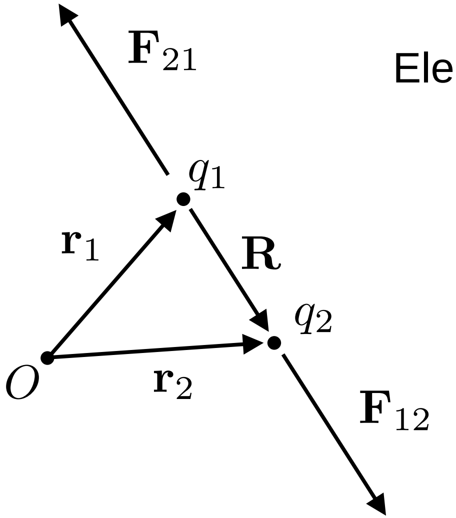
Divergence theorem

$$\iiint \nabla \cdot \mathbf{F}\; dV = \oiint \mathbf{F} \cdot \mathbf{dS}$$



Electrostatics

What is an Electric Field ?



Electric force between two charged particles
(Coulomb law)

$$\boxed{\mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{R}|^2}\hat{\mathbf{R}}}$$

where $\begin{cases} \mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1 \\ \hat{\mathbf{R}} = \dfrac{\mathbf{R}}{|\mathbf{R}|} \end{cases}$

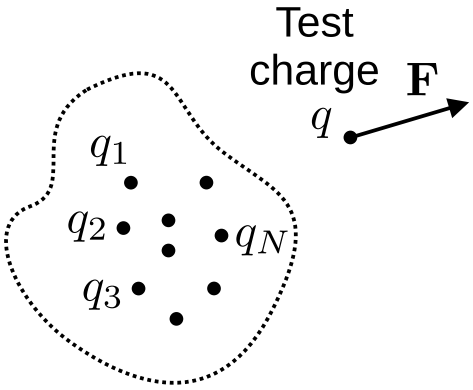
$$\mathbf{F}_{12} = \frac{q_1 q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\epsilon_0 |\mathbf{r}_2 - \mathbf{r}_1|^3}$$

$\begin{cases} \epsilon_0 = 8.854\ 187\ 8\ldots \times 10^{-12}\ \mathrm{F/m} \\ q \text{ is expressed in Coulomb [C]} \end{cases}$

Generalization to *N* charges

$$\boxed{\mathbf{F} = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N} q_k \frac{(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}} \longrightarrow \boxed{\mathbf{F} = q\mathbf{E}}$$

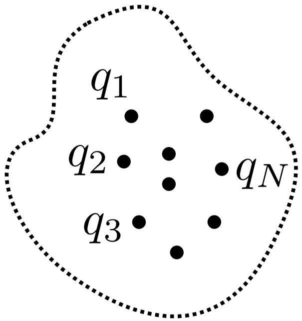
$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N} q_k \frac{(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}}$$



Collection of *N* charges

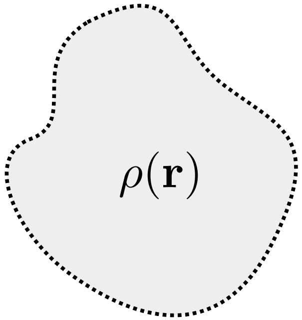
We consider that *q* is a “test charge” used to measure **F** everywhere in space

From Discrete to Continuous Distribution of Charges



Collection of *N* charges

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{N} q_k \frac{(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}$$



Continuous
distribution of charge

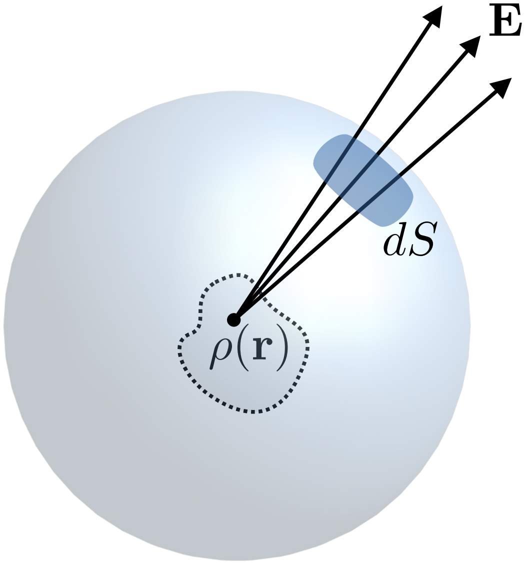
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{V'} \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \; dV'$$

The total charge is

ρ is expressed in [C/m³]

$$Q = \iiint_{V} \rho(\mathbf{r}) \; dV$$

Gauss Law



Electric flux through a closed surface

$$\boxed{\Phi_\mathrm{E} = \oiint_S \mathbf{E}(\mathbf{r}) \cdot \mathbf{n}\, dS}$$

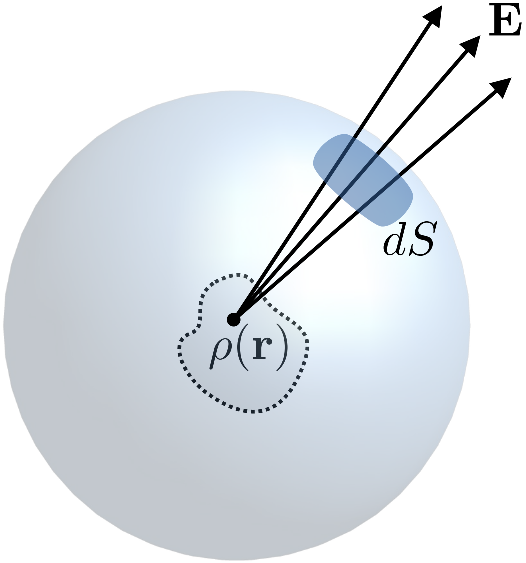
$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{V'} \rho(\mathbf{r}') \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dV'}$$

$$\Phi_\mathrm{E} = \frac{1}{4\pi\epsilon_0} \iiint_{V'} \rho(\mathbf{r}') \underbrace{\left[\oiint_S \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \cdot \mathbf{n}\, dS\right]} dV'$$

Solid angle: $\Omega = \oiint_S \frac{\mathbf{R}}{R^3} \cdot \mathbf{n}\, dS = 4\pi$

$$\Phi_\mathrm{E} = \frac{1}{\epsilon_0} \iiint_{V'} \rho(\mathbf{r}') dV' = \frac{Q}{\epsilon_0} \longrightarrow \boxed{\oiint_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0}}$$

Gauss Law



$$\boxed{\oiint_S \mathbf{E}\cdot\mathbf{dS} = \frac{Q}{\epsilon_0}}$$

alternatively $\oiint_S \mathbf{E}(\mathbf{r})\cdot\mathbf{dS} = \frac{1}{\epsilon_0}\iiint_V \rho(\mathbf{r})dV$

Using the divergence theorem

$$\iiint_V \nabla\cdot\mathbf{E}(\mathbf{r})\ dV = \frac{1}{\epsilon_0}\iiint_V \rho(\mathbf{r})dV$$

Note that this assumes that we are in vacuum $\boxed{\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}}$

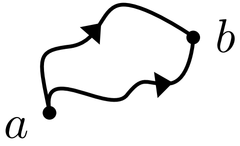
If the background material has a dielectric response, then $\boxed{\nabla\cdot\mathbf{D} = \rho}$ where $\mathbf{D} = \epsilon_0\mathbf{E}$

Scalar and Vector Potentials

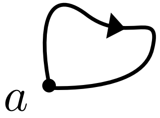
Scalar potential: An arbitrary vector field **F** is irrotational and conservative if

$\nabla \times \mathbf{F} = 0$ everywhere

$\int_c \mathbf{F} \cdot d\mathbf{l}$ is independent of the path



$\oint_c \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed path



$\mathbf{F} = -\nabla V$ where *V* is a scalar potential $\longrightarrow$ $\nabla \times (\nabla V) = 0$

In electrostatics, the electric field may be expressed as

$$\boxed{\mathbf{E} = -\nabla V}$$

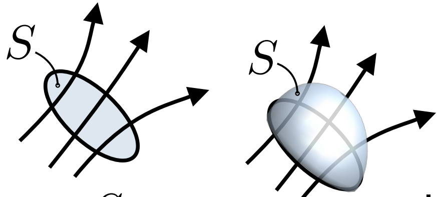
In general, we have

$$\boxed{\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}}$$

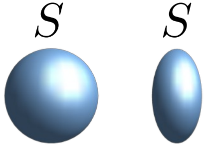
Vector potential: An arbitrary vector field **F** is solenoidal or divergence-less if

$\nabla \cdot \mathbf{F} = 0$ everywhere

$\iint_S \mathbf{F} \cdot d\mathbf{S}$ is independent of the surface for any boundary line



$\oint\!\!\oint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed surface



$\mathbf{F} = \nabla \times \mathbf{A}$ where **A** is a vector potential $\longrightarrow$ $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

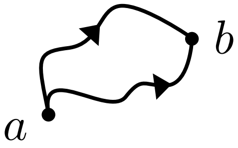
In magnetostatics, the magnetic field may be expressed as

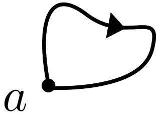
$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

The Scalar Potential

Scalar potential: An arbitrary vector field **F** is irrotational and conservative if

$$\begin{cases} \nabla \times \mathbf{F} = 0 & \text{everywhere} \\ \displaystyle\int_c \mathbf{F} \cdot d\mathbf{l} & \text{is independent of the path} \\ \displaystyle\oint_c \mathbf{F} \cdot d\mathbf{l} = 0 & \text{for any closed path} \\ \mathbf{F} = -\nabla V & \text{where } V \text{ is a scalar potential} \end{cases}$$

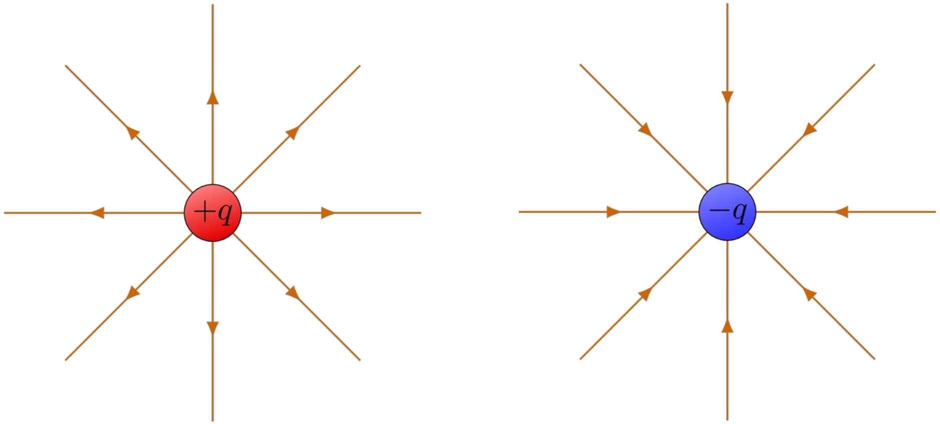




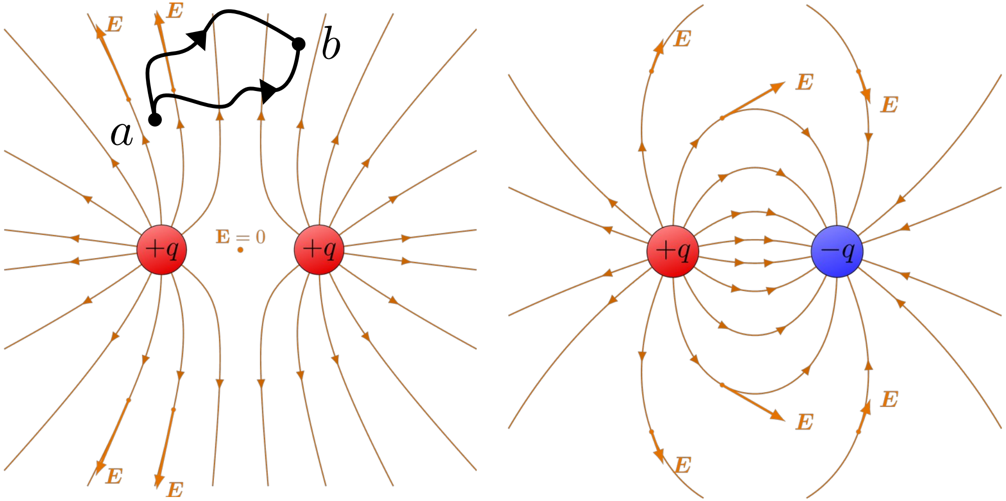
→ $\nabla \times (\nabla V) = 0$

In electrostatics, the electric field may be expressed as

$$\boxed{\mathbf{E} = -\nabla V}$$

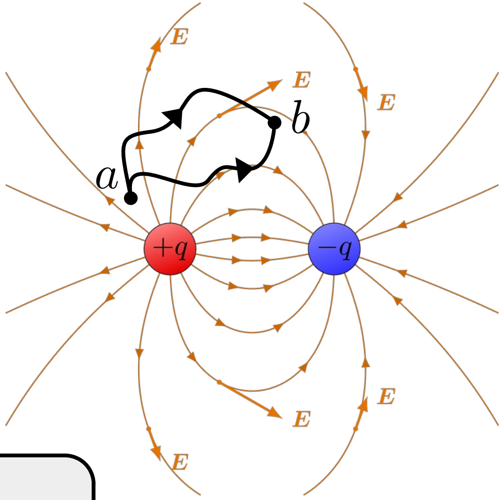


https://tikz.net/electric_fieldlines



https://tikz.net/electric_fieldlines2/

The Electric Potential



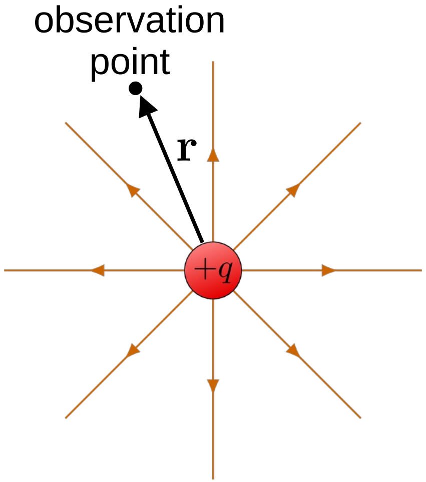
$$\boxed{\mathbf{E} = -\nabla V} \longrightarrow V(\mathbf{r}) = -\int_c \mathbf{E}\cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = V(\mathbf{b}) - V(\mathbf{a})$$

Gauss law

Poisson's equation

$$\boxed{\nabla\cdot\mathbf{D} = \rho} \longrightarrow \nabla\cdot\mathbf{E} = \nabla\cdot(-\nabla V) = \frac{\rho}{\epsilon} \longrightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon}}$$

General solution to Poisson's equation $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon}\iiint_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV'}$



Electric field of a point charge

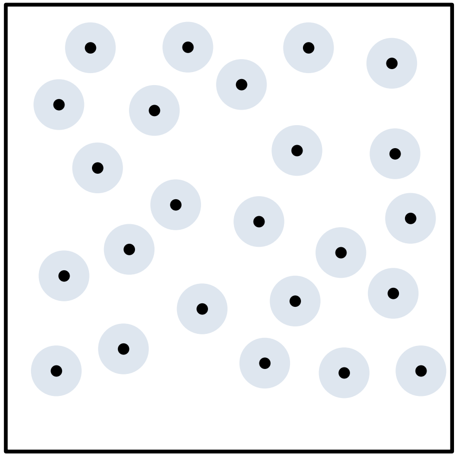
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon}\iiint_{V'} \rho(\mathbf{r}')\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dV' \xrightarrow{\rho(\mathbf{r}) = q\delta(\mathbf{r})} \boxed{\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3}}$$

Electric potential of a point charge

$$V(\mathbf{r}) = -\int_r^\infty \mathbf{E}\cdot d\mathbf{l} = \int_\infty^r \frac{q}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot d\mathbf{r} \longrightarrow \boxed{V(\mathbf{r}) = \frac{q}{4\pi\epsilon}\frac{1}{r}}$$

Simple Model of Dielectrics

Unpolarized medium



Total field in the medium

Free-space field

Polarization response of the medium

$$\mathbf{D} = \epsilon_0 \mathbf{E} + {\color{blue}\mathbf{P}}$$

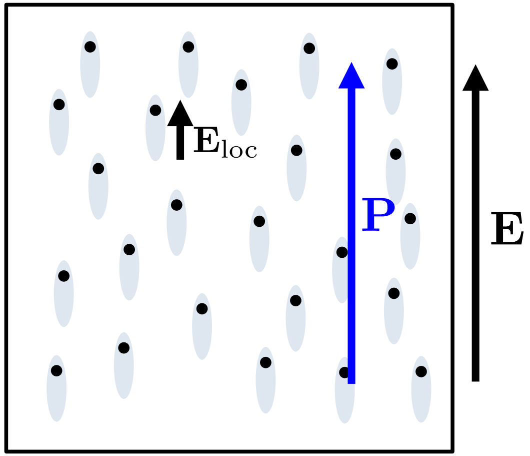
D and **P** are in [C/m^2]

we express the effective response of the medium using a susceptibility

$$\mathbf{P} = \epsilon_0 \chi_{\text{ee}} \mathbf{E}$$

where χ_{ee} is unitless

Polarized medium



$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_{\text{ee}} \mathbf{E}$$

$$\downarrow$$

$$\mathbf{D} = \epsilon_0 (1 + \chi_{\text{ee}}) \mathbf{E}$$

$$\downarrow$$

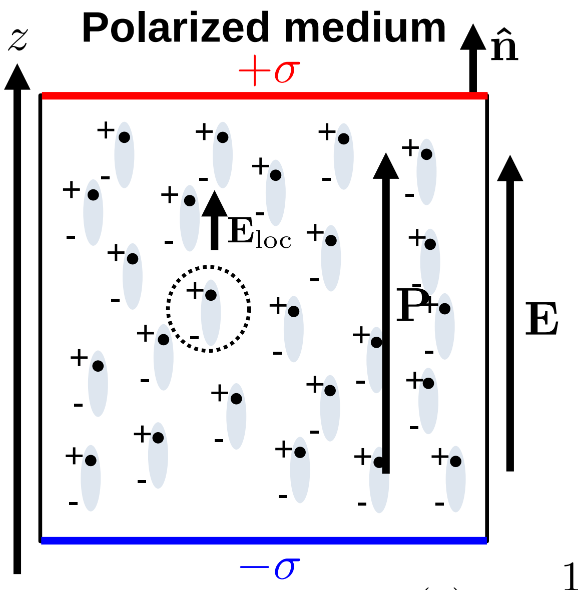
$$\mathbf{D} = \epsilon_0 \epsilon_{\text{r}} \mathbf{E}$$

$$\downarrow$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

The relative permittivity ϵ_r is unitless but ϵ_0 and ϵ are in [F/m]

Clausius-Mosotti Equation



When a medium is polarized, the polarization within the material cancels out leaving only the surface charge density

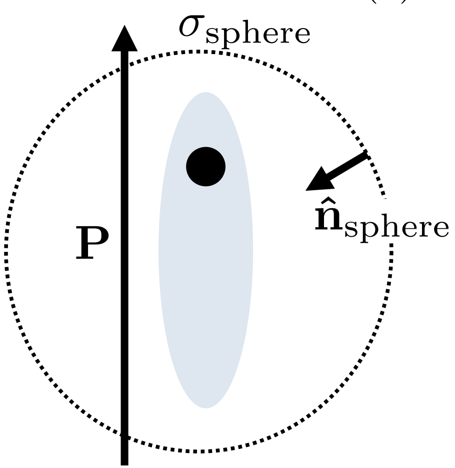
$$\sigma = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \text{where} \quad \mathbf{P} = P_0\hat{\mathbf{z}}$$

To find the local field **E$_{\mathrm{loc}}$** acting on a atom, we consider a small sphere around the atom and assume it is homogeneously polarized. This means that the sphere has a surface charge

$$\sigma_{\mathrm{sphere}} = \mathbf{P}\cdot\hat{\mathbf{n}}_{\mathrm{sphere}} = -P_0\cos\theta$$

The potential of the sphere is now (note that the sphere is in vacuum!)

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_{V'}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' = \frac{1}{4\pi\epsilon_0}\oiint_{S'}\frac{\sigma_{\mathrm{sphere}}}{|\mathbf{r}-\mathbf{r}'|}dS' = -\frac{1}{4\pi\epsilon_0}\oiint_{S'}\frac{P_0\cos\theta'}{|\mathbf{r}-\mathbf{r}'|}dS' = -\frac{P_0}{3\epsilon_0}z$$



The electric field inside the sphere is given by

$$\mathbf{E}_{\mathrm{sphere}} = -\nabla V = -\nabla\left(-\frac{P_0}{3\epsilon_0}z\right) = \frac{P_0}{3\epsilon_0}\hat{\mathbf{z}} = \frac{\mathbf{P}}{3\epsilon_0}$$

E$_{\mathrm{sphere}}$ represents the effect of all the atoms

$$\boxed{\mathbf{E}_{\mathrm{loc}} = \mathbf{E} + \mathbf{E}_{\mathrm{sphere}} = \mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0}}$$

The local field is the sum of the excitation **E** and the field produced by the sphere **E$_{\mathrm{sphere}}$**

Rothwell, “Electromagnetics”, 2018

The Electric Potential Energy

Potential energy

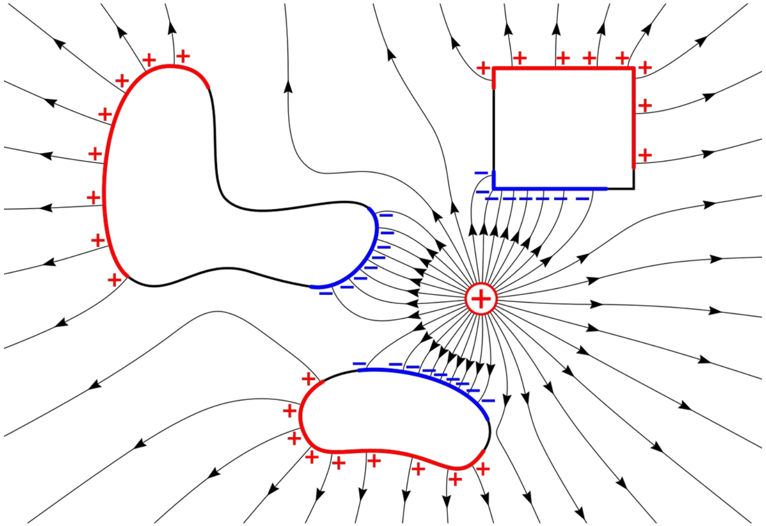
Force and potential energy

$$\boxed{U = qV} \longrightarrow \boxed{\mathbf{F} = -\nabla U} \longrightarrow U_{ab}(\mathbf{r}) = U(\mathbf{b}) - U(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$$

$$\boxed{\mathbf{F} = -q\nabla V}$$

$$\boxed{\mathbf{E} = -\nabla V} \longrightarrow \boxed{\mathbf{F} = q\mathbf{E}}$$

Electric charge near conductors



https://commons.wikimedia.org/wiki/File:Electrostatic_induction.svg#/media/File:Electrostatic_induction.svg

In electrostatics there is no variation in time

↓

Currents do not exist because charges do not move

↓

Inside a conductor, the force is zero

↓

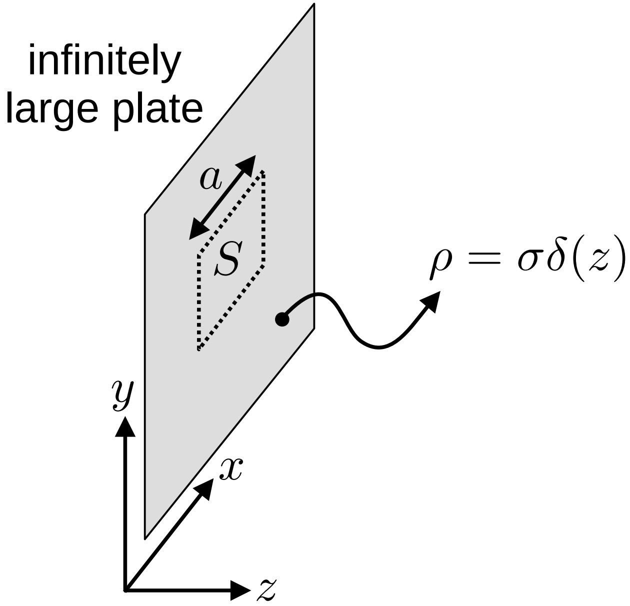
$$\boxed{\mathbf{E} = 0} \longrightarrow \boxed{V = \text{const}}$$

On the surface of a conductor, we must also have

$$\boxed{\mathbf{E}_{||} = 0}$$

So that the tangential force is zero on the conductor surface preventing the existence of surface currents

Example: Field from a Charged Plate



σ is a surface charge density in [C/m²]

$$\boxed{\nabla \cdot \mathbf{D} = \rho} \longrightarrow \oiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_V \rho dV = a^2\sigma$$

By symmetry of the problem, only the D_z component is non-zero

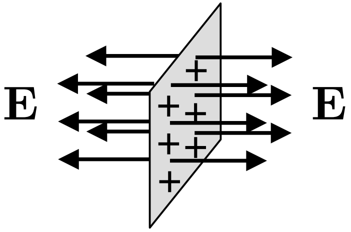
$$a^2\sigma = \oiint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^a \int_0^a D_z dxdy = 2a^2 D_z$$

$$\Big\downarrow$$

$$\mathbf{D} = \frac{\sigma}{2}\hat{\mathbf{z}} \longrightarrow \boxed{\mathbf{E} = \frac{\sigma}{2\epsilon}\hat{\mathbf{z}}}$$

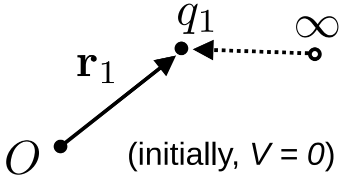
The corresponding electric potential is

$$V(z) = -\int_0^z \mathbf{E} \cdot d\mathbf{l} = \int_z^0 \frac{\sigma}{2\epsilon} dz' = -\frac{\sigma}{2\epsilon} z$$



Energy in Electrostatic Fields

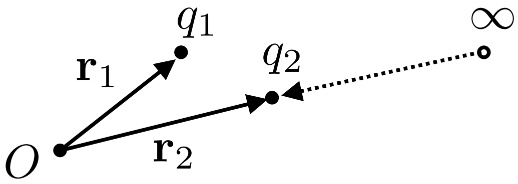
How much energy to bring q_1 from infinity to position $\mathbf{r}_1$?



(initially, $V = 0$)

$$U_1 = 0$$

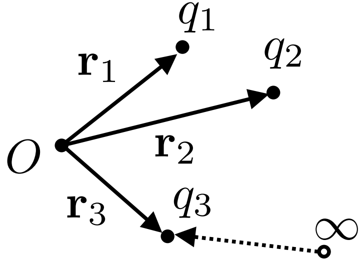
Now, we bring q_2 from infinity to position $\mathbf{r}_2$



$$U_2 = q_2 V_{21}$$

V_{21} is the potential difference between $\mathbf{r}_1$ and $\mathbf{r}_2$

Now we bring a third charge from infinity



$$U_3 = q_3 \left(V_{31} + V_{32} \right)$$

The total energy is $U = U_1 + U_2 + U_3 = q_2 V_{12} + q_3 \left(V_{31} + V_{32} \right)$

If the charges were brought together in the reverse order $U = U_3 + U_2 + U_1 = q_2 V_{23} + q_1 \left(V_{12} + V_{13} \right)$

Summing both equations and re-arranging the terms

$$2U = q_1 \left(V_{12} + V_{13} \right) + q_2 \left(V_{21} + V_{23} \right) + q_3 \left(V_{31} + V_{32} \right)$$

total potential at point $\mathbf{r}_1$

Potential energy

$$\boxed{U = qV}$$

$$U = \frac{1}{2} \sum_{i}^{N} q_i V_i \xrightarrow{\text{continuous charge}} \boxed{W_\mathrm{e} = \frac{1}{2} \iiint_{V'} \rho V \, dV'}$$

Energy in Electric Fields

$$\boxed{W_\mathrm{e} = \frac{1}{2}\iiint_{V'} \rho V\, dV'}$$

Gauss law

$$\boxed{\nabla \cdot \mathbf{D} = \rho}$$

$$\longrightarrow W_\mathrm{e} = \frac{1}{2}\iiint_{V'} (\nabla \cdot \mathbf{D}) V\ dV'$$

Mathematical identity

$$(\nabla \cdot \mathbf{D})V = \nabla \cdot (V\mathbf{D}) - \mathbf{D} \cdot \nabla V$$

$$\longrightarrow W_\mathrm{e} = \frac{1}{2}\iiint_{V'} \nabla \cdot (V\mathbf{D}) dV' - \frac{1}{2}\iiint_{V'} \mathbf{D} \cdot \nabla V\ dV'$$

Using the divergence theorem and $\boxed{\mathbf{E} = -\nabla V}$

$$W_\mathrm{e} = \frac{1}{2}\oiint_S (V\mathbf{D}) \cdot \mathbf{dS} + \frac{1}{2}\iiint_V \mathbf{D} \cdot \mathbf{E}\ dV$$

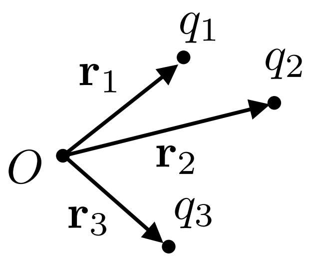
Typically V ∝ 1/r and D ∝ $1/r^2$, so by taking the surface of integration to infinity, the integral goes to zero !

$$\boxed{W_\mathrm{e} = \frac{1}{2}\iiint_V \mathbf{D} \cdot \mathbf{E}\ dV} \xrightarrow{\mathbf{D} = \epsilon \mathbf{E}} W_\mathrm{e} = \frac{\epsilon_0 \epsilon_\mathrm{r}}{2}\iiint_V |\mathbf{E}|^2\ dV$$

Notice the role of ϵ_0 !

Interpretation of Electric Energy

Electric energy stored in a system of electric charges



Two formulas to compute the same thing

$$W_\mathrm{e} = \frac{1}{2}\sum_i^N q_i V_i$$

finite value

$$W_\mathrm{e} = \frac{\epsilon_0}{2}\iiint_V |\mathbf{E}|^2 \; dV$$

$$W_\mathrm{e} \to \infty$$

The electric field of a charge diverges at *r = 0* !

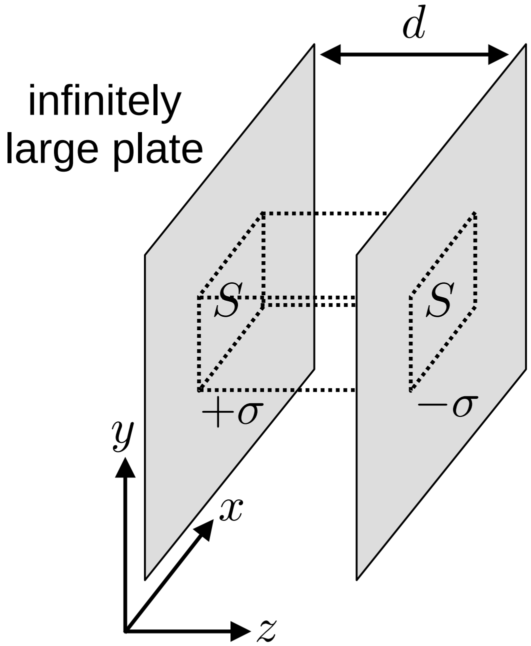
$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3}}$$

$$\begin{aligned} |\mathbf{E}|^2 &= |\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3|^2 \\ &= |\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 + |\mathbf{E}_3|^2 + \underbrace{\mathbf{E}_1 \cdot \mathbf{E}_2 + \mathbf{E}_1 \cdot \mathbf{E}_3 + \mathbf{E}_2 \cdot \mathbf{E}_3} \end{aligned}$$

Fields of the charges themselves that are indeed infinite because it would take an infinite amount of energy to “create” point sources

Leads to a finite amount of energy

Parallel Plate Capacitor

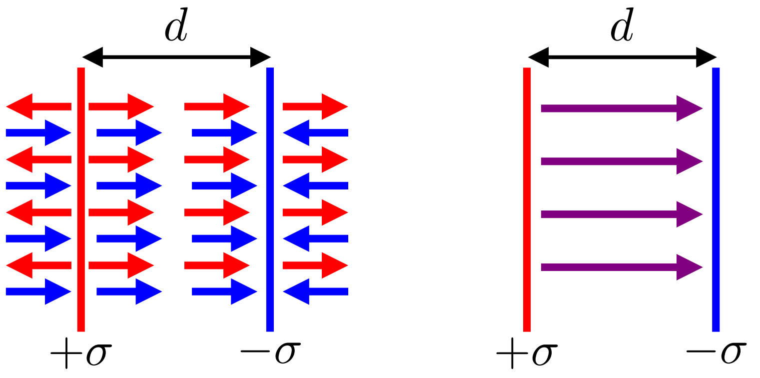


The field of a single plate is

$$\boxed{\mathbf{E} = \frac{\sigma}{2\epsilon}\hat{\mathbf{z}}}$$

$$\downarrow$$

$$\boxed{\mathbf{E} = \frac{\sigma}{\epsilon}\hat{\mathbf{z}}} \longleftarrow$$



The field outside the capacitor is zero and it is continuous and constant inside

The potential difference between the plates is

$$V = -\int_d^0 \mathbf{E}\cdot d\mathbf{l} = \int_0^d \frac{\sigma}{\epsilon}\,dz = \frac{\sigma}{\epsilon}d \quad\longrightarrow\quad \boxed{E = \frac{V}{d}}$$

where $E = |\mathbf{E}|$

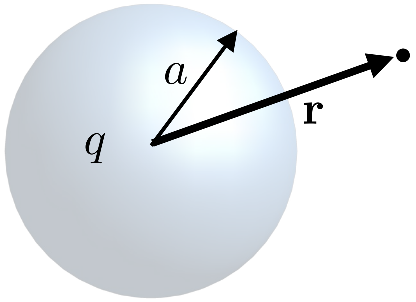
The stored energy is

$$W_\mathrm{e} = \frac{\epsilon}{2}\iiint_V |\mathbf{E}|^2\,dV = \frac{\epsilon}{2}SdE^2 = \frac{1}{2}\epsilon\frac{S}{d}V^2 = \frac{1}{2}CV^2 \longrightarrow \boxed{C = \epsilon\frac{S}{d}}$$

where C is the capacitance in [F] and S is an area on the plates

Capacitance of a Charged Sphere

Conductive sphere with charge *q* and radius *a*



$$\begin{cases} \mathbf{E}(\mathbf{r}) = \dfrac{q}{4\pi\epsilon r^2}\hat{\mathbf{r}} \\ V(\mathbf{r}) = \dfrac{q}{4\pi\epsilon r} \end{cases}$$

for $r > a$

The energy stored in the field outside the sphere is

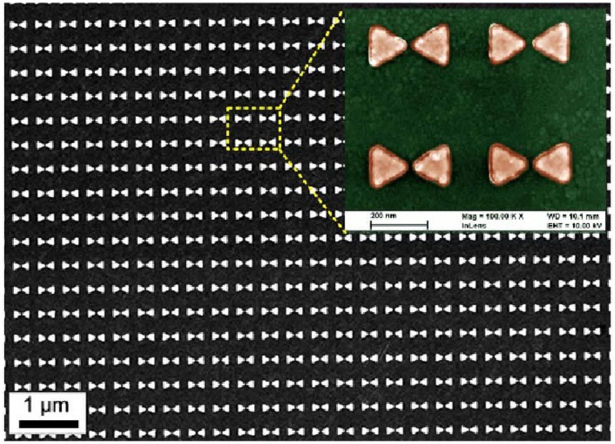
$$W_\mathrm{e} = \frac{\epsilon}{2}\iiint_V |\mathbf{E}|^2 \, dV = \frac{\epsilon}{2}\iiint_V \left(\frac{q}{4\pi\epsilon r^2}\right)^2 \, dV = \frac{q^2}{32\pi^2\epsilon}\int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\frac{1}{r^4}\, r^2 \sin\theta dr d\theta d\phi = \frac{q^2}{8\pi\epsilon a}$$

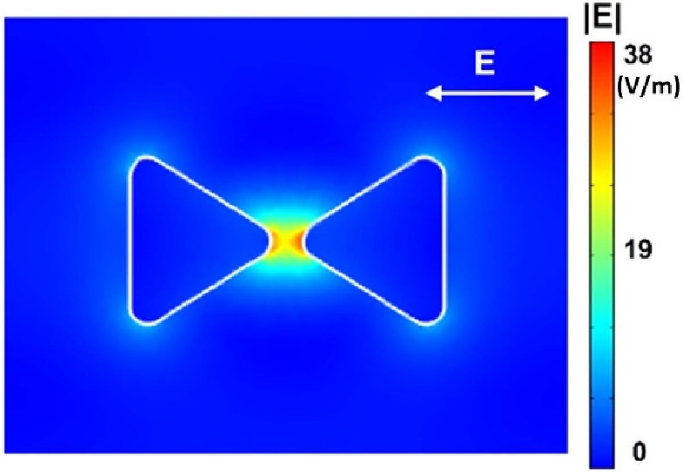
The capacitance of the sphere is found using $C = \dfrac{2W_\mathrm{e}}{V^2} = 4\pi\epsilon a$ or simply as $C = \dfrac{q}{V} = 4\pi\epsilon a$

$$\boxed{C = 4\pi\epsilon a}$$

Why is the Electric Field Strong at Sharp Edges ?

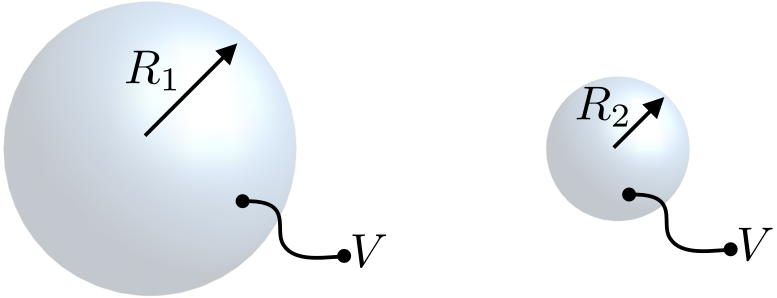
Excitation of gold bow tie antennas





https://doi.org/10.1038/srep18567

Consider two isolated spheres with the same potential *V*



$$C_1 = 4\pi\epsilon R_1 \qquad C_2 = 4\pi\epsilon R_2$$

The total charge is given by $Q = CV$

Since they are at the same potential $\dfrac{Q_1}{C_1} = V = \dfrac{Q_2}{C_2}$

$$\boxed{\frac{R_2}{R_1} = \frac{Q_2}{Q_1}}$$

Bigger sphere holds more charge

surface charge

$$\sigma = \frac{Q}{4\pi R^2}$$

$$\boxed{\frac{\sigma_2}{\sigma_1} = \frac{R_1}{R_2}}$$

Smaller sphere has a higher surface charge => stronger field

Why Do We Want Strong Fields ?

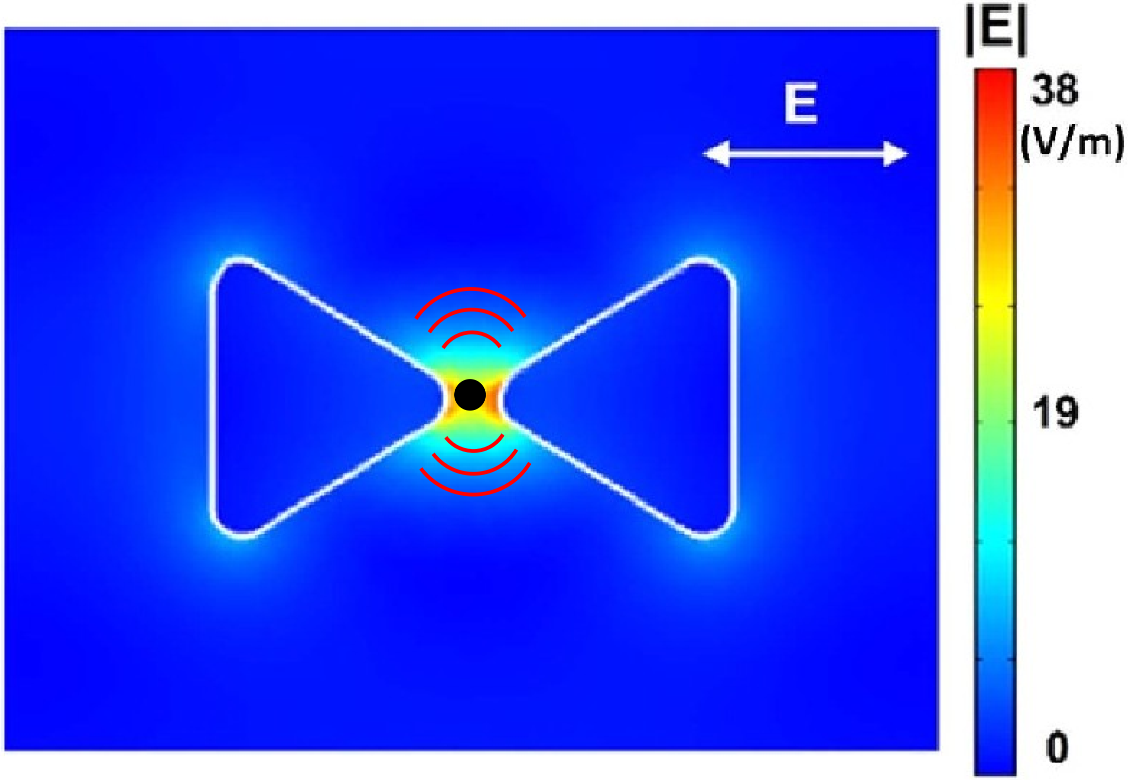
Strong field means lots of electric energy => increases particle interactions

$$\boxed{W_\mathrm{e} = \frac{\epsilon_0 \epsilon_\mathrm{r}}{2} \iiint_V |\mathbf{E}|^2 \, dV}$$

Force is the gradient of the energy

$$\boxed{\mathbf{F} = \nabla W}$$

Particles will feel a force !



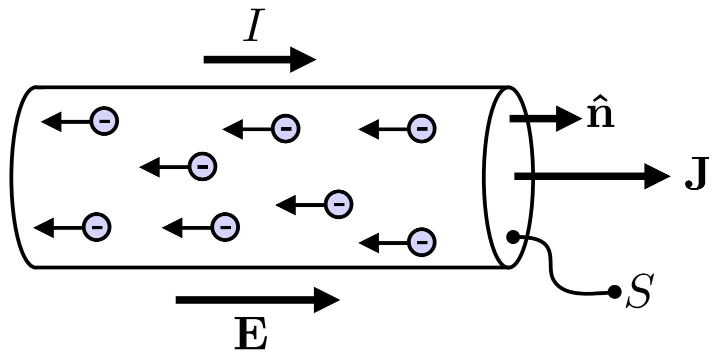
https://doi.org/10.1038/srep18567

What Have We Learned So Far….

- The concept of fields comes from the desire to explain “forces acting at a distance”
- A static electric field is irrotational and conservative and may be expressed as the gradient of a scalar potential.
- A static magnetic field is divergence-less or solenoidal and may be expressed as the curl of a vector potential
- The electric field of a charge decreases as $1/r^2$. It is in fact simply the ratio of the charge by the surface area of a sphere since the field lines spread in 3D.
- The electric energy in a volume is proportional to $\mathbf{D} \cdot \mathbf{E}$. The permittivity increases the stored energy.
- The vacuum permittivity may be seen as a mean to convert electric field intensity in a volume into electric energy
- The electric field of a charged plate is independent of the position from the plate. The field lines do not spread in space in contrast to the case of a charged sphere
- The capacitance is a measure of the ability of a system to store electric energy
- The capacitance of a parallel plate capacitor is $C = \epsilon S/d$
- Sharp conductive edges have higher surface charge density than smooth ones leading to stronger fields
- Strong fields are interesting because they increase interactions with nanoparticles (useful for sensing)
- Strong field gradients lead to optical force (useful for trapping particles)

Magnetostatics

What is a Current ?



The current is a variation
of charges in time

$$I = \frac{\partial Q}{\partial t} = \iint_S \mathbf{J} \cdot \hat{\mathbf{n}} dS$$

The units are

- *I* is in [A]
- *Q* is in [C]
- **J** is in [A/m^2]

Convection currents: flow of charges in an insulator $\mathbf{J} = \rho_v \mathbf{u}$

velocity in [m/s]

volume charge density in [C/m^3]

Conduction currents: flow of charges in a conductor

Charges move due to electric force $\mathbf{F} = q\mathbf{E} \xrightarrow{\text{for electron}} \mathbf{F} = -e\mathbf{E}$

$$\mathbf{F} = \frac{m\mathbf{u}}{\tau} = -e\mathbf{E} \longrightarrow$$ the drift velocity of the electrons is $\mathbf{u} = -\frac{e\tau}{m}\mathbf{E}$

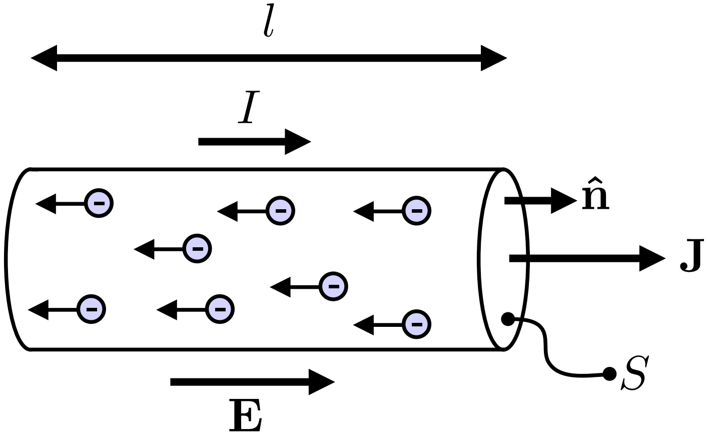
mean
collision time

$\rho_v = -ne$ where *n* is the density of electrons

$$\boxed{\mathbf{J} = \rho_v \mathbf{u} = \frac{ne^2\tau}{m}\mathbf{E} = \sigma \mathbf{E}}$$

the conductivity σ is in [1/(Ωm)]

Ohm Law



Let’s model this piece of wire of length *l* as a parallel plate capacitor. We have that

$$E = \frac{V}{l}$$

Assuming a uniform current within the wire

$$\frac{I}{S} = J = \sigma E$$

where σ is in [1/(Ωm)]

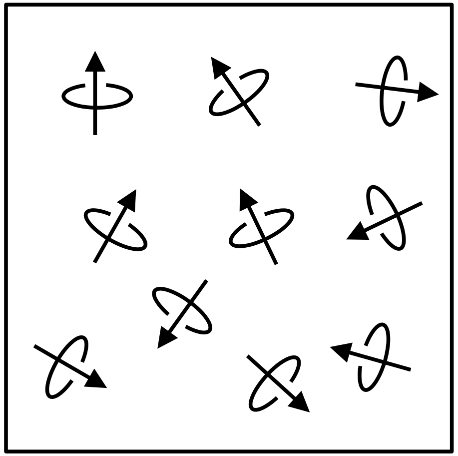
Combining both, we get

$$\sigma \frac{V}{l} = \frac{I}{S} \longrightarrow V = \frac{l}{\sigma S} I \longrightarrow \boxed{V = RI}$$

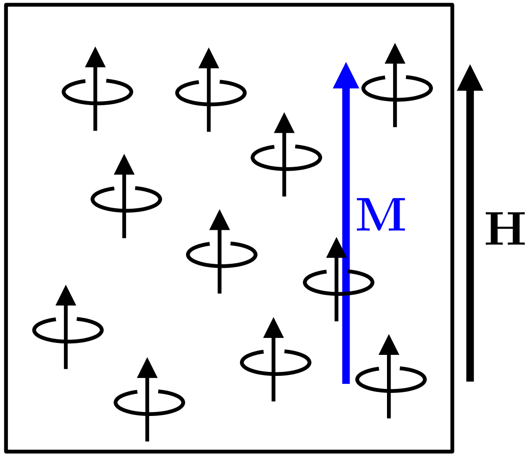
where the resistance is $$\boxed{R = \frac{l}{\sigma S}}$$

Simple Model of Magnetic Materials

Unmagnetized medium



Magnetized medium



Total field in the medium

Free-space field

Magnetization response of the medium

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right)$$

B is in $[\mathrm{Vs/m^2}]$, **H** and **M** are in [A/m]

we express the effective response of the medium using a susceptibility

$$\mathbf{M} = \chi_{\mathrm{mm}} \mathbf{H}$$

where χ_{mm} is unitless

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \chi_{\mathrm{mm}} \mathbf{H} \right)$$

$$\downarrow$$

$$\mathbf{B} = \mu_0 (1 + \chi_{\mathrm{mm}}) \mathbf{H}$$

$$\downarrow$$

$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}} \mathbf{H}$$

$$\downarrow$$

$$\mathbf{B} = \mu \mathbf{H}$$

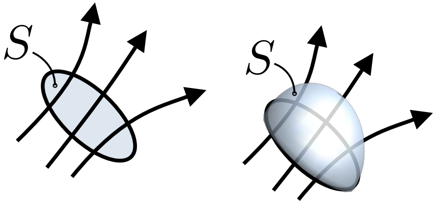
The relative permeability μ_r is unitless but μ_0 and μ are in [H/m]

Main Concept of Magnetostatics

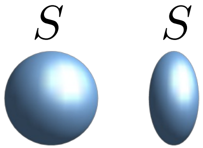
Vector potential: An arbitrary vector field **F** is solenoidal or divergence-less if

$\nabla \cdot \mathbf{F} = 0$ everywhere

$\iint_S \mathbf{F} \cdot d\mathbf{S}$ is independent of the surface for any boundary line



$\oint\!\!\!\oint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed surface



$\mathbf{F} = \nabla \times \mathbf{A}$ where **A** is a vector potential $\longrightarrow \nabla \cdot (\nabla \times \mathbf{A}) = 0$

In magnetostatics, the magnetic field may be expressed as

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

Ampère circuit law

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

Magnetic fields exist around currents

Magnetic Gauss law

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

In the absence of magnetic charges, magnetic fields can only form loops

Constitutive relation

$$\boxed{\mathbf{B} = \mu\mathbf{H}}$$

Magnetic energy

$$\boxed{W_\mathrm{m} = \frac{1}{2} \iiint_V \mathbf{B} \cdot \mathbf{H}\, dV = \frac{\mu}{2} \iiint_V |\mathbf{H}|^2\, dV}$$

Static Vector Potential

Ampère circuit law

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

Constitutive relation

$$\boxed{\mathbf{B} = \mu \mathbf{H}}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

Taking the curl and using Ampère law

$$\nabla \times \mathbf{H} = \frac{1}{\mu} \nabla \times \nabla \times \mathbf{A}$$

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J}$$

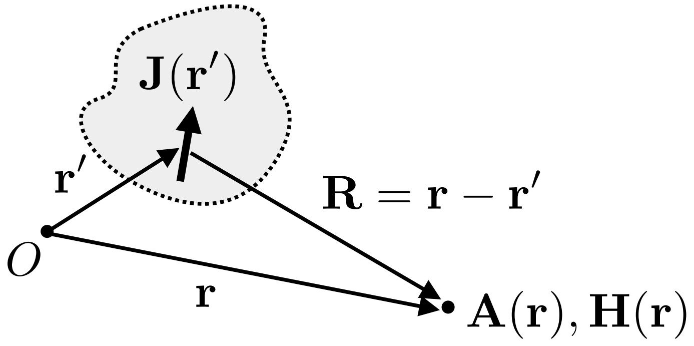
$$(\nabla \cdot \mathbf{A})\nabla - \nabla^2 \mathbf{A} = \mu \mathbf{J}$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

We are free to define **A** such that its divergence is 0 since we know that

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

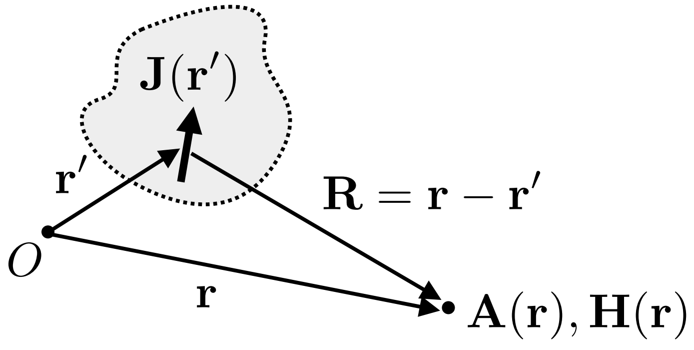
Continuous distribution of current



General solution

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{1}{R} \, dV'}$$

The Biot-Savart Law



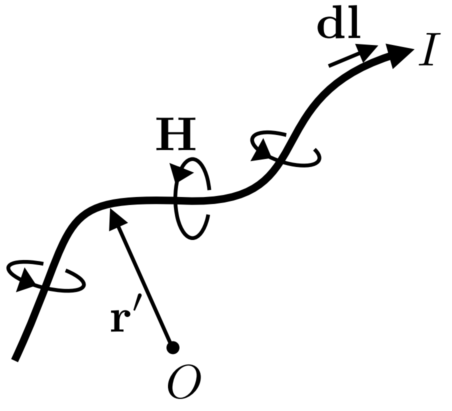
General solution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\frac{1}{R}\ dV'$$

The magnetic field is directly obtained using

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}$$

$$\mathbf{H} = \frac{1}{4\pi}\int_V \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{R}}{R^3}\ dV'$$



For a line current

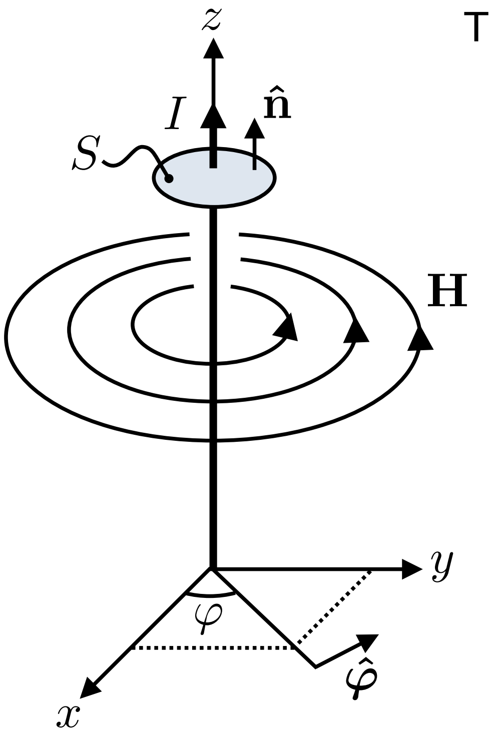
$$\mathbf{J}(\mathbf{r}') = I\delta(\mathbf{r}')\mathbf{dl}$$

$$\mathbf{H} = \frac{I}{4\pi}\int_{L'} \mathbf{dl}' \times \frac{\mathbf{R}}{R^3}$$

where we have used the identity

$$\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{R}\right) = \frac{1}{R}\nabla \times \mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r}') \times \nabla\frac{1}{R} = \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{R}}{R^3}$$

Example: Infinite Current Line



The current density is

$$\mathbf{J} = I\delta(x)\delta(y)\hat{\mathbf{z}} \longrightarrow I = \iint_S \mathbf{J}\cdot\hat{\mathbf{n}}dS$$

Ampère circuit law

$$\boxed{\nabla\times\mathbf{H} = \mathbf{J}}$$

Integrating over the surface *S*

$$\iint_S (\nabla\times\mathbf{H})\cdot\hat{\mathbf{n}}dS = \iint_S \mathbf{J}\cdot\hat{\mathbf{n}}dS$$

Using Stokes theorem

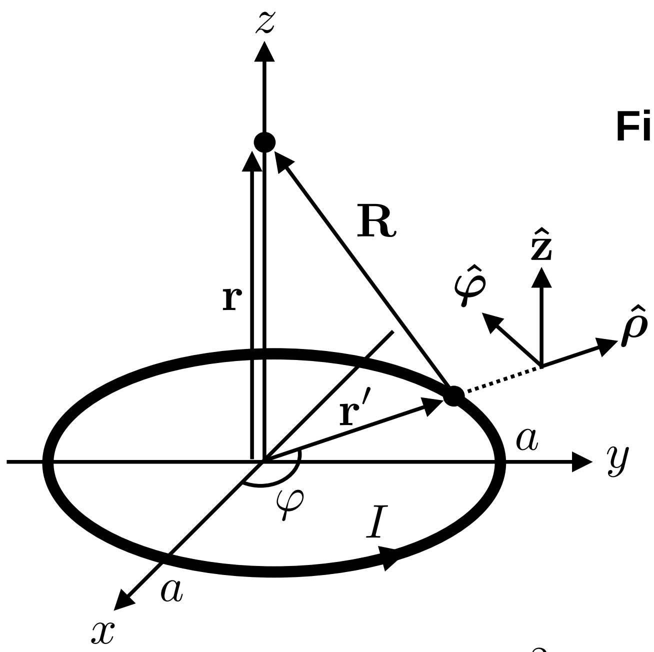
$$\oint_c \mathbf{H}\cdot d\mathbf{l} = I$$

By the symmetry of the problem

$$\boxed{\mathbf{H}(\mathbf{r}) = \frac{I}{2\pi r}\hat{\boldsymbol{\varphi}}}$$

Example: Current Loop

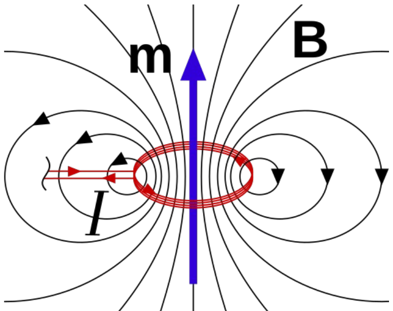
Find the magnetic field along the z-axis



$$\boxed{\mathbf{H} = \frac{I}{4\pi}\int_{L'} d\mathbf{l}' \times \frac{\mathbf{R}}{R^3}} \quad \left\{ \begin{aligned} \mathbf{r} &= z\hat{\mathbf{z}} \\ \mathbf{r}' &= a\hat{\boldsymbol{\rho}} \\ \mathbf{R} &= z\hat{\mathbf{z}} - a\hat{\boldsymbol{\rho}} \\ d\mathbf{l}' &= a\,d\varphi\,\hat{\boldsymbol{\varphi}} \end{aligned} \right.$$

$$\Big\downarrow \; R = |\mathbf{R}| = \sqrt{\mathbf{R}\cdot\mathbf{R}}$$

$$\mathbf{H}(z) = \frac{Ia}{4\pi}\int_0^{2\pi} \hat{\boldsymbol{\varphi}} \times \frac{z\hat{\mathbf{z}} - a\hat{\boldsymbol{\rho}}}{\left(a^2+z^2\right)^{3/2}}\, d\varphi = \frac{Ia}{4\pi\left(a^2+z^2\right)^{3/2}}\int_0^{2\pi} \left(\overset{\nearrow\,0}{z\hat{\boldsymbol{\rho}}} + a\hat{\boldsymbol{z}}\right)\, d\varphi$$



$$\boxed{\mathbf{H}(z) = \frac{Ia^2}{2\left(a^2+z^2\right)^{3/2}}\hat{\boldsymbol{z}}} \xrightarrow[z=0]{\text{In the center of the loop}} \boxed{\mathbf{H}(0) = \frac{I}{2a}\hat{\boldsymbol{z}}}$$

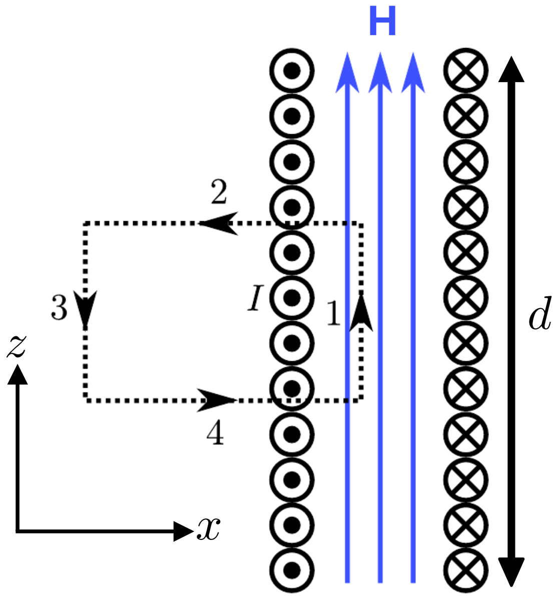
if $z \gg a$

$$\boxed{\mathbf{H}(z) \approx \frac{a^2 I}{2z^3}\hat{\boldsymbol{z}}}$$

https://en.m.wikipedia.org/wiki/File:Magnetic_field_due_to_current.svg

Example: Solenoid

Coil with current *I* and *N* turns



Ampère circuit law

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

By symmetry, the contributions from sides 2 and 4 cancel out. By taking side 3 to infinity, the field is zero.

$$\int_1 \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{S} \longleftarrow \mathbf{J} = \frac{NI}{d}\delta(x)$$

variation of current per unit length

$$\boxed{\mathbf{H} = \frac{NI}{d}\hat{\mathbf{z}}}$$

Magnetic energy

$$\boxed{W_\mathrm{m} = \frac{\mu}{2}\iiint_V |\mathbf{H}|^2 \, dV} \longrightarrow W_\mathrm{m} = \frac{\mu}{2}Sd|\mathbf{H}|^2 = \frac{1}{2}\mu\frac{SN^2}{d}I^2 = \frac{1}{2}LI^2 \longrightarrow$$

Coil inductance

$$\boxed{L = \mu\frac{S}{d}N^2}$$

S is the surface area of the coil

What Have We Learned So Far….

- The magnetic energy in a volume is proportional to **B·H**. The permeability increases the stored energy.
- The vacuum permeability may be seen as a mean to convert magnetic field intensity in a volume into magnetic energy
- If the distribution of current density **J** inside a volume is known, we can integrate it to find the vector potential **A**, which allows us to find the magnetic field everywhere in space
- The inductance is a measure of the ability of a system to store magnetic energy
- The magnetic field of a current line is simply the current divided by the perimeter of a circle since the field lines spread over a circle (cylindrical symmetry)
- The magnetic field of a current loop forms an equivalent magnetic dipole (like a magnet)
- Remember the formula for the inductance of a coil

Maxwell Equations

Summary from Statics

Electrostatics

Ampère circuit law

$$\boxed{\nabla \times \mathbf{E} = 0}$$

Gauss law

$$\boxed{\nabla \cdot \mathbf{D} = \rho}$$

Constitutive relation

$$\boxed{\mathbf{D} = \epsilon \mathbf{E}}$$

Electric energy

$$\boxed{W_{\mathrm{e}} = \frac{1}{2} \iiint_V \mathbf{D} \cdot \mathbf{E} \, dV}$$

Magnetostatics

Ampère circuit law

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

Magnetic Gauss law

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

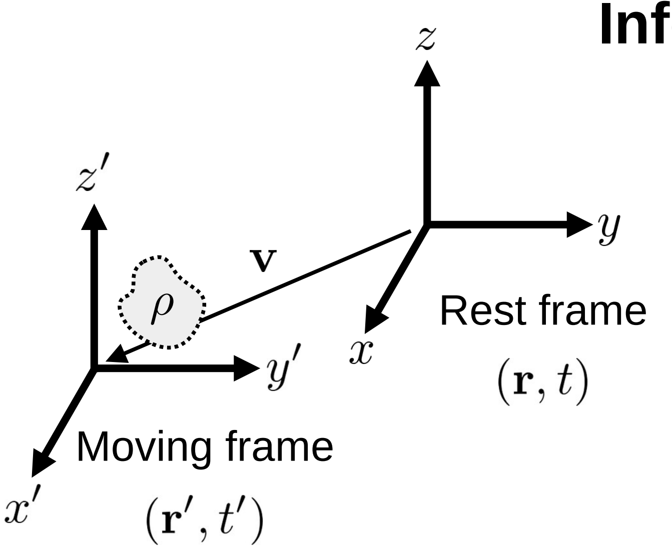
Constitutive relation

$$\boxed{\mathbf{B} = \mu \mathbf{H}}$$

Magnetic energy

$$\boxed{W_{\mathrm{m}} = \frac{1}{2} \iiint_V \mathbf{B} \cdot \mathbf{H} \, dV}$$

Inferring Maxwell Equations



A charge distribution is moving with velocity **v** such that

$$|\mathbf{v}| \ll c$$

Gallilean transform

$$\begin{cases} t = t' \\ \mathbf{r}' = \mathbf{r} - \mathbf{v}t \end{cases} \longrightarrow \mathbf{r} = \mathbf{r}' + \mathbf{v}t$$

Electric field in the rest frame

$$\mathbf{E}(\mathbf{r}, t)$$

$$\Big\downarrow \text{ Change of variables}$$

$$\mathbf{E}(\mathbf{r}' + \mathbf{v}t', t')$$

What is the time derivative of **E** ?

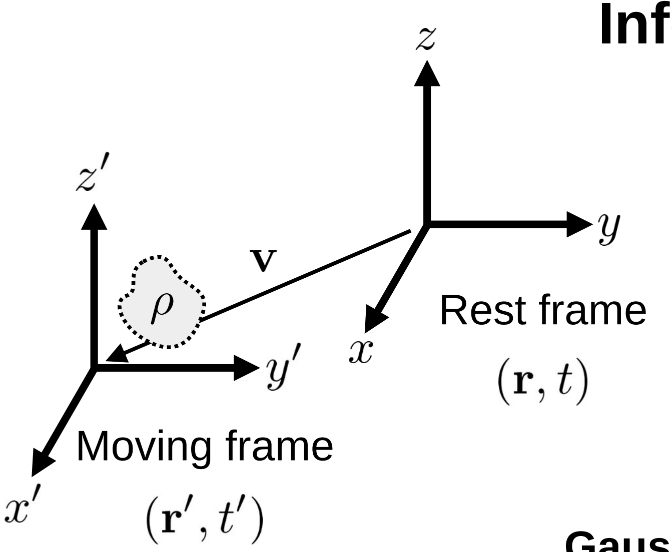
$$\frac{d}{dt'}\mathbf{E}(\mathbf{r}' + \mathbf{t}', t') = \frac{\partial \mathbf{E}}{\partial t}\frac{dt}{dt'} + \sum_{i=1}^{3}\frac{\partial \mathbf{E}}{\partial r_i}\frac{dr_i}{dt} \longleftarrow \frac{dt}{dt'} = 1$$

$$\frac{d}{dt'}\mathbf{E}(\mathbf{r}' + \mathbf{t}', t') = \frac{\partial \mathbf{E}}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \nabla \mathbf{E} \longleftarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}$$

$$\frac{d}{dt'}\mathbf{E}(\mathbf{r}' + \mathbf{t}', t') = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{E}$$

Julian Schwinger, et al. “Classical electrodynamics”, 1998.

Inferring Maxwell Equations



A charge distribution is moving with velocity **v** such that

$$|\mathbf{v}| \ll c$$

$$\underbrace{\frac{d}{dt'}\mathbf{E}(\mathbf{r}' + \mathbf{t}', t')}_{=0} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{E}$$

The time variation of **E** w.r.t the moving frame variables is zero

$$\nabla \times (\mathbf{v} \times \mathbf{E}) = \mathbf{v}(\nabla\cdot\mathbf{E}) - \underbrace{\mathbf{E}(\nabla\cdot\mathbf{v}) + \mathbf{E}\cdot\nabla\mathbf{v}}_{=0} - \mathbf{v}\cdot\nabla\mathbf{E}$$

Gauss law

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

The velocity vector is constant with position

$$\nabla \times (\epsilon_0 \mathbf{v} \times \mathbf{E}) = \mathbf{v}\rho + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} \\ \mathbf{J} = \mathbf{v}\rho \end{cases}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

Vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B}$$

Maxwell Equations in the Time Domain

Differential form

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mathbf{K} - \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_\mathrm{e}$$

$$\nabla \cdot \mathbf{B} = \rho_\mathrm{m}$$

Integral form

$$\oint_c \mathbf{H} \cdot \mathbf{dl} = \iint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$$

$$\oint_c \mathbf{E} \cdot \mathbf{dl} = -\iint_S \left(\mathbf{K} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S}$$

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_V \rho_\mathrm{e} dV = Q_\mathrm{e}$$

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = \iiint_V \rho_\mathrm{m} dV = Q_\mathrm{m}$$

Units

E is in [V/m]
H is in [A/m]
D is in [C/m^2]
B is in [Vs/m^2]
J is in [A/m^2]
K is in [V/m^2]
ρ_e is in [C/m^3]
ρ_m is in [Vs/m^3]

Note that [C] = [As]

Lorentz force

$$\mathbf{F} = q \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Constitutive relations

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

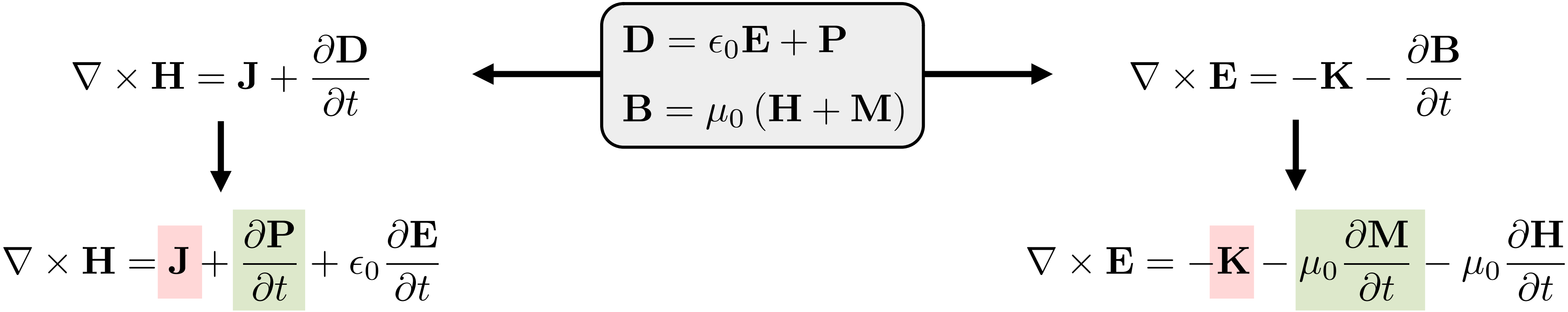
$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right)$$

For simple isotropic media

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

Currents and Polarizations



Constitutive relations

$$\boxed{\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{B} &= \mu_0 \left(\mathbf{H} + \mathbf{M}\right)\end{aligned}}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mathbf{K} - \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mathbf{K} - \mu_0 \frac{\partial \mathbf{M}}{\partial t} - \mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

🟥 = impressed

🟩 = induced

$$\boxed{\mathbf{J}_{\mathrm{ind}} = \frac{\partial \mathbf{P}}{\partial t}} \qquad \boxed{\mathbf{K}_{\mathrm{ind}} = \mu_0 \frac{\partial \mathbf{M}}{\partial t}}$$

Electric and magnetic polarizations may be seen as equivalent induced currents

The Continuity Equations

Differential form

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mathbf{K} - \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{e}}$$

$$\nabla \cdot \mathbf{B} = \rho_{\mathrm{m}}$$

Pre-multiplying by $\nabla\cdot$

$$\xrightarrow{\qquad} \left\{ \begin{aligned} \nabla \cdot (\nabla \times \mathbf{H}) &= \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} \\ \nabla \cdot (\nabla \times \mathbf{E}) &= -\nabla \cdot \mathbf{K} - \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} \end{aligned} \right.$$

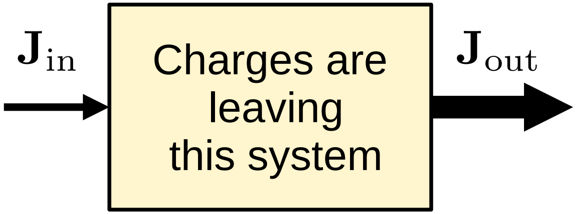
$$\downarrow$$

$$0$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} \rho_{\mathrm{e}}$$

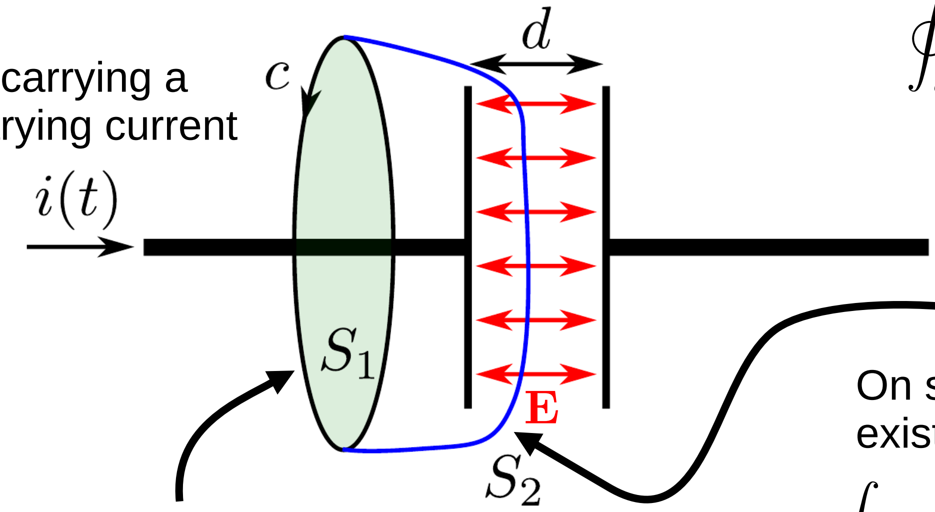
$$\nabla \cdot \mathbf{K} = -\frac{\partial}{\partial t} \rho_{\mathrm{m}}$$

Divergence of the currents implies a variation in time of the charge densities



$$\nabla \cdot \mathbf{J} > 0$$

Understanding Capacitors



parallel plate capacitor
of surface area *A*

wire carrying a
time-varying current

Ampère law in integral form

$$\oint_c \mathbf{H}(t) \cdot d\mathbf{l} = \iint_S \mathbf{J}(t) \cdot d\mathbf{S} + \frac{\partial}{\partial t} \iint_S \mathbf{D}(t) \cdot d\mathbf{S}$$

where $\mathbf{D} = \epsilon \mathbf{E}$

For a fixed contour *c*, we are free to choose any surface of integration *S*

On surface S_2, **J** = 0, and assuming that **E = v/d** exist only between the plates, we have

$$\oint_c \mathbf{H}(t) \cdot d\mathbf{l} = \frac{\partial}{\partial t} \iint_{S_2} \epsilon \mathbf{E}(t) \cdot d\mathbf{S}$$

$$= \frac{\partial}{\partial t} \left(\epsilon A \frac{v(t)}{d} \right) = \epsilon \frac{A}{d} \frac{\partial}{\partial t} v(t) = C \frac{\partial}{\partial t} v(t)$$

On surface S_1, **E** = **D** = 0, and so

$$\oint_c \mathbf{H}(t) \cdot d\mathbf{l} = \iint_{S_1} \mathbf{J}(t) \cdot d\mathbf{S} = i(t)$$

Equating both equations

$$\boxed{i(t) = C \frac{\partial}{\partial t} v(t)}$$

where *C* is the capacitance of the plates

For time-harmonic signals with $e^{j\omega t}$

$$\longrightarrow I = j\omega C V = \frac{V}{Z}$$

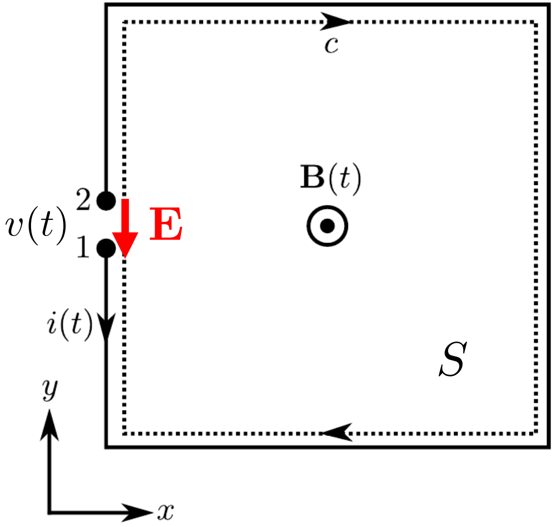
where

Impedance of a capacitor

$$\boxed{Z = \frac{1}{j\omega C}}$$

Understanding Inductors

Loop of wire with time-varying magnetic field



Faraday law in integral form

$$\oint_c \mathbf{E}(t)\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}(t)\cdot d\mathbf{S} \longrightarrow \Phi_\mathrm{B}\text{: magnetic flux}$$

We only consider the electric field in the gap between the electrodes

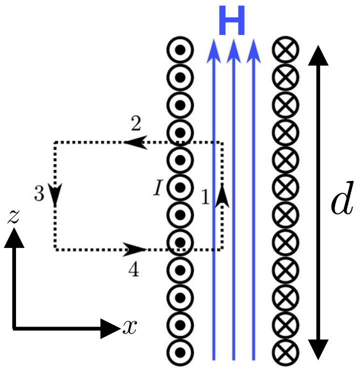
$$\oint_c \mathbf{E}(t)\cdot d\mathbf{l} = v(t) = \frac{\partial}{\partial t}\Phi_\mathrm{B} \longrightarrow \boxed{v(t) = N\frac{\partial}{\partial t}\Phi_\mathrm{B}}$$

For a coil with *N* loops

The magnetic field in a coil is

$$\mathbf{H}(t) = \frac{Ni(t)}{d}\hat{\boldsymbol{z}} \longrightarrow \Phi_\mathrm{B} = \iint_S \mathbf{B}(t)\cdot d\mathbf{S} = \frac{\mu N i(t) S}{d}$$

where *S* is the surface area of the coil and *d* is its length



Combining these equations yields

$$v(t) = N\frac{\partial}{\partial t}\frac{\mu N i(t) S}{d} = \mu\frac{S}{d}N^2\frac{\partial}{\partial t}i(t) \longrightarrow \boxed{v(t) = L\frac{\partial}{\partial t}i(t)}$$

where *L* is the inductance of the coil

For time-harmonic signals with $e^{j\omega t}$

Impedance of an inductor

$$V = j\omega L I = ZI \longrightarrow \boxed{Z = j\omega L}$$

Maxwell Equations for Time-Harmonic Fields

Time domain: *f(t)*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mathbf{K} - \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{e}}$$

$$\nabla \cdot \mathbf{B} = \rho_{\mathrm{m}}$$

From now on, we will generally consider the time-harmonic function

$$e^{j\omega t}$$

Frequency domain: *f(ω)*

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D}$$

$$\nabla \times \mathbf{E} = -\mathbf{K} - j\omega\mathbf{B}$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{e}}$$

$$\nabla \cdot \mathbf{B} = \rho_{\mathrm{m}}$$

This is a typical engineering convention. Physicists tend to use

$$e^{-i\omega t}$$

To convert from one to the other, simply use $j \leftrightarrow -i$ but be careful when considering material loss!

Duality Principle

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\left(\epsilon_0\mathbf{E} + \mathbf{P}\right) \quad \longleftrightarrow \quad \nabla \times \mathbf{E} = -\mathbf{K} - j\omega\mu_0\left(\mathbf{H} + \mathbf{M}\right)$$

$$\nabla \cdot \left(\epsilon_0\mathbf{E} + \mathbf{P}\right) = \rho_\mathrm{e} \quad \longleftrightarrow \quad \mu_0\nabla \cdot \left(\mathbf{H} + \mathbf{M}\right) = \rho_\mathrm{m}$$

$$\mathbf{E} \rightarrow \mathbf{H}$$

$$\mathbf{H} \rightarrow -\mathbf{E}$$

$$\mathbf{J} \rightarrow \mathbf{K}$$

$$\mathbf{K} \rightarrow -\mathbf{J}$$

$$\mathbf{P} \rightarrow \mu_0\mathbf{M}$$

$$\mathbf{M} \rightarrow -\frac{\mathbf{P}}{\epsilon_0}$$

$$\rho_\mathrm{e} \rightarrow \rho_\mathrm{m}$$

$$\rho_\mathrm{m} \rightarrow -\rho_\mathrm{e}$$

$$\epsilon_0 \leftrightarrow \mu_0$$

What Have We Learned So Far….

- In a time varying case, electric and magnetic field are connected together via Maxwell equations
- We can consider that time-varying electric and magnetic polarization density **P** and **M** are equivalent to electric and magnetic current densities **J** and **K**, respectively
- Understand how Maxwell equations may be cleverly used to compute the response of capacitors and inductors
- The impedance of a capacitor is Z = 1/(jωC), that of an inductor is Z = jωL
- Duality is a very useful concept that may be used to derive solutions for magnetic problems if the solutions for the corresponding electric problems are known