Lecture 13

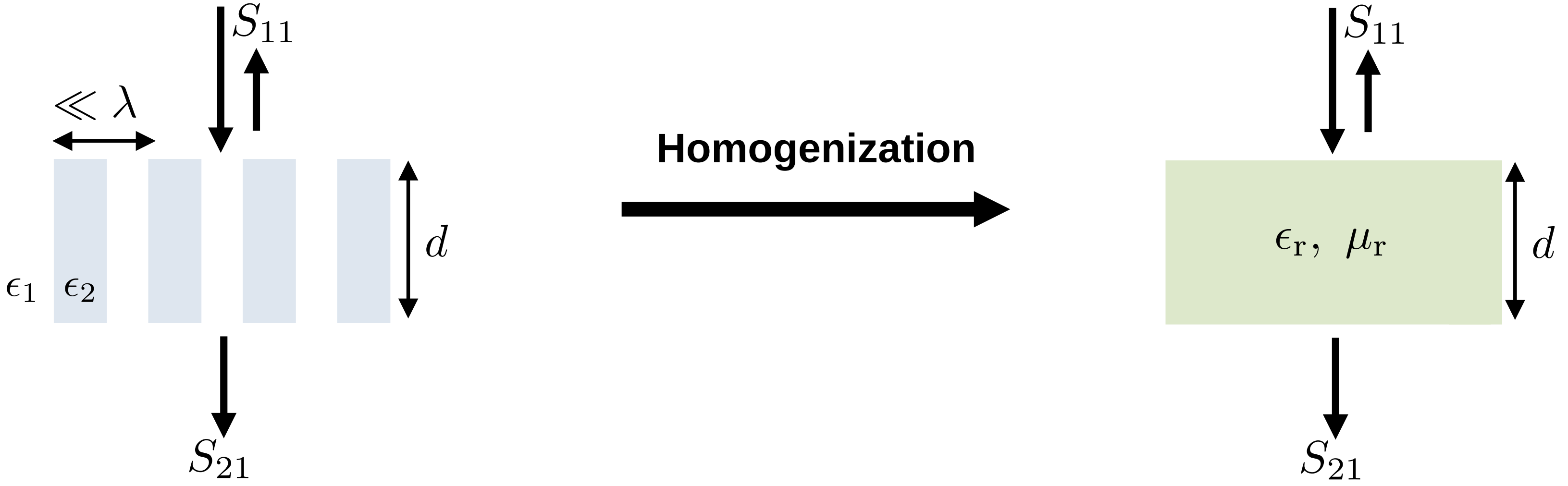
Homogenization Techniques

Slab Homogenization

Slab Homogenization Problem

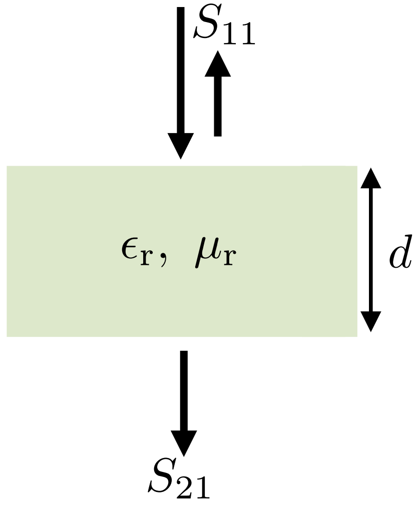
<u>Initial situation</u>: subwavelength periodic slab of dielectric material illuminated at normal incidence

<u>Final situation</u>: uniform and homogeneous material slab that produces the same scattering as in the original situation



The scattering parameters S11 and S21 are obtained by numerical simulations. The goal of the homogenization procedure is to obtain the effective parameters ϵ_r and μ_r

Slab Homogenization Problem



The refractive index and the impedance contain our unknowns

$$n = \sqrt{\epsilon_\mathrm{r}\mu_\mathrm{r}}$$

$$\eta = \sqrt{\frac{\mu_\mathrm{r}}{\epsilon_\mathrm{r}}}$$

The total scattering parameters are

$$S_{11} = \frac{r\left(1 - e^{-j2k_0nd}\right)}{1 - r^2e^{-j2k_0nd}} \qquad S_{21} = \frac{\left(1 - r^2\right)e^{-jk_0nd}}{1 - r^2e^{-j2k_0nd}}$$

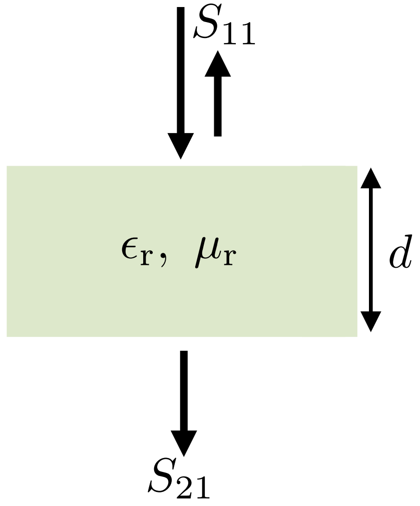
where *r* is the reflection coefficient at the first interface and is defined in terms of the normalized impedance as

$$r = \frac{\eta - 1}{\eta + 1}$$

We solve these three equations to get

$$\eta = \pm\sqrt{\frac{(1 + S_{11})^2 - S_{21}^2}{(1 - S_{11})^2 - S_{21}^2}}$$

Determining the Slab Impedance



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Impedance as function of scattering parameters

$$\eta = \pm\sqrt{\frac{(1+S_{11})^2 - S_{21}^2}{(1-S_{11})^2 - S_{21}^2}}$$

We next assume that the system is passive (no gain) but potentially lossy. This implies the following conditions

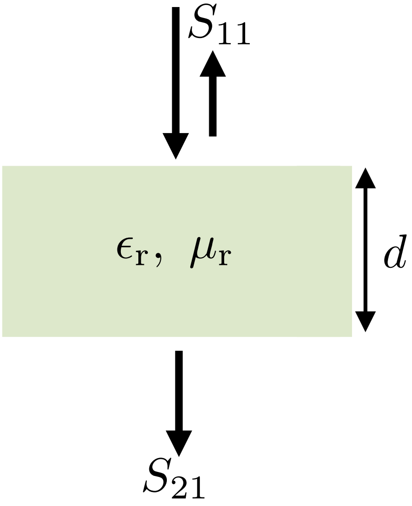
$$\mathrm{Re}[\eta] \geq 0 \qquad \mathrm{Im}[n] \geq 0 \qquad \mathrm{Im}[\epsilon_\mathrm{r}] \geq 0 \qquad \mathrm{Im}[\mu_\mathrm{r}] \geq 0$$

We can now determine the sign of the impedance

$$\eta = \mathrm{sgn}\left[\mathrm{Re}\left\{\sqrt{\frac{(1+S_{11})^2 - S_{21}^2}{(1-S_{11})^2 - S_{21}^2}}\right\}\right]\sqrt{\frac{(1+S_{11})^2 - S_{21}^2}{(1-S_{11})^2 - S_{21}^2}}$$

Knowing the impedance, the reflection is $r = \dfrac{\eta - 1}{\eta + 1}$

Determining the Complex Refractive Index



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Solve for the exponential function

$$e^{jk_0nd} = \frac{S_{21}}{1 - S_{11}r} = e^{jk_0n'd}e^{-k_0n''d}$$

where we have separated the real and imaginary parts of the refractive index

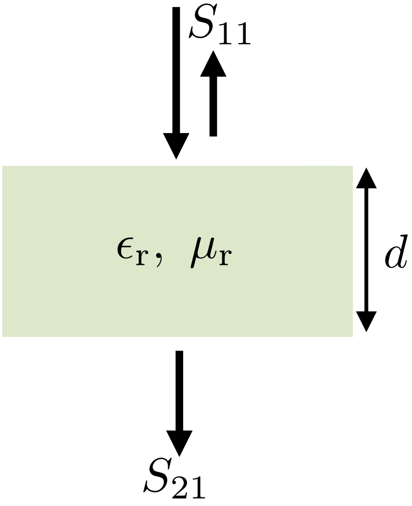
$$n = n' + jn''$$

Taking the log en both sides leads to

$$\ln\left(\frac{S_{21}}{1 - S_{11}r}\right) = \ln\left(e^{jk_0n'd}e^{-k_0n''d}\right) = j\left(k_0n'd - 2\pi m\right) - k_0n''d$$

where $m \in \mathbb{N}$

Determining the Complex Refractive Index



The refractive index and the impedance contain our unknowns

$$n = \sqrt{\epsilon_{\mathrm{r}}\mu_{\mathrm{r}}}$$

$$\eta = \sqrt{\frac{\mu_{\mathrm{r}}}{\epsilon_{\mathrm{r}}}}$$

$$\ln\left(\frac{S_{21}}{1 - S_{11}r}\right) = j\left(k_0 n' d - 2\pi m\right) - k_0 n'' d$$

We can now split this equation to get the real and imaginary parts of the refractive index

$$n = n' + jn'' \begin{cases} n' = \dfrac{1}{k_0 d}\left[\mathrm{Im}\left\{\ln\left(\dfrac{S_{21}}{1 - S_{11}r}\right)\right\} + 2\pi m\right] \\ n'' = -\dfrac{1}{k_0 d}\mathrm{Re}\left\{\ln\left(\dfrac{S_{21}}{1 - S_{11}r}\right)\right\} \end{cases}$$

Since we know both the refractive index and the normalized impedance, we have that

$$\boxed{\mu_{\mathrm{r}} = n\eta} \qquad \boxed{\epsilon_{\mathrm{r}} = \frac{n}{\eta}}$$

Satisfying the Passivity Conditions

$$\boxed{\mu_\mathrm{r} = n\eta}$$

$$\boxed{\epsilon_\mathrm{r} = \frac{n}{\eta}}$$

Separating real and imaginary parts

$$\mu'_\mathrm{r} + j\mu''_\mathrm{r} = (n' + jn'')(\eta' + j\eta'')$$

$$\mu'_\mathrm{r} + j\mu''_\mathrm{r} = n'\eta' - n''\eta'' + j(n''\eta' + n'\eta'')$$

$$\epsilon'_\mathrm{r} + j\epsilon''_\mathrm{r} = \frac{n' + jn''}{\eta' + j\eta''}$$

$$\epsilon'_\mathrm{r} + j\epsilon''_\mathrm{r} = \frac{(n' + jn'')(\eta' - j\eta'')}{(\eta' + j\eta'')(\eta' - j\eta'')}$$

$$\epsilon'_\mathrm{r} + j\epsilon''_\mathrm{r} = \frac{n'\eta' + n''\eta'' + j(n''\eta' - n'\eta'')}{|\eta|^2}$$

Keeping only the imaginary parts

$$\mu''_\mathrm{r} = n''\eta' + n'\eta''$$

$$\epsilon''_\mathrm{r} = \frac{1}{|\eta|^2}\,(n''\eta' - n'\eta'')$$

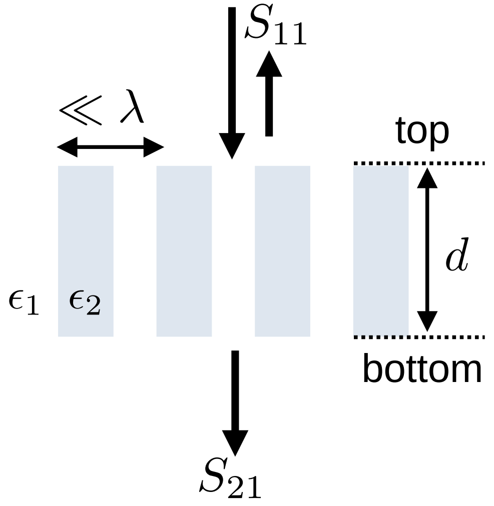
The system is passive if

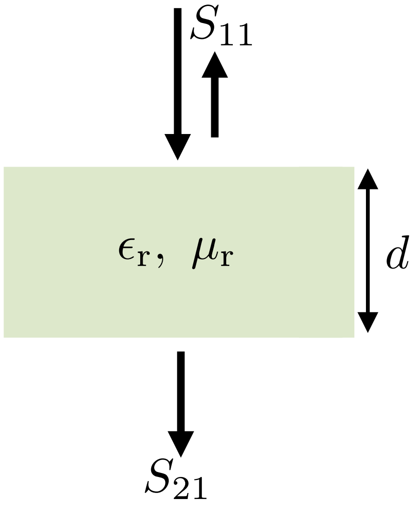
$$\mathrm{Im}[\mu_\mathrm{r}] \geq 0$$

$$\mathrm{Im}[\epsilon_\mathrm{r}] \geq 0$$

$$\boxed{n''\eta' \geq |n'\eta''|}$$

Slab Homogenization Summary





1) Numerically simulate (e.g., RCWA) the scattering parameters. Make sure to measure the scattering phase with respect to the top and bottom of the structure.

2) Compute the normalized impedance and reflection coefficient

$$\boxed{\eta = \mathrm{sgn}\left[\mathrm{Re}\left\{\sqrt{\frac{(1+S_{11})^2 - S_{21}^2}{(1-S_{11})^2 - S_{21}^2}}\right\}\right]\sqrt{\frac{(1+S_{11})^2 - S_{21}^2}{(1-S_{11})^2 - S_{21}^2}}} \rightarrow r = \frac{\eta - 1}{\eta + 1}$$

3) Compute the refractive index

$$\boxed{n = \frac{1}{k_0 d}\left[\mathrm{Im}\left\{\ln\left(\frac{S_{21}}{1 - S_{11} r}\right)\right\} + 2\pi m\right] - \frac{j}{k_0 d}\mathrm{Re}\left\{\ln\left(\frac{S_{21}}{1 - S_{11} r}\right)\right\}}$$

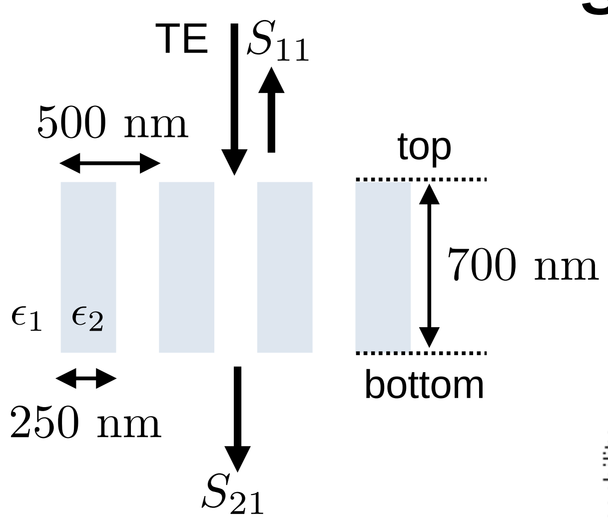
4) Compute the effective permittivity and permeability

$$\boxed{\epsilon_{\mathrm{r}} = \frac{n}{\eta}} \qquad \boxed{\mu_{\mathrm{r}} = n\eta}$$

5) Choose the value of *m* such that the passivity condition is mostly satisfied

$$\boxed{n''\eta' \geq |n'\eta''|}$$

Slab Homogenization Example



ridge parameters

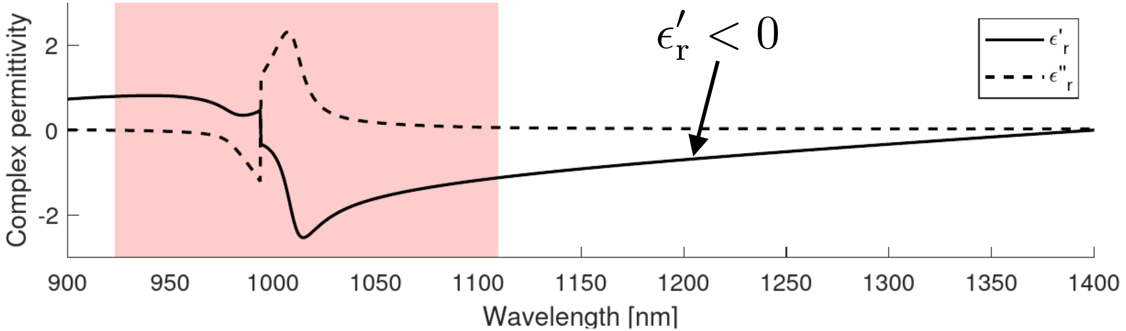
$\epsilon_2 = 6$

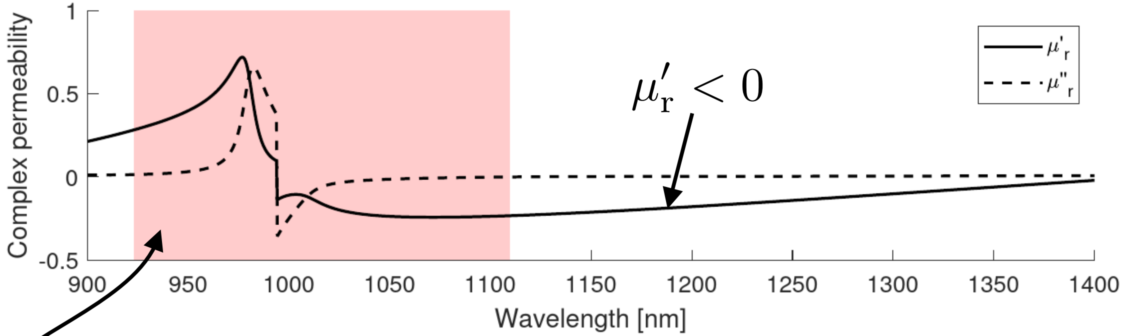
$\sigma_2 = 1000 \text{ S/m}$

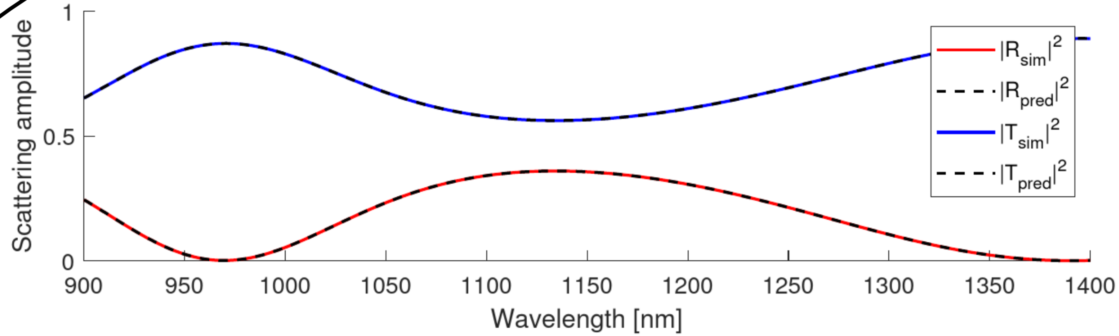
Region where the passivity condition is <u>not</u> satisfied

$$\boxed{n''\eta' \geq |n'\eta''|}$$

$m = 0$







Even though the initial system is purely dielectric, its homogenized version has a magnetic response!

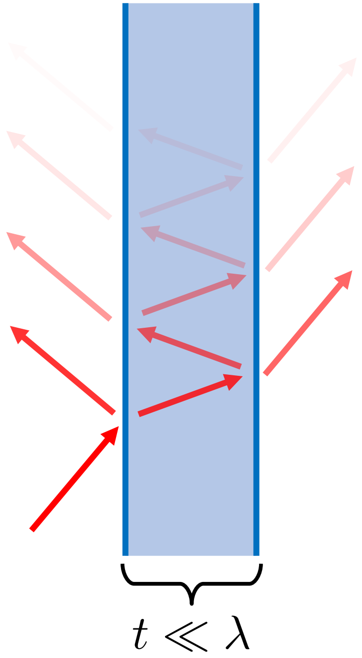
What Have We Learned So Far….

- Homogenization consists in modeling an electromagnetic structure in terms of effective material parameters that produce the same scattering response as the original structure. This works only in a non-diffraction regime (only 0^{th} diffraction orders exist).
- Slab homogenization consists in replacing a spatially modulated slab by a slab of effective material parameters with the same thickness. This means that we need to account for multiple scattering within the slab.
- Due to the thickness of the homogenized slab and the resulting multiple scattering inside, there is an infinite number of possible effective refractive indices. The correct solution should typically be selected in accordance with the passivity condition.
- This homogenization shows that a purely dielectric spatially varying structure is modeled by a combination of both permittivity and permeability. This shows one possible mechanism to engineer the magnetic response of a non-magnetic material.

Boundary Modeling Approach

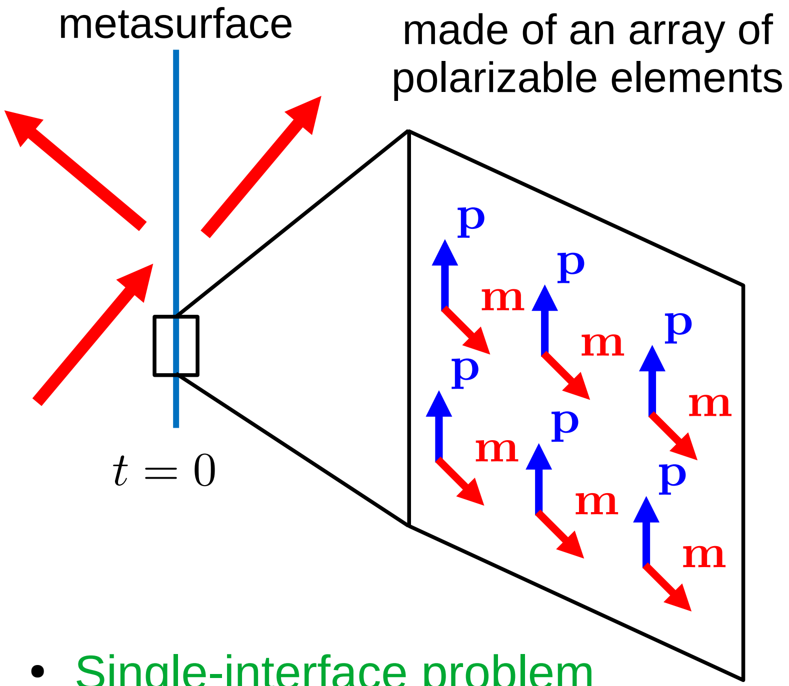
Motivation for Boundary Approach

Thin slab



- Double-interface problem
- 3D material parameters
- Multiple scattering inside
- **Very difficult to model**

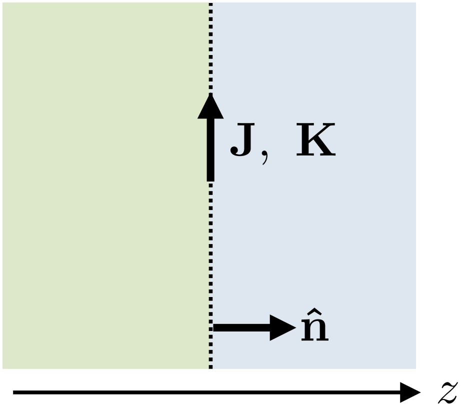
Zero-thickness metasurface



- Single-interface problem
- 2D material parameters
- No scattering inside
- **Exact and rigorous model**

Limitations of Conventional Boundary Conditions

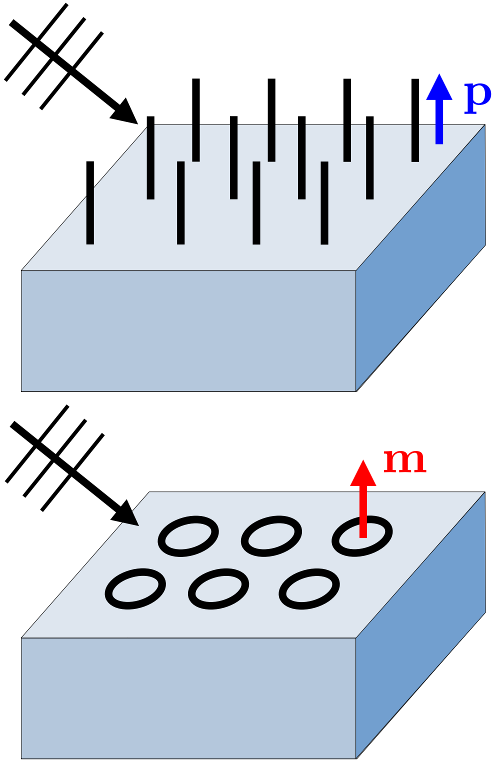
Conventional boundary conditions assume the presence of <u>tangential</u> current sources at the interface



$$\hat{\mathbf{n}} \times \Delta \mathbf{H}_{\mathrm{t}} = +\mathbf{J}_{\mathrm{t}}$$
$$\hat{\mathbf{n}} \times \Delta \mathbf{E}_{\mathrm{t}} = -\mathbf{K}_{\mathrm{t}}$$

Does not account for normal currents !

What if there are subwavelength rods or rings lying at the interface that would induce normal electric and magnetic polarizations ?

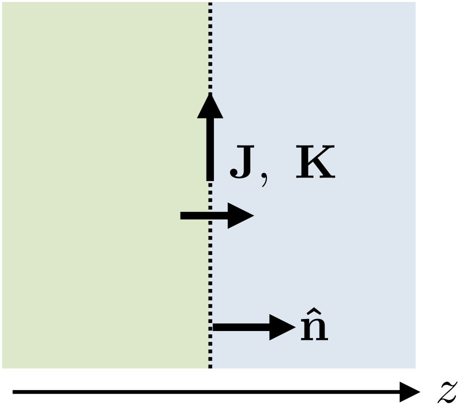


Derivation of Generalized Boundary Conditions

We consider an interface that can support both tangential and normal current densities

$$\nabla \times \mathbf{H} = +\mathbf{J}\delta(z) + j\omega\epsilon\mathbf{E}$$

$$\nabla \times \mathbf{E} = -\mathbf{K}\delta(z) - j\omega\mu\mathbf{H}$$



Split all quantities into tangential and normal components

$$\begin{cases} \mathbf{E} = \mathbf{E}_\mathrm{t} + \hat{\mathbf{n}}E_n \\ \mathbf{H} = \mathbf{H}_\mathrm{t} + \hat{\mathbf{n}}H_n \\ \mathbf{K} = \mathbf{K}_\mathrm{t} + \hat{\mathbf{n}}K_n \\ \mathbf{J} = \mathbf{J}_\mathrm{t} + \hat{\mathbf{n}}J_n \\ \nabla = \nabla_\mathrm{t} + \hat{\mathbf{n}}\dfrac{\partial}{\partial z} \end{cases}$$

$$\left(\nabla_\mathrm{t} + \hat{\mathbf{n}}\frac{\partial}{\partial z}\right) \times (\mathbf{H}_\mathrm{t} + \hat{\mathbf{n}}H_n) = +(\mathbf{J}_\mathrm{t} + \hat{\mathbf{n}}J_n)\,\delta(z) + j\omega\epsilon\,(\mathbf{E}_\mathrm{t} + \hat{\mathbf{n}}E_n)$$

$$\left(\nabla_\mathrm{t} + \hat{\mathbf{n}}\frac{\partial}{\partial z}\right) \times (\mathbf{E}_\mathrm{t} + \hat{\mathbf{n}}E_n) = -(\mathbf{K}_\mathrm{t} + \hat{\mathbf{n}}K_n)\,\delta(z) - j\omega\mu\,(\mathbf{H}_\mathrm{t} + \hat{\mathbf{n}}H_n)$$

Eliminating Normal Field Components

$$\left(\nabla_{\mathrm{t}} + \hat{\mathbf{n}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{H}_{\mathrm{t}} + \hat{\mathbf{n}}H_n\right) = +\left(\mathbf{J}_{\mathrm{t}} + \hat{\mathbf{n}}J_n\right)\delta(z) + j\omega\epsilon\left(\mathbf{E}_{\mathrm{t}} + \hat{\mathbf{n}}E_n\right)$$

$$\left(\nabla_{\mathrm{t}} + \hat{\mathbf{n}}\frac{\partial}{\partial z}\right) \times \left(\mathbf{E}_{\mathrm{t}} + \hat{\mathbf{n}}E_n\right) = -\left(\mathbf{K}_{\mathrm{t}} + \hat{\mathbf{n}}K_n\right)\delta(z) - j\omega\mu\left(\mathbf{H}_{\mathrm{t}} + \hat{\mathbf{n}}H_n\right)$$

Only tangential components

$$\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}H_n + \frac{\partial}{\partial z}\left(\hat{\mathbf{n}} \times \mathbf{H}_{\mathrm{t}}\right) = +\mathbf{J}_{\mathrm{t}}\delta(z) + j\omega\epsilon\mathbf{E}_{\mathrm{t}}$$

substituting the normal field components

$$\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}E_n + \frac{\partial}{\partial z}\left(\hat{\mathbf{n}} \times \mathbf{E}_{\mathrm{t}}\right) = -\mathbf{K}_{\mathrm{t}}\delta(z) - j\omega\mu\mathbf{H}_{\mathrm{t}}$$

Only normal components

$$\nabla_{\mathrm{t}} \times \mathbf{H}_{\mathrm{t}} = +\hat{\mathbf{n}}J_n\delta(z) + j\omega\epsilon\hat{\mathbf{n}}E_n \quad \longrightarrow \quad \hat{\mathbf{n}}E_n = +\frac{1}{j\omega\epsilon}\left[\nabla_{\mathrm{t}} \times \mathbf{H}_{\mathrm{t}} - \hat{\mathbf{n}}J_n\delta(z)\right]$$

$$\nabla_{\mathrm{t}} \times \mathbf{E}_{\mathrm{t}} = -\hat{\mathbf{n}}K_n\delta(z) - j\omega\mu\hat{\mathbf{n}}H_n \quad \longrightarrow \quad \hat{\mathbf{n}}H_n = -\frac{1}{j\omega\mu}\left[\nabla_{\mathrm{t}} \times \mathbf{E}_{\mathrm{t}} + \hat{\mathbf{n}}K_n\delta(z)\right]$$

Equations with only tangential fields

$$\frac{\partial}{\partial z}\left(\hat{\mathbf{n}} \times \mathbf{H}_{\mathrm{t}}\right) = \frac{1}{j\omega\mu}\nabla_{\mathrm{t}} \times \nabla_{\mathrm{t}} \times \mathbf{E}_{\mathrm{t}} + j\omega\epsilon\mathbf{E}_{\mathrm{t}} + \mathbf{J}_{\mathrm{t}}\delta(z) + \frac{1}{j\omega\mu}\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}K_n\delta(z)$$

$$\frac{\partial}{\partial z}\left(\hat{\mathbf{n}} \times \mathbf{E}_{\mathrm{t}}\right) = -\frac{1}{j\omega\epsilon}\nabla_{\mathrm{t}} \times \nabla_{\mathrm{t}} \times \mathbf{H}_{\mathrm{t}} - j\omega\mu\mathbf{H}_{\mathrm{t}} - \mathbf{K}_{\mathrm{t}}\delta(z) + \frac{1}{j\omega\epsilon}\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}J_n\delta(z)$$

Generalized Sheet Transition Conditions (GSTC)

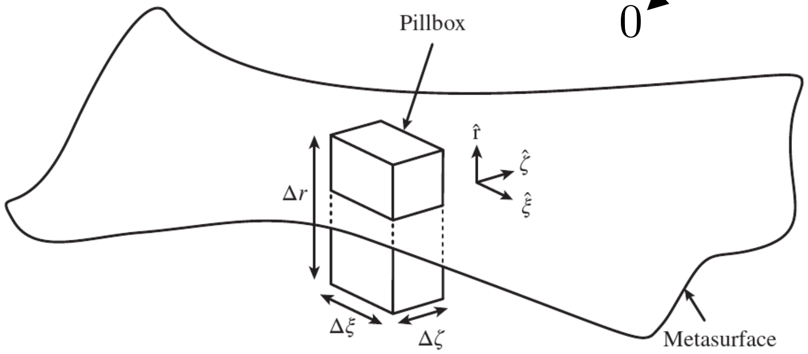
Leads to differences of the fields between both sides

$$\frac{\partial}{\partial z}(\hat{\mathbf{n}} \times \mathbf{H}_{\mathrm{t}}) = \frac{1}{j\omega\mu_0}\nabla_{\mathrm{t}} \times \nabla_{\mathrm{t}} \times \mathbf{E}_{\mathrm{t}} + j\omega\epsilon_0\mathbf{E}_{\mathrm{t}} + \mathbf{J}_{\mathrm{t}}\delta(z) + \frac{1}{j\omega\mu_0}\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}K_n\delta(z)$$

$$\frac{\partial}{\partial z}(\hat{\mathbf{n}} \times \mathbf{E}_{\mathrm{t}}) = -\frac{1}{j\omega\epsilon_0}\nabla_{\mathrm{t}} \times \nabla_{\mathrm{t}} \times \mathbf{H}_{\mathrm{t}} - j\omega\mu_0\mathbf{H}_{\mathrm{t}} - \mathbf{K}_{\mathrm{t}}\delta(z) + \frac{1}{j\omega\epsilon_0}\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}J_n\delta(z)$$

These terms survive and the integration absorbs the Dirac deltas

0



Integrate over a small volume around the interface (pillbox) and make it tend to 0

$$\hat{\mathbf{n}} \times \Delta\mathbf{H}_{\mathrm{t}} = +\mathbf{J}_{\mathrm{t}} + \frac{1}{j\omega\mu_0}\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}K_n$$

$$\hat{\mathbf{n}} \times \Delta\mathbf{E}_{\mathrm{t}} = -\mathbf{K}_{\mathrm{t}} + \frac{1}{j\omega\epsilon_0}\nabla_{\mathrm{t}} \times \hat{\mathbf{n}}J_n$$

We consider dipolar responses and a flat metasurface lying in the xy-plane at z=0

$$\mathbf{J} = j\omega\mathbf{P}$$

$$\mathbf{K} = j\omega\mu_0\mathbf{M}$$

$$\hat{\mathbf{n}} = \hat{\mathbf{z}}$$

GSTC

$$\hat{\mathbf{z}} \times \Delta\mathbf{H} = +j\omega\mathbf{P}_{\|} - \hat{\mathbf{n}} \times \nabla_{\|}M_z$$

$$\hat{\mathbf{z}} \times \Delta\mathbf{E} = -j\omega\mathbf{M}_{\|} - \frac{1}{\epsilon_0}\hat{\mathbf{n}} \times \nabla_{\|}P_z$$

where $\Delta\mathbf{E} = \mathbf{E}_{z=0^+} - \mathbf{E}_{z=0^-}$

GSTC – Normal Components

$$\nabla \times \mathbf{H} = +\mathbf{J}\delta(z) + j\omega\mathbf{D}$$

$$\nabla \times \mathbf{E} = -\mathbf{K}\delta(z) - j\omega\mathbf{B}$$

Applying $\nabla\cdot$

..and adding surface charges

$$\nabla \cdot \mathbf{D} = \tfrac{j}{\omega}\nabla \cdot [\mathbf{J}\delta(z)] + \rho_{\mathrm{e}}\delta(z)$$

$$\nabla \cdot \mathbf{B} = \frac{j}{\omega}\nabla \cdot [\mathbf{K}\delta(z)] + \rho_{\mathrm{m}}\delta(z)$$

After splitting into tangential and normal parts and simplifying the equations

$$\frac{\partial}{\partial z}D_n = \frac{j}{\omega}\left(\nabla_{\mathrm{t}} \cdot \mathbf{J}_{\mathrm{t}} + \frac{\partial}{\partial z}J_n\right)\delta(z) - \nabla_{\mathrm{t}} \cdot \mathbf{D}_{\mathrm{t}} + \rho_{\mathrm{e}}\delta(z)$$

$$\frac{\partial}{\partial z}B_n = \frac{j}{\omega}\left(\nabla_{\mathrm{t}} \cdot \mathbf{K}_{\mathrm{t}} + \frac{\partial}{\partial z}K_n\right)\delta(z) - \nabla_{\mathrm{t}} \cdot \mathbf{B}_{\mathrm{t}} + \rho_{\mathrm{m}}\delta(z)$$

After pillbox integration

$$\boxed{\begin{aligned}\Delta D_n &= \frac{j}{\omega}\nabla_{\mathrm{t}} \cdot \mathbf{J}_{\mathrm{t}} + \rho_{\mathrm{e}} \\ \Delta B_n &= \frac{j}{\omega}\nabla_{\mathrm{t}} \cdot \mathbf{K}_{\mathrm{t}} + \rho_{\mathrm{m}}\end{aligned}}$$

Complete Formulation of Dipolar GSTC

$$\hat{\mathbf{z}} \times \Delta\mathbf{H} = +j\omega\mathbf{P}_{\|} - \hat{\mathbf{z}} \times \nabla_{\|} M_z$$

$$\hat{\mathbf{z}} \times \Delta\mathbf{E} = -j\omega\mathbf{M}_{\|} - \frac{1}{\epsilon_0}\hat{\mathbf{z}} \times \nabla_{\|} P_z$$

$$\Delta D_z = -\nabla_{\|} \cdot \mathbf{P}_{\|} + \rho_{\mathrm{e}}$$

$$\Delta B_z = -\mu_0 \nabla_{\|} \cdot \mathbf{M}_{\|} + \rho_{\mathrm{m}}$$

Gradient of normal polarizations are equivalent to tangential polarizations

Divergence of tangential polarizations are equivalent to surface charges

In the absence of impressed surface charges, these two equations are redundant with respect to the two first ones

GSTC and Effective Susceptibilities

GSTC

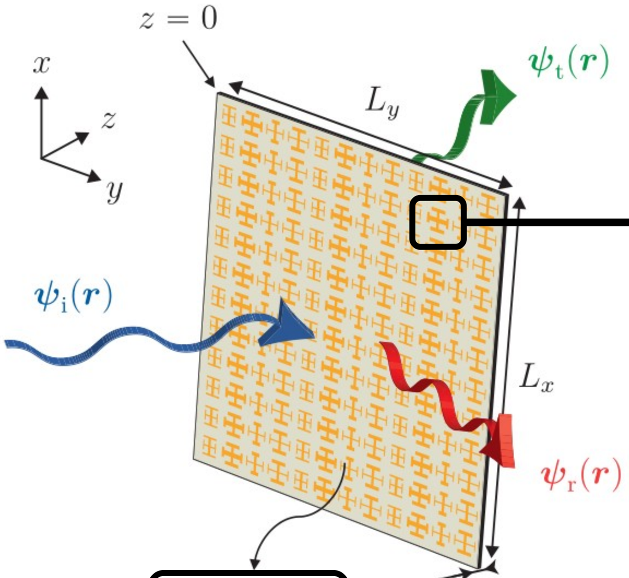
$$\hat{\mathbf{z}} \times \Delta\mathbf{H} = +j\omega\mathbf{P}_{\|} - \hat{\mathbf{n}} \times \nabla_{\|} M_z$$

$$\hat{\mathbf{z}} \times \Delta\mathbf{E} = -j\omega\mathbf{M}_{\|} - \frac{1}{\epsilon_0}\hat{\mathbf{n}} \times \nabla_{\|} P_z$$

Bianisotropic polarization densities

$$\begin{cases} \mathbf{P} = \epsilon_0 \overline{\overline{\chi}}_{\mathrm{ee}} \cdot \mathbf{E}_{\mathrm{av}} + \dfrac{1}{c_0}\overline{\overline{\chi}}_{\mathrm{em}} \cdot \mathbf{H}_{\mathrm{av}} \\ \mathbf{M} = \overline{\overline{\chi}}_{\mathrm{mm}} \cdot \mathbf{H}_{\mathrm{av}} + \dfrac{1}{\eta_0}\overline{\overline{\chi}}_{\mathrm{me}} \cdot \mathbf{E}_{\mathrm{av}} \end{cases}$$

3D susceptibilities are unitless. However, 2D susceptibilities have unit of [m]



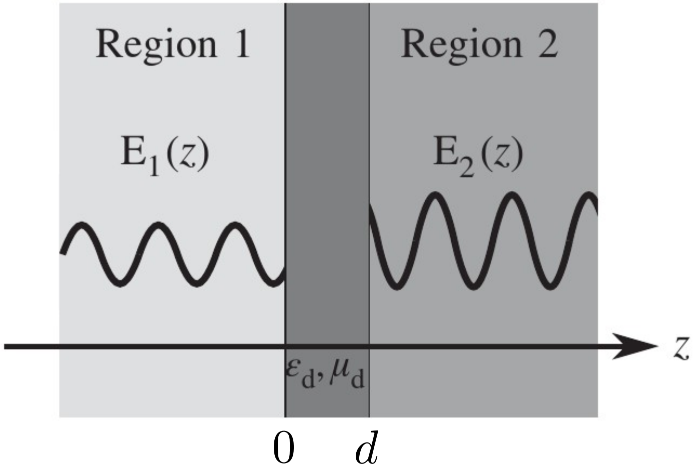
$\overline{\overline{\chi}}(\boldsymbol{\rho}) = ?$

Find the shapes of the scattering particles that correspond to the susceptibilities

Find the material parameters (susceptibilities) as functions of the incident, reflected and transmitted fields. Alternatively, we can also find the scattered from known susceptibilities

How to Define the Average Field

Material slab with arbitrary permittivity and permeability



Boundary conditions

$$\mathbf{E}_{\mathrm{d},\|}(0) = \mathbf{E}_{1,\|}(0) \qquad \mathbf{E}_{\mathrm{d},\|}(d) = \mathbf{E}_{2,\|}(d)$$

$$\epsilon_{\mathrm{d}} E_{\mathrm{d},z}(0) = \epsilon_1 \mathbf{E}_{1,z}(0) \qquad \epsilon_{\mathrm{d}} E_{\mathrm{d},z}(d) = \epsilon_2 \mathbf{E}_{1,z}(d)$$

Field inside the slab

$$\mathbf{E}_{\mathrm{d}}(z) = \mathbf{A}e^{-j\beta z} + \mathbf{B}e^{+j\beta z}$$

Splitting into tangential and normal parts and applying the boundary conditions

$$\mathbf{A} = \mathbf{A}_{\|} + \hat{\mathbf{z}}A_z$$

$$\mathbf{B} = \mathbf{B}_{\|} + \hat{\mathbf{z}}B_z$$

$$\mathbf{A}_{\|} = \frac{\mathbf{E}_{2,\|}(d) - \mathbf{E}_{1,\|}(0)e^{j\beta d}}{e^{-j\beta d} - e^{j\beta d}}$$

$$\mathbf{B}_{\|} = -\frac{\mathbf{E}_{2,\|}(d) - \mathbf{E}_{1,\|}(0)e^{-j\beta d}}{e^{-j\beta d} - e^{j\beta d}}$$

$$A_z = \frac{\epsilon_2 E_{2,z}(d) - \epsilon_1 E_{1,z}(0)e^{j\beta d}}{\epsilon_{\mathrm{d}}\left(e^{-j\beta d} - e^{j\beta d}\right)}$$

$$B_z = -\frac{\epsilon_2 E_{2,z}(d) - \epsilon_1 E_{1,z}(0)e^{-j\beta d}}{\epsilon_{\mathrm{d}}\left(e^{-j\beta d} - e^{j\beta d}\right)}$$

Definition of the Average Field

For the tangential field

$$\mathbf{E}_{\mathrm{av},\parallel} = \frac{1}{d}\int_0^d \mathbf{E}_{\mathrm{d},\parallel}(z)dz = \left[\frac{\mathbf{E}_{1,\parallel}(0) + \mathbf{E}_{2,\parallel}(d)}{\beta d}\right]\tan\left(\frac{\beta d}{2}\right)$$

Taylor expansion

$$\mathbf{E}_{\mathrm{av},\parallel} = \frac{1}{2}\left(\mathbf{E}_{1,\parallel}(0) + \mathbf{E}_{2,\parallel}(d)\right)\left[1 + \frac{1}{12}(\beta d)^2 + \frac{1}{120}(\beta d)^4 + \dots\right]$$

$$\lim_{d\to 0}\mathbf{E}_{\mathrm{av},\parallel} = \frac{1}{2}\left(\mathbf{E}_{1,\parallel} + \mathbf{E}_{2,\parallel}\right) \longrightarrow \boxed{\mathbf{E}_{\mathrm{av},\parallel} = \frac{1}{2}\left(\mathbf{E}_{1,\parallel} + \mathbf{E}_{2,\parallel}\right)}$$

For the normal field

$$E_{\mathrm{av},z} = \frac{1}{d}\int_0^d E_{\mathrm{d},z}(z)dz = \left[\frac{\epsilon_1 E_{1,z}(0) + \epsilon_2 E_{2,z}(d)}{\beta d}\right]\tan\left(\frac{\epsilon_{\mathrm{d}}\beta d}{2}\right)$$

$$\lim_{d\to 0} E_{\mathrm{av},z} = \frac{1}{2\epsilon_{\mathrm{d}}}\left(\epsilon_1 E_{1,z} + \epsilon_2 E_{2,z}\right) \longrightarrow \boxed{E_{\mathrm{av},z} = \frac{1}{2}\left(\epsilon_{\mathrm{r},1}E_{1,z} + \epsilon_{\mathrm{r},2}E_{2,z}\right)}$$

the value of $\epsilon_{\mathrm{r,d}}$ may be absorbed within the susceptibility

and similarly for the magnetic field

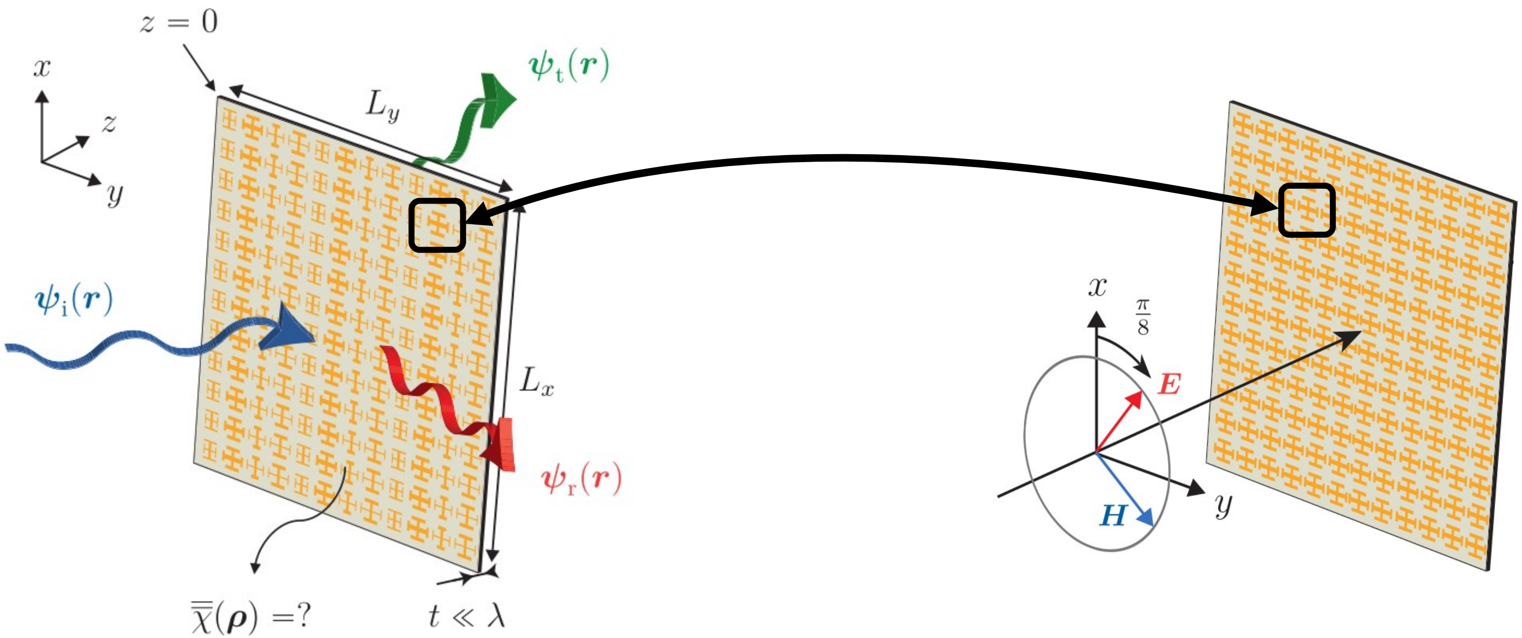
Homogenization Approach

Non-uniform metasurface

$$\boxed{\chi \propto f(x, y)}$$

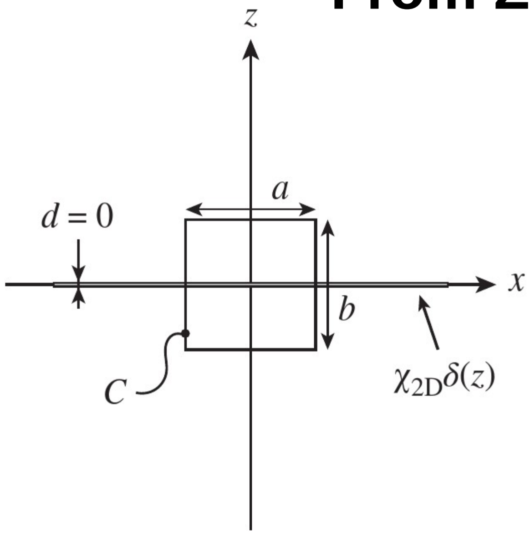
Uniform metasurface

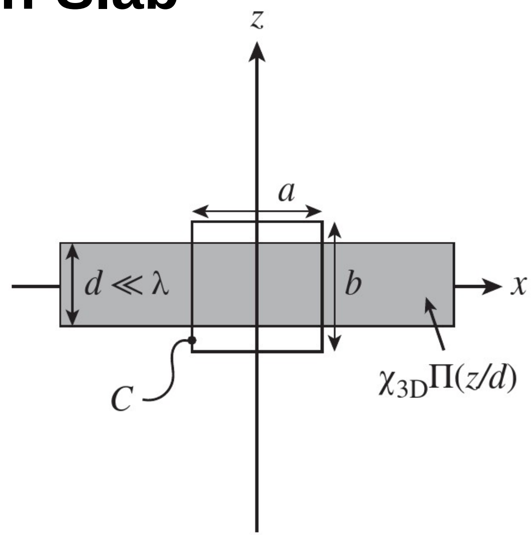
$$\boxed{\chi \not\propto f(x, y)}$$



Homogenization is typically performed on a unifrom metasurface. If the goal is to implement a non-uniform metasurface, then each unit cell is first treated individually assuming it forms a periodic structure. It is then incorporated into the final metasurface.

From Zero-Thickness to Thin-Slab





$$\nabla \times \mathbf{H} = j\omega\epsilon_0(1+\chi_{\mathrm{ee}})\mathbf{E}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = j\omega\epsilon_0 \iint_S \left[1+\chi_{\mathrm{2D}}\delta(z)\right]\mathbf{E}\cdot d\mathbf{S}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = j\omega\epsilon_0 \iint_S \left[1+\chi_{\mathrm{3D}}\Pi(z/d)\right]\mathbf{E}\cdot d\mathbf{S}$$

$b \to 0$

$b \to 0$

$$\left(H_x^+ - H_y^-\right)a \approx j\omega\epsilon_0\chi_{\mathrm{2D}}E_y a$$

$$\left(H_x^+ - H_y^-\right)a \approx j\omega\epsilon_0\chi_{\mathrm{3D}}dE_y a$$

Unitless $\rightarrow$ $\boxed{\chi_{\mathrm{3D}} \approx \dfrac{\chi_{\mathrm{2D}}}{d}}$ $\leftarrow$ unit: [m]

About Normal Polarizations

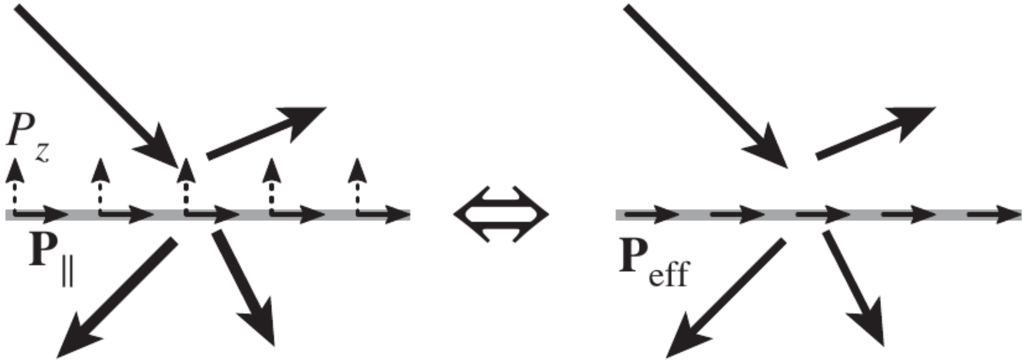
GSTC

$$\hat{\mathbf{z}} \times \Delta \mathbf{H} = +j\omega \mathbf{P}_{\|} - \hat{\mathbf{n}} \times \nabla_{\|} M_z$$
$$\hat{\mathbf{z}} \times \Delta \mathbf{E} = -j\omega \mathbf{M}_{\|} - \frac{1}{\epsilon_0} \hat{\mathbf{n}} \times \nabla_{\|} P_z$$

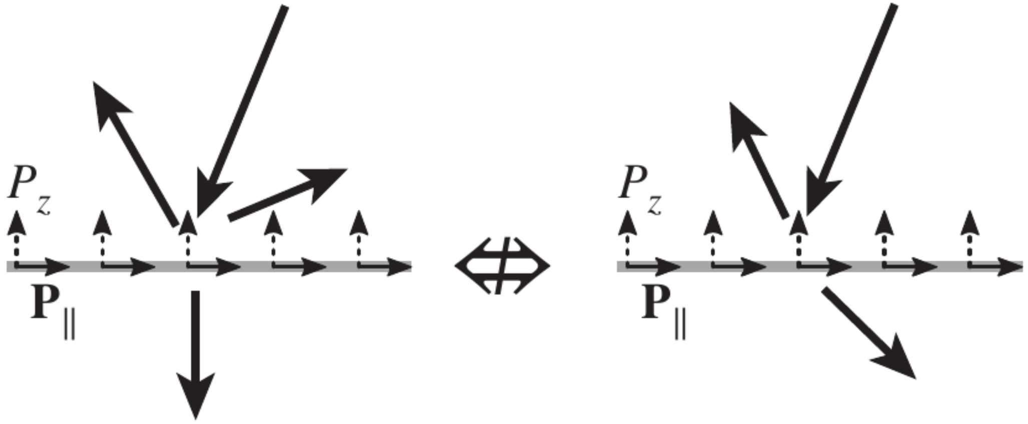
$$\hat{\mathbf{z}} \times \Delta \mathbf{H} = +j\omega \mathbf{P}_{\mathrm{eff},\|}$$
$$\hat{\mathbf{z}} \times \Delta \mathbf{E} = -j\omega \mathbf{M}_{\mathrm{eff},\|}$$

We often ignore the presence of normal polarizations for simplicity. This is usually acceptable as long as the incidence angle is fixed.

Normal polarizations can be modeled by effective tangential ones



This is valid only for a single incidence angle



GSTC at Normal Incidence

GSTC

$$\hat{\mathbf{z}} \times \Delta \mathbf{H} = +j\omega \mathbf{P}_{\parallel} - \cancel{\hat{\mathbf{n}} \times \nabla_{\parallel} M_z}$$

$$\hat{\mathbf{z}} \times \Delta \mathbf{E} = -j\omega \mathbf{M}_{\parallel} - \cancel{\frac{1}{\epsilon_0}\hat{\mathbf{n}} \times \nabla_{\parallel} P_z}$$

Bianisotropic polarization densities

$$\begin{cases} \mathbf{P} = \epsilon_0 \overline{\overline{\chi}}_{\mathrm{ee}} \cdot \mathbf{E}_{\mathrm{av}} + \dfrac{1}{c_0}\overline{\overline{\chi}}_{\mathrm{em}} \cdot \mathbf{H}_{\mathrm{av}} \\ \mathbf{M} = \overline{\overline{\chi}}_{\mathrm{mm}} \cdot \mathbf{H}_{\mathrm{av}} + \dfrac{1}{\eta_0}\overline{\overline{\chi}}_{\mathrm{me}} \cdot \mathbf{E}_{\mathrm{av}} \end{cases}$$

We assume the case of a <u>uniform</u> metasurface. The susceptibilities are <u>not</u> functions of (x,y)

$$\hat{\mathbf{z}} \times \Delta \mathbf{H} = +j\omega\epsilon_0 \overline{\overline{\chi}}_{\mathrm{ee}} \cdot \mathbf{E}_{\mathrm{av}} + jk_0 \overline{\overline{\chi}}_{\mathrm{em}} \cdot \mathbf{H}_{\mathrm{av}}$$

$$\hat{\mathbf{z}} \times \Delta \mathbf{E} = -j\omega\mu_0 \overline{\overline{\chi}}_{\mathrm{mm}} \cdot \mathbf{H}_{\mathrm{av}} - jk_0 \overline{\overline{\chi}}_{\mathrm{me}} \cdot \mathbf{E}_{\mathrm{av}}$$

Matrix formulation

$$\begin{bmatrix} \Delta H_y \\ \Delta H_x \\ \Delta E_y \\ \Delta E_x \end{bmatrix} = \begin{bmatrix} \hat{\chi}_{\mathrm{ee}}^{xx} & \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{em}}^{xx} & \hat{\chi}_{\mathrm{em}}^{xy} \\ \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{ee}}^{yy} & \hat{\chi}_{\mathrm{em}}^{yx} & \hat{\chi}_{\mathrm{em}}^{yy} \\ \hat{\chi}_{\mathrm{me}}^{xx} & \hat{\chi}_{\mathrm{me}}^{xy} & \hat{\chi}_{\mathrm{mm}}^{xx} & \hat{\chi}_{\mathrm{mm}}^{xy} \\ \hat{\chi}_{\mathrm{me}}^{yx} & \hat{\chi}_{\mathrm{me}}^{yy} & \hat{\chi}_{\mathrm{mm}}^{yx} & \hat{\chi}_{\mathrm{mm}}^{yy} \end{bmatrix} \cdot \begin{bmatrix} E_{x,\mathrm{av}} \\ E_{y,\mathrm{av}} \\ H_{x,\mathrm{av}} \\ H_{y,\mathrm{av}} \end{bmatrix}$$

where

$$\overline{\overline{\mathrm{N}}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

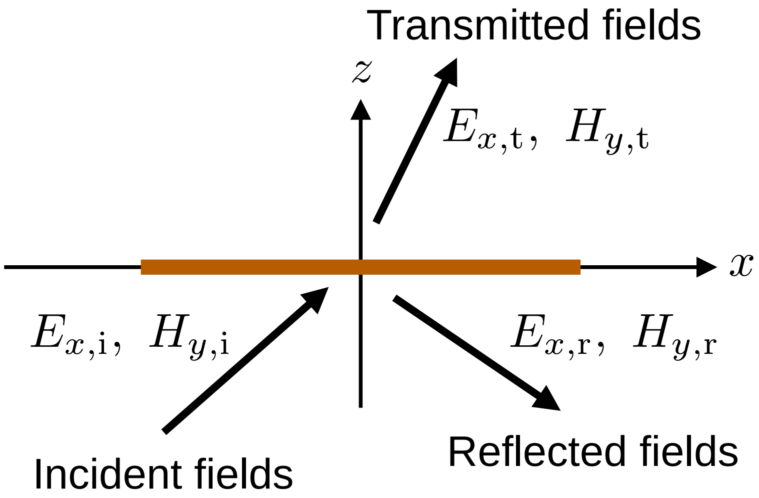
$$\begin{cases} \overline{\overline{\hat{\chi}}}_{\mathrm{ee}} = \dfrac{j}{\omega\epsilon_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{ee}} \\ \overline{\overline{\hat{\chi}}}_{\mathrm{mm}} = -\dfrac{j}{\omega\mu_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{mm}} \\ \overline{\overline{\hat{\chi}}}_{\mathrm{em}} = \dfrac{j}{k_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{em}} \\ \overline{\overline{\hat{\chi}}}_{\mathrm{me}} = -\dfrac{j}{k_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{me}} \end{cases}$$

Achouri K, “Electromagnetic metasurfaces: Theory and applications”, 2021

Example: Birefringent Metasurface

Birefringent metasurface

$$\begin{bmatrix} \Delta H_y \\ \Delta H_x \\ \Delta E_y \\ \Delta E_x \end{bmatrix} = \begin{bmatrix} \hat{\chi}_{\text{ee}}^{xx} & 0 & 0 & 0 \\ 0 & \hat{\chi}_{\text{ee}}^{yy} & 0 & 0 \\ 0 & 0 & \hat{\chi}_{\text{mm}}^{xx} & 0 \\ 0 & 0 & 0 & \hat{\chi}_{\text{mm}}^{yy} \end{bmatrix} \cdot \begin{bmatrix} E_{x,\text{av}} \\ E_{y,\text{av}} \\ H_{x,\text{av}} \\ H_{y,\text{av}} \end{bmatrix}$$



TE polarization

$$\chi_{\text{ee}}^{yy} = \frac{\Delta H_x}{j\omega\epsilon_0 E_{y,\text{av}}}$$

$$\chi_{\text{mm}}^{xx} = \frac{\Delta E_y}{j\omega\mu_0 H_{x,\text{av}}}$$

TM polarization

$$\chi_{\text{ee}}^{xx} = \frac{-\Delta H_y}{j\omega\epsilon_0 E_{x,\text{av}}}$$

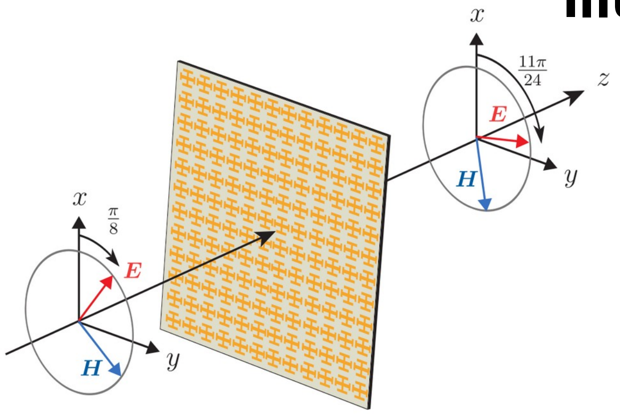
$$\chi_{\text{mm}}^{yy} = \frac{-\Delta E_x}{j\omega\mu_0 H_{y,\text{av}}}$$

$$\Delta H_y = H_{y,\text{t}} - (H_{y,\text{i}} + H_{y,\text{r}})$$

$$E_{x,\text{av}} = \frac{1}{2}\left(E_{x,\text{t}} + E_{x,\text{i}} + E_{x,\text{r}}\right)$$

Not limited to plane waves.
Valid for any fields

Illustration: Polarization Rotation



Synthesis of a π/3 reflectionless polarization rotation

$$\begin{bmatrix} \Delta H_y \\ \Delta H_x \\ \Delta E_y \\ \Delta E_x \end{bmatrix} = \begin{bmatrix} \hat{\chi}_{\mathrm{ee}}^{xx} & \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{em}}^{xx} & \hat{\chi}_{\mathrm{em}}^{xy} \\ \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{ee}}^{yy} & \hat{\chi}_{\mathrm{em}}^{yx} & \hat{\chi}_{\mathrm{em}}^{yy} \\ \hat{\chi}_{\mathrm{me}}^{xx} & \hat{\chi}_{\mathrm{me}}^{xy} & \hat{\chi}_{\mathrm{mm}}^{xx} & \hat{\chi}_{\mathrm{mm}}^{xy} \\ \hat{\chi}_{\mathrm{me}}^{yx} & \hat{\chi}_{\mathrm{me}}^{yy} & \hat{\chi}_{\mathrm{mm}}^{yx} & \hat{\chi}_{\mathrm{mm}}^{yy} \end{bmatrix} \cdot \begin{bmatrix} E_{x,\mathrm{av}} \\ E_{y,\mathrm{av}} \\ H_{x,\mathrm{av}} \\ H_{y,\mathrm{av}} \end{bmatrix}$$

Birefringent

$$\chi_{\mathrm{ee}}^{xx} = \chi_{\mathrm{mm}}^{yy} = -\frac{1.5048}{k_0} j$$

$$\chi_{\mathrm{ee}}^{yy} = \chi_{\mathrm{mm}}^{xx} = \frac{0.88063}{k_0} j$$

- Reciprocal
- Gain and loss
- Limited to spec.

Gyrotropic

$$\chi_{\mathrm{ee}}^{xy} = \chi_{\mathrm{mm}}^{xy} = -\frac{2}{\sqrt{3}k_0} j$$

$$\chi_{\mathrm{ee}}^{yx} = \chi_{\mathrm{mm}}^{yx} = \frac{2}{\sqrt{3}k_0} j$$

- Nonreciprocal
- Passive/lossless
- Any angle

Chiral

$$\chi_{\mathrm{em}}^{xx} = \chi_{\mathrm{em}}^{yy} = -\frac{2}{\sqrt{3}k_0} j$$

$$\chi_{\mathrm{me}}^{xx} = \chi_{\mathrm{me}}^{yy} = \frac{2}{\sqrt{3}k_0} j$$

- Reciprocal
- Passive/lossless
- Any angle

Moving

$$\chi_{\mathrm{em}}^{xy} = \chi_{\mathrm{me}}^{yx} = -\frac{1.5048}{k_0} j$$

$$\chi_{\mathrm{em}}^{yx} = \chi_{\mathrm{me}}^{xy} = -\frac{0.88063}{k_0} j$$

- Nonreciprocal
- Gain and loss
- Limited to spec.

Reciprocity conditions

$$\overline{\overline{\chi}}_{\mathrm{ee}} = \overline{\overline{\chi}}_{\mathrm{ee}}^{T} \qquad \overline{\overline{\chi}}_{\mathrm{mm}} = \overline{\overline{\chi}}_{\mathrm{mm}}^{T} \qquad \overline{\overline{\chi}}_{\mathrm{em}} = -\overline{\overline{\chi}}_{\mathrm{me}}^{T}$$

Lossless/gainless conditions

$$\overline{\overline{\chi}}_{\mathrm{ee}} = \overline{\overline{\chi}}_{\mathrm{ee}}^{\dagger} \qquad \overline{\overline{\chi}}_{\mathrm{mm}} = \overline{\overline{\chi}}_{\mathrm{mm}}^{\dagger} \qquad \overline{\overline{\chi}}_{\mathrm{em}} = \overline{\overline{\chi}}_{\mathrm{me}}^{\dagger}$$

Multiple Transformations

16 susceptibilities, yet only 1 set of 4 fields

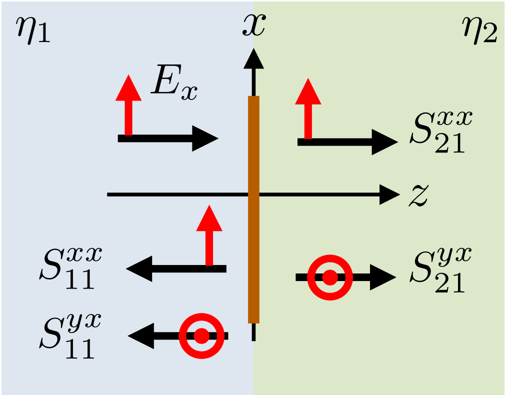
$$\begin{bmatrix} \Delta H_y \\ \Delta H_x \\ \Delta E_y \\ \Delta E_x \end{bmatrix} = \begin{bmatrix} \hat{\chi}_{\mathrm{ee}}^{xx} & \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{em}}^{xx} & \hat{\chi}_{\mathrm{em}}^{xy} \\ \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{ee}}^{yy} & \hat{\chi}_{\mathrm{em}}^{yx} & \hat{\chi}_{\mathrm{em}}^{yy} \\ \hat{\chi}_{\mathrm{me}}^{xx} & \hat{\chi}_{\mathrm{me}}^{xy} & \hat{\chi}_{\mathrm{mm}}^{xx} & \hat{\chi}_{\mathrm{mm}}^{xy} \\ \hat{\chi}_{\mathrm{me}}^{yx} & \hat{\chi}_{\mathrm{me}}^{yy} & \hat{\chi}_{\mathrm{mm}}^{yx} & \hat{\chi}_{\mathrm{mm}}^{yy} \end{bmatrix} \cdot \begin{bmatrix} E_{x,\mathrm{av}} \\ E_{y,\mathrm{av}} \\ H_{x,\mathrm{av}} \\ H_{y,\mathrm{av}} \end{bmatrix}$$

16 susceptibilities and 4 set of 4 fields

$$\begin{bmatrix} \Delta H_{y1} & \Delta H_{y2} & \Delta H_{y3} & \Delta H_{y4} \\ \Delta H_{x1} & \Delta H_{x2} & \Delta H_{x3} & \Delta H_{x4} \\ \Delta E_{y1} & \Delta E_{y2} & \Delta E_{y3} & \Delta E_{y4} \\ \Delta E_{x1} & \Delta E_{x2} & \Delta E_{x3} & \Delta E_{x4} \end{bmatrix} = \begin{bmatrix} \hat{\chi}_{\mathrm{ee}}^{xx} & \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{em}}^{xx} & \hat{\chi}_{\mathrm{em}}^{xy} \\ \hat{\chi}_{\mathrm{ee}}^{xy} & \hat{\chi}_{\mathrm{ee}}^{yy} & \hat{\chi}_{\mathrm{em}}^{yx} & \hat{\chi}_{\mathrm{em}}^{yy} \\ \hat{\chi}_{\mathrm{me}}^{xx} & \hat{\chi}_{\mathrm{me}}^{xy} & \hat{\chi}_{\mathrm{mm}}^{xx} & \hat{\chi}_{\mathrm{mm}}^{xy} \\ \hat{\chi}_{\mathrm{me}}^{yx} & \hat{\chi}_{\mathrm{me}}^{yy} & \hat{\chi}_{\mathrm{mm}}^{yx} & \hat{\chi}_{\mathrm{mm}}^{yy} \end{bmatrix} \cdot \begin{bmatrix} E_{x1,\mathrm{av}} & E_{x2,\mathrm{av}} & E_{x3,\mathrm{av}} & E_{x4,\mathrm{av}} \\ E_{y1,\mathrm{av}} & E_{y2,\mathrm{av}} & E_{y3,\mathrm{av}} & E_{y4,\mathrm{av}} \\ H_{x1,\mathrm{av}} & H_{x2,\mathrm{av}} & H_{x3,\mathrm{av}} & H_{x4,\mathrm{av}} \\ H_{y1,\mathrm{av}} & H_{y2,\mathrm{av}} & H_{y3,\mathrm{av}} & H_{y4,\mathrm{av}} \end{bmatrix}$$

$$\overline{\overline{\Delta}} = \overline{\overline{\hat{\chi}}} \cdot \overline{\overline{\mathrm{A}}}_{\mathrm{v}}$$

Relation to Scattering Parameters



Incident fields	Reflected fields	Transmitted fields
$\mathbf{E}_\mathrm{i} = \hat{\mathbf{x}}$	$\mathbf{E}_\mathrm{r} = S_{11}^{xx}\hat{\mathbf{x}} + S_{11}^{xy}\hat{\mathbf{y}}$	$\mathbf{E}_\mathrm{t} = S_{21}^{xx}\hat{\mathbf{x}} + S_{21}^{xy}\hat{\mathbf{y}}$
$\mathbf{H}_\mathrm{i} = \frac{1}{\eta_1}\hat{\mathbf{y}}$	$\mathbf{H}_\mathrm{r} = \frac{1}{\eta_1}\left(S_{11}^{yx}\hat{\mathbf{x}} - S_{11}^{xx}\hat{\mathbf{y}}\right)$	$\mathbf{H}_\mathrm{t} = \frac{1}{\eta_2}\left(-S_{21}^{yx}\hat{\mathbf{x}} + S_{21}^{xx}\hat{\mathbf{y}}\right)$

$$\boxed{\overline{\overline{\Delta}} = \overline{\overline{\hat{\chi}}} \cdot \overline{\overline{\mathrm{A}}}_\mathrm{v}} \xrightarrow{\text{The susceptibilities are found using}} \boxed{\overline{\overline{\hat{\chi}}} = \overline{\overline{\mathrm{A}}}_\mathrm{v}^{-1} \cdot \overline{\overline{\Delta}}}$$

The 16 tangential susceptibilities may be found by numerically computing the scattering parameters for x and y polarizations and by illuminating the metasurface on both sides.

$$\overline{\overline{\Delta}} = \begin{bmatrix} -\overline{\overline{\mathrm{N}}}/\eta_1 + \overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{11}/\eta_1 + \overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{21}/\eta_2 & -\overline{\overline{\mathrm{N}}}/\eta_2 + \overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{12}/\eta_1 + \overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{22}/\eta_2 \\ -\overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{N}}} - \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{11} + \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{21} & \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{N}}} - \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{12} + \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{N}}}\cdot\overline{\overline{\mathrm{S}}}_{22} \end{bmatrix}$$

$$\overline{\overline{\mathrm{A}}}_\mathrm{v} = \frac{1}{2}\begin{bmatrix} \overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{S}}}_{11} + \overline{\overline{\mathrm{S}}}_{21} & \overline{\overline{\mathrm{I}}} + \overline{\overline{\mathrm{S}}}_{12} + \overline{\overline{\mathrm{S}}}_{22} \\ \overline{\overline{\mathrm{J}}}/\eta_1 - \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{S}}}_{11}/\eta_1 + \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{S}}}_{21}/\eta_2 & -\overline{\overline{\mathrm{J}}}/\eta_2 - \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{S}}}_{12}/\eta_1 + \overline{\overline{\mathrm{J}}}\cdot\overline{\overline{\mathrm{S}}}_{22}/\eta_2 \end{bmatrix}$$

$$\overline{\overline{\mathrm{S}}}_{ab} = \begin{bmatrix} S_{ab}^{xx} & S_{ab}^{xy} \\ S_{ab}^{yx} & S_{ab}^{yy} \end{bmatrix} \qquad \overline{\overline{\mathrm{I}}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \overline{\overline{\mathrm{J}}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \overline{\overline{\mathrm{N}}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Scattering Parameters from Susceptibilities

Knowing the susceptibilities allows us to compute the scattering parameters

$$\boxed{\overline{\overline{\mathrm{S}}} = \begin{bmatrix} \overline{\overline{\mathrm{S}}}_{11} & \overline{\overline{\mathrm{S}}}_{12} \\ \overline{\overline{\mathrm{S}}}_{21} & \overline{\overline{\mathrm{S}}}_{22} \end{bmatrix} = \overline{\overline{\mathrm{M}}}_1^{-1} \cdot \overline{\overline{\mathrm{M}}}_2}$$

$$\overline{\overline{\mathrm{M}}}_1 = \begin{bmatrix} \overline{\overline{\mathrm{N}}}/\eta_1 - \overline{\overline{\hat{\chi}}}_{\mathrm{ee}}/2 + \overline{\overline{\hat{\chi}}}_{\mathrm{em}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_1) & \overline{\overline{\mathrm{N}}}/\eta_2 - \overline{\overline{\hat{\chi}}}_{\mathrm{ee}}/2 - \overline{\overline{\hat{\chi}}}_{\mathrm{em}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_2) \\ -\overline{\overline{\mathrm{J}}} \cdot \overline{\overline{\mathrm{N}}} - \overline{\overline{\hat{\chi}}}_{\mathrm{me}}/2 + \overline{\overline{\hat{\chi}}}_{\mathrm{mm}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_1) & \overline{\overline{\mathrm{J}}} \cdot \overline{\overline{\mathrm{N}}} - \overline{\overline{\hat{\chi}}}_{\mathrm{me}}/2 - \overline{\overline{\hat{\chi}}}_{\mathrm{mm}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_2) \end{bmatrix}$$

$$\overline{\overline{\mathrm{M}}}_2 = \begin{bmatrix} \overline{\overline{\hat{\chi}}}_{\mathrm{ee}}/2 + \overline{\overline{\mathrm{N}}}/\eta_1 + \overline{\overline{\hat{\chi}}}_{\mathrm{em}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_1) & \overline{\overline{\hat{\chi}}}_{\mathrm{ee}}/2 + \overline{\overline{\mathrm{N}}}/\eta_2 - \overline{\overline{\hat{\chi}}}_{\mathrm{em}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_2) \\ \overline{\overline{\hat{\chi}}}_{\mathrm{me}}/2 + \overline{\overline{\mathrm{J}}} \cdot \overline{\overline{\mathrm{N}}} + \overline{\overline{\hat{\chi}}}_{\mathrm{mm}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_1) & \overline{\overline{\hat{\chi}}}_{\mathrm{me}}/2 - \overline{\overline{\mathrm{J}}} \cdot \overline{\overline{\mathrm{N}}} - \overline{\overline{\hat{\chi}}}_{\mathrm{mm}} \cdot \overline{\overline{\mathrm{J}}}/(2\eta_2) \end{bmatrix}$$

$$\begin{cases} \overline{\overline{\hat{\chi}}}_{\mathrm{ee}} = \dfrac{j}{\omega\epsilon_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{ee}} \\ \overline{\overline{\hat{\chi}}}_{\mathrm{mm}} = -\dfrac{j}{\omega\mu_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{mm}} \\ \overline{\overline{\hat{\chi}}}_{\mathrm{em}} = \dfrac{j}{k_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{em}} \\ \overline{\overline{\hat{\chi}}}_{\mathrm{me}} = -\dfrac{j}{k_0}\overline{\overline{\mathrm{N}}} \cdot \overline{\overline{\chi}}_{\mathrm{me}} \end{cases}$$

$$\overline{\overline{\mathrm{S}}}_{ab} = \begin{bmatrix} S_{ab}^{xx} & S_{ab}^{xy} \\ S_{ab}^{yx} & S_{ab}^{yy} \end{bmatrix} \qquad \overline{\overline{\mathrm{J}}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \overline{\overline{\mathrm{N}}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Scattering Parameters of a Birefringent Metasurface

$$S_{21}^{xx} = S_{12}^{xx} = t_x \qquad S_{21}^{yy} = S_{12}^{yy} = t_y \qquad S_{11}^{xx} = S_{22}^{xx} = r_x \qquad S_{11}^{yy} = S_{22}^{yy} = r_y$$

TE polarization

$$\chi_{\mathrm{ee}}^{yy} = \frac{\Delta H_x}{j\omega\epsilon_0 E_{y,\mathrm{av}}}$$

$$\chi_{\mathrm{mm}}^{xx} = \frac{\Delta E_y}{j\omega\mu_0 H_{x,\mathrm{av}}}$$

$\rightarrow$

$$\chi_{\mathrm{ee}}^{xx} = \frac{2j\left(t_x + r_x - 1\right)}{k\left(t_x + r_x + 1\right)}$$

$$\chi_{\mathrm{mm}}^{yy} = \frac{2j\left(t_x - r_x - 1\right)}{k\left(t_x - r_x + 1\right)}$$

$\leftrightarrow$

$$t_x = \frac{4 + \chi_{\mathrm{ee}}^{xx}\chi_{\mathrm{mm}}^{yy}k^2}{\left(2 + jk\chi_{\mathrm{ee}}^{xx}\right)\left(2 + jk\chi_{\mathrm{mm}}^{yy}\right)}$$

$$r_x = \frac{2jk\left(\chi_{\mathrm{mm}}^{yy} - \chi_{\mathrm{ee}}^{xx}\right)}{\left(2 + jk\chi_{\mathrm{ee}}^{xx}\right)\left(2 + jk\chi_{\mathrm{mm}}^{yy}\right)}$$

TM polarization

$$\chi_{\mathrm{ee}}^{xx} = \frac{-\Delta H_y}{j\omega\epsilon_0 E_{x,\mathrm{av}}}$$

$$\chi_{\mathrm{mm}}^{yy} = \frac{-\Delta E_x}{j\omega\mu_0 H_{y,\mathrm{av}}}$$

$\rightarrow$

$$\chi_{\mathrm{ee}}^{yy} = \frac{2j\left(t_y + r_y - 1\right)}{k\left(t_y + r_y + 1\right)}$$

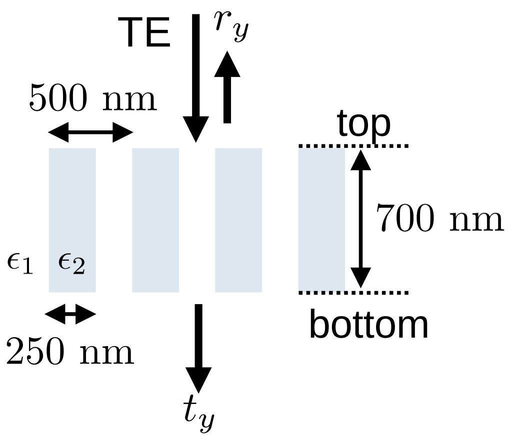
$$\chi_{\mathrm{mm}}^{xx} = \frac{2j\left(t_y - r_y - 1\right)}{k\left(t_y - r_y + 1\right)}$$

$\leftrightarrow$

$$t_y = \frac{4 + \chi_{\mathrm{ee}}^{yy}\chi_{\mathrm{mm}}^{xx}k^2}{\left(2 + jk\chi_{\mathrm{ee}}^{yy}\right)\left(2 + jk\chi_{\mathrm{mm}}^{xx}\right)}$$

$$r_y = \frac{2jk\left(\chi_{\mathrm{mm}}^{xx} - \chi_{\mathrm{ee}}^{yy}\right)}{\left(2 + jk\chi_{\mathrm{ee}}^{yy}\right)\left(2 + jk\chi_{\mathrm{mm}}^{xx}\right)}$$

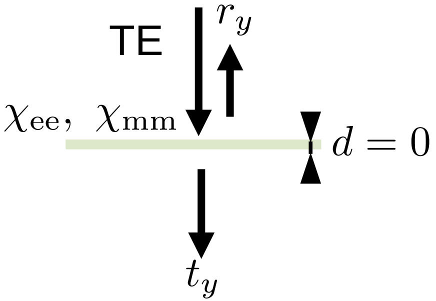
Slab vs GSTC Homogenization

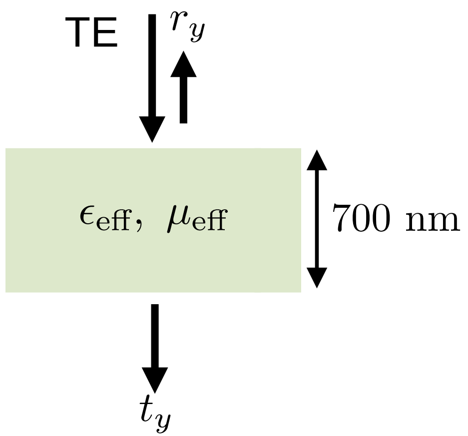


ridge parameters

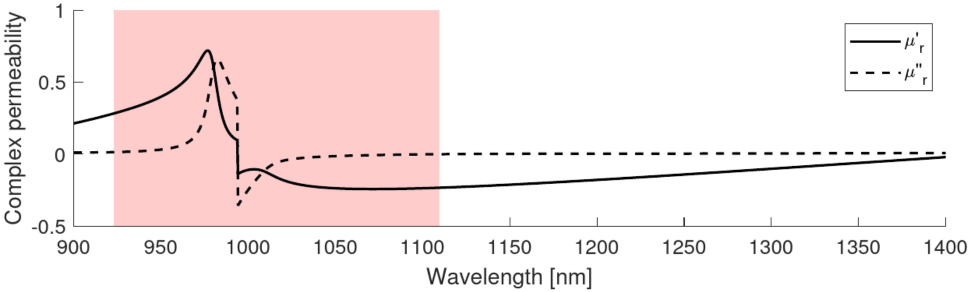
$\epsilon_2 = 6$

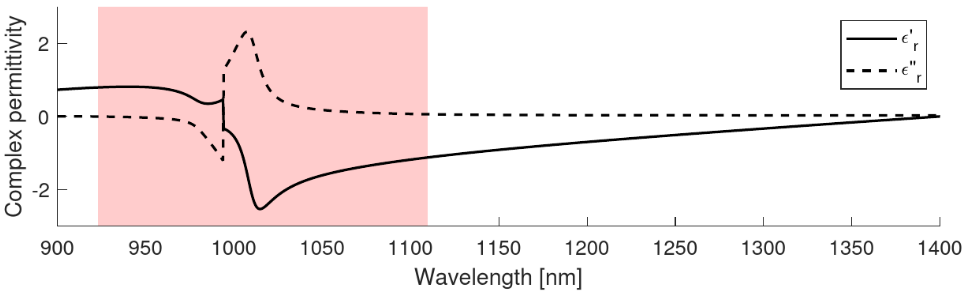
$\sigma_2 = 1000$ S/m





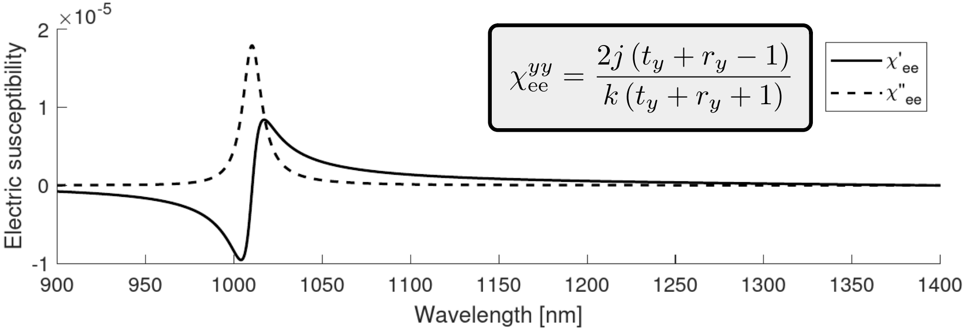
Slab homogenization



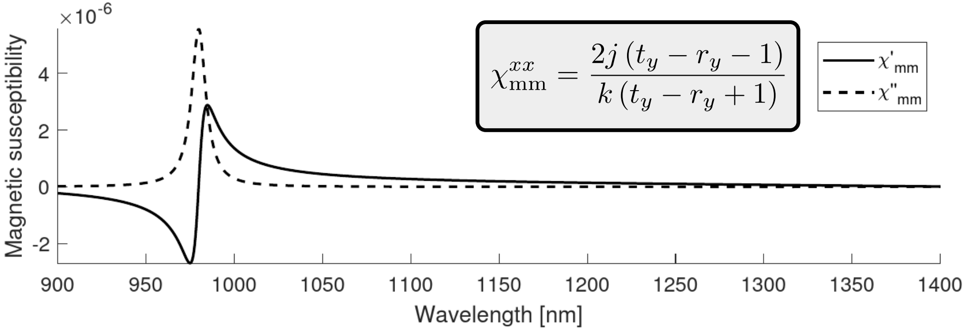


GSTC homogenization

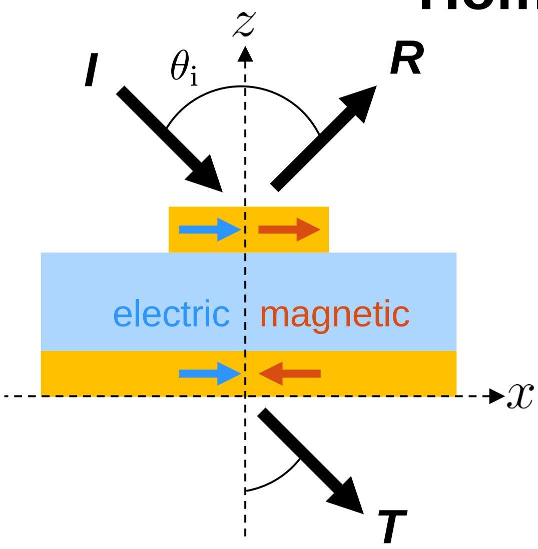
$$\chi_{\mathrm{ee}}^{yy} = \frac{2j\left(t_y + r_y - 1\right)}{k\left(t_y + r_y + 1\right)}$$



$$\chi_{\mathrm{mm}}^{xx} = \frac{2j\left(t_y - r_y - 1\right)}{k\left(t_y - r_y + 1\right)}$$



Homogenization and Scattering Prediction

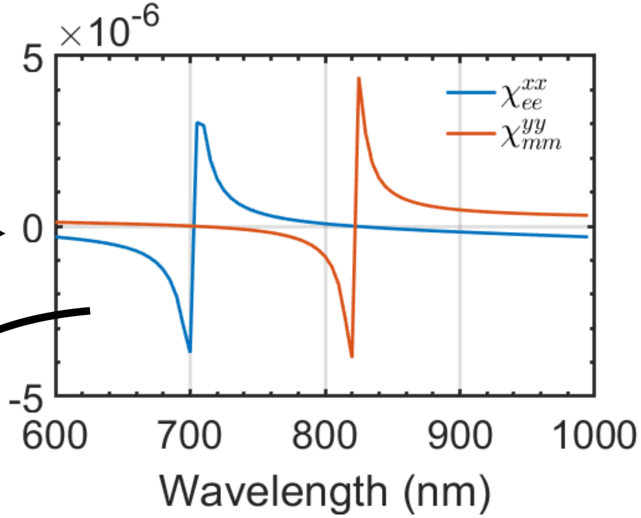


Homogenization

$$\chi_\mathrm{ee}^{xx} = \frac{2j}{k_z}\left(\frac{t_\mathrm{TM} + r_\mathrm{TM} - 1}{t_\mathrm{TM} + r_\mathrm{TM} + 1}\right)$$

$$\chi_\mathrm{mm}^{yy} = \frac{2jk_z}{k^2}\left(\frac{t_\mathrm{TM} - r_\mathrm{TM} - 1}{t_\mathrm{TM} - r_\mathrm{TM} + 1}\right)$$

$\theta_\mathrm{i} = 0°$

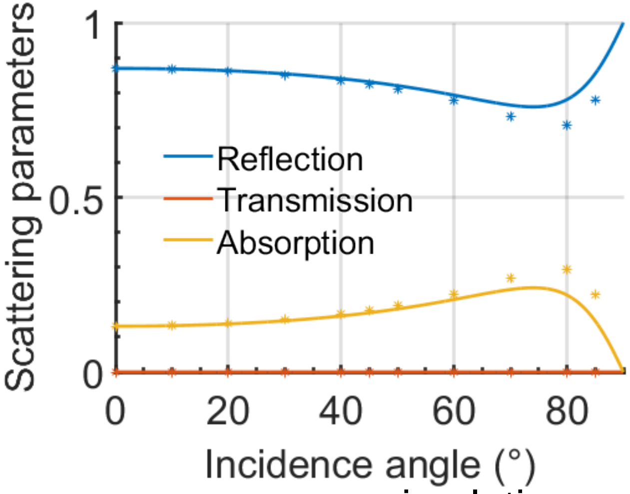


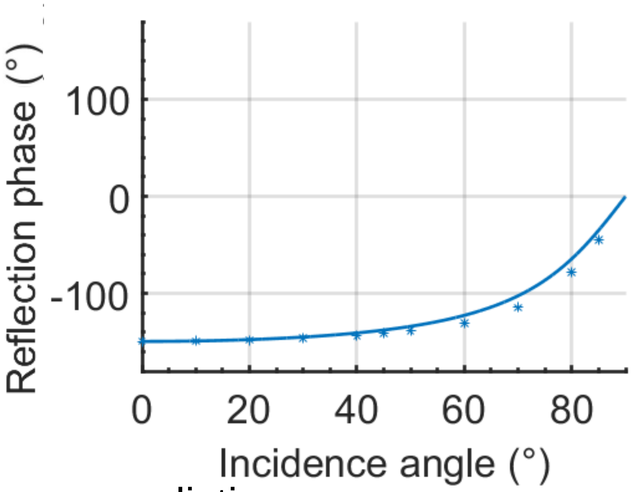
Predict angular scattering using homogenized susceptibilities

Scattering prediction

$$t_\mathrm{TM} = \frac{k_z\left(4 + k^2\chi_\mathrm{ee}^{xx}\chi_\mathrm{mm}^{yy}\right)}{\left(2 + jk_z\chi_\mathrm{ee}^{xx}\right)\left(2k_z + jk^2\chi_\mathrm{mm}^{yy}\right)}$$

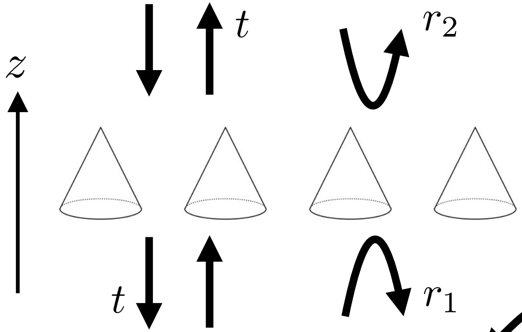
$$r_\mathrm{TM} = \frac{2j\left(k^2\chi_\mathrm{mm}^{yy} - k_z^2\chi_\mathrm{ee}^{xx}\right)}{\left(2 + jk_z\chi_\mathrm{ee}^{xx}\right)\left(2k_z + jk^2\chi_\mathrm{mm}^{yy}\right)}$$





* = simulation — = prediction

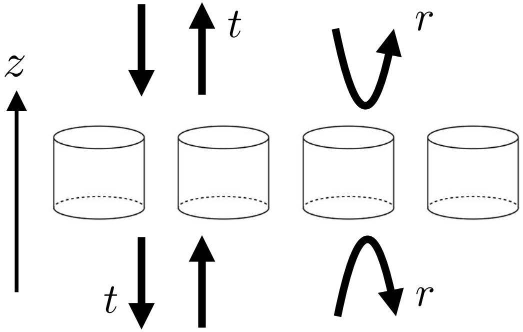
Symmetric vs Asymmetric Structures



By reciprocity, the transmission is identical from both sides

Different reflection coefficients

Represents the asymmetry of the structure



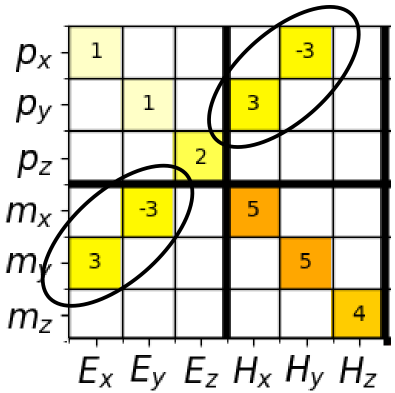
$$r_{1,2} = \frac{2jk(\chi_{\mathrm{mm}} - \chi_{\mathrm{ee}} \pm 2\chi_{\mathrm{em}})}{(2 + jk\chi_{\mathrm{ee}})(2 + jk\chi_{\mathrm{mm}}) - k^2\chi_{\mathrm{em}}^2}$$

$$t = \frac{4 + \chi_{\mathrm{mm}}\chi_{\mathrm{ee}}k^2 + k^2\chi_{\mathrm{em}}^2}{(2 + jk\chi_{\mathrm{ee}})(2 + jk\chi_{\mathrm{mm}}) - k^2\chi_{\mathrm{em}}^2}$$

$$\chi_{\mathrm{ee}} = \frac{2j}{k}\left[\frac{(r_1 - 1)(r_2 - 1) - t^2}{r_1 r_2 - (1 + t)^2}\right]$$

$$\chi_{\mathrm{mm}} = \frac{2j}{k}\left[\frac{(r_1 + 1)(r_2 + 1) - t^2}{r_1 r_2 - (1 + t)^2}\right]$$

$$\chi_{\mathrm{em}} = \frac{2j}{k}\left[\frac{r_2 - r_1}{r_1 r_2 - (1 + t)^2}\right]$$



Structure with broken σ_z symmetry

$$r = \frac{2jk\left(\chi_{\mathrm{mm}} - \chi_{\mathrm{ee}}\right)}{\left(2 + jk\chi_{\mathrm{ee}}\right)\left(2 + jk\chi_{\mathrm{mm}}\right)}$$

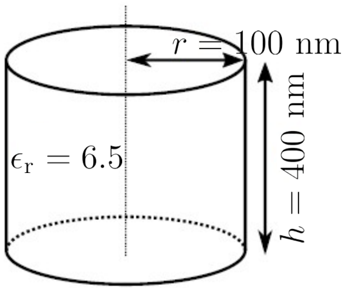
$$t = \frac{4 + \chi_{\mathrm{ee}}\chi_{\mathrm{mm}}k^2}{\left(2 + jk\chi_{\mathrm{ee}}\right)\left(2 + jk\chi_{\mathrm{mm}}\right)}$$

$$\chi_{\mathrm{ee}} = \frac{2j\left(t + r - 1\right)}{k\left(t + r + 1\right)}$$

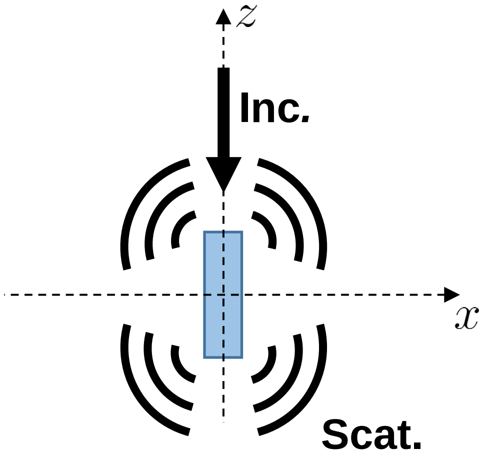
$$\chi_{\mathrm{mm}} = \frac{2j\left(t - r - 1\right)}{k\left(t - r + 1\right)}$$

Presence of Higher-Order Multipolar Responses in Metasurfaces

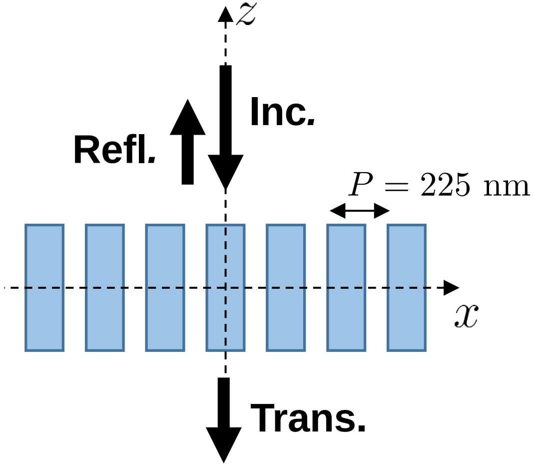
Particle parameters

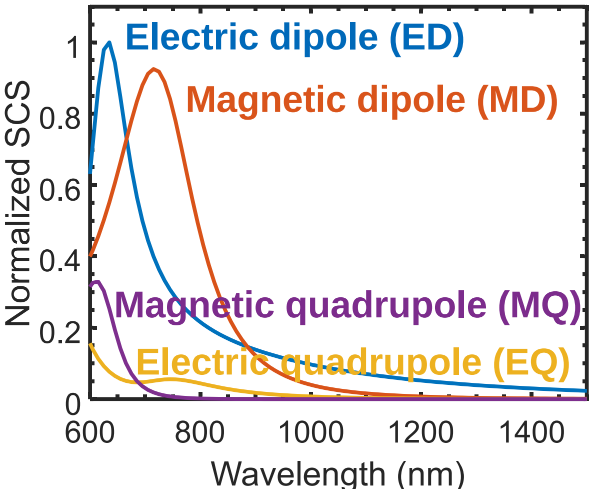


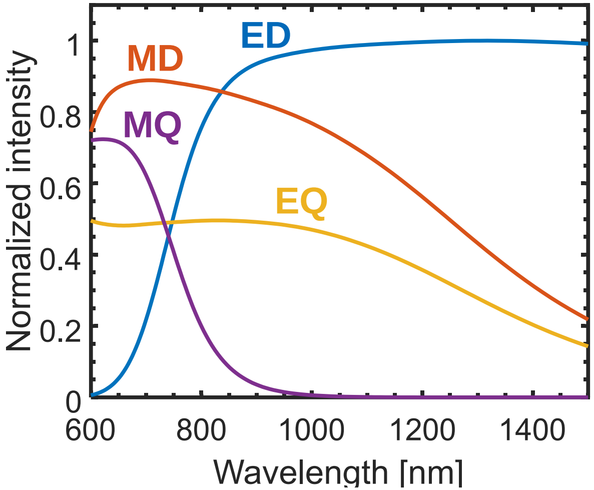
Isolated scatterer



Metasurface





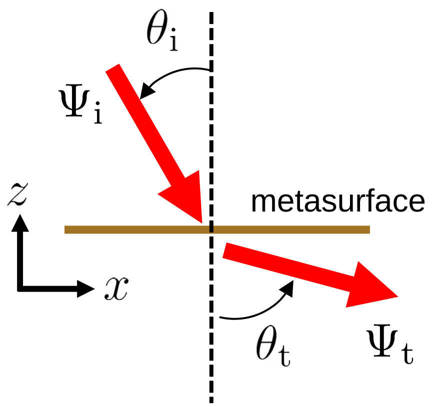


What Have We Learned So Far….

- Thin (compared to the wavelength) electromagnetic structures may be conveniently modeled using boundary conditions rather than thin slabs. This makes the homogenization process easier since we do not have to take into account the multiple scattering within the slab.
- Conventional boundary conditions are not well suited for that task since they do not include normal polarization currents, which are easily excitable with metamaterial structures (e.g., array of rings leading to normal magnetic moments).
- The Generalized Sheet Transition Conditions (GSTC) accommodate that issue by including normal polarizations.
- The GSTC directly connect the metasurface susceptibilities to the difference and average of the fields. This means that they can be used to either field the susceptibilities in terms of specified fields or vice versa.
- Very often we either ignore normal polarizations or embed them inside effective purely tangential polarizations. This approach works well as long as we are not trying to model the angular scattering response of the metasurface.

Refraction with Metasurfaces

Light Refraction with Metasurfaces



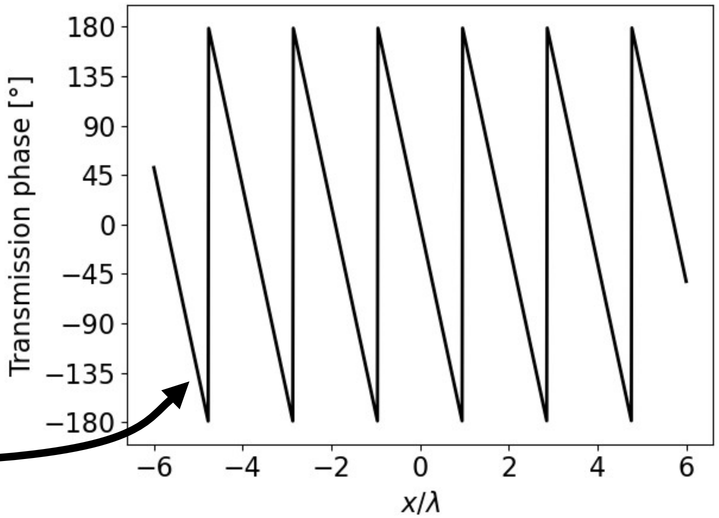
Intuitive design approach based on optical transfer function

$$\boxed{\Psi_\mathrm{t} = T \cdot \Psi_\mathrm{i}}$$

$$\Psi_\mathrm{t} = e^{-jk_{x,\mathrm{t}}x} \qquad \Psi_\mathrm{i} = e^{-jk_{x,\mathrm{i}}x}$$

$$\boxed{T = e^{-j(k_{x,\mathrm{t}} - k_{x,\mathrm{i}})x}}$$

Example with θ_i = 20° and θ_t = 60°



This approach does not account for the polarization of the wave! It is not a rigorous approach but works decently well. It is the same as the "Generalized law of refraction"

$$\boxed{\sin\theta_\mathrm{i} - \sin\theta_\mathrm{t} = \frac{1}{k}\frac{d\Phi}{dx}}$$

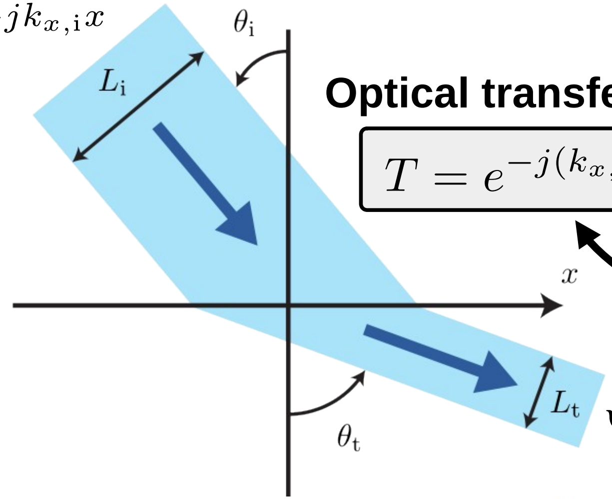
phase function of the transmission

Notice that the periodicity is larger than the wavelength! This is a diffraction grating optimized to send the incident energy into a specified diffraction order.

We need 2π phase variations

Efficiency of Refractive Metasurfaces

$$\Psi_{\mathrm{i}} = e^{-jk_{x,\mathrm{i}}x}$$



$$\Psi_{\mathrm{t}} = e^{-jk_{x,\mathrm{t}}x}$$

Optical transfer function

$$\boxed{T = e^{-j(k_{x,\mathrm{t}} - k_{x,\mathrm{i}})x}}$$

Incompatible with power conservation

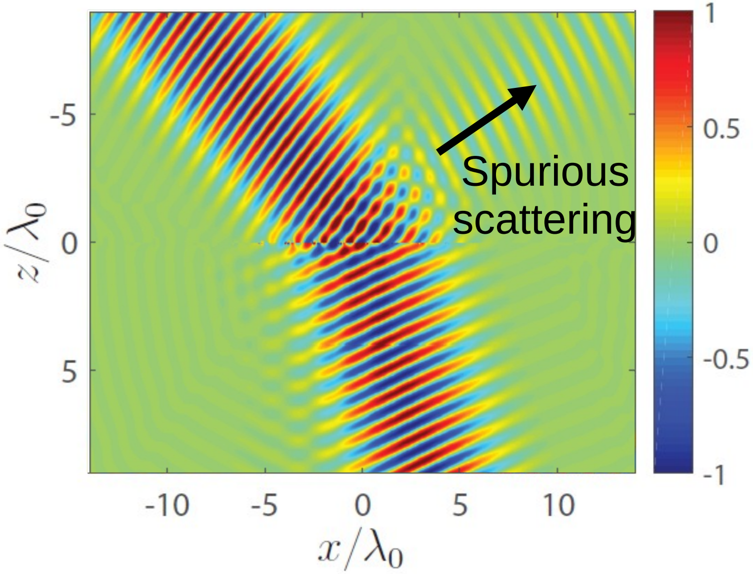
Power conservation

$$P_{\mathrm{i}} = P_{\mathrm{t}}$$

$$S_{\mathrm{i}} L_{\mathrm{i}} = S_{\mathrm{t}} L_{\mathrm{t}}$$

$$|\mathbf{E}_{\mathrm{i}}|^2 L_{\mathrm{i}} = |\mathbf{E}_{\mathrm{t}}|^2 L_{\mathrm{t}}$$

$$L_{\mathrm{i}} > L_{\mathrm{t}} \longrightarrow |\mathbf{E}_{\mathrm{t}}| > |\mathbf{E}_{\mathrm{i}}|$$



Wave impedance

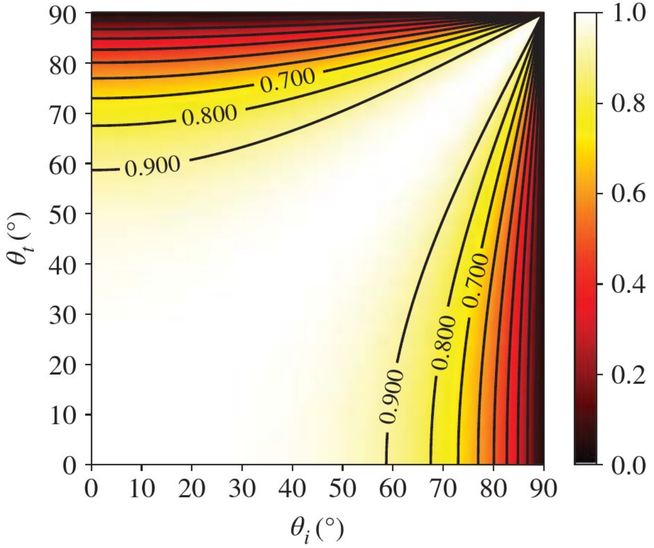
$$Z_{\mathrm{i}} = \frac{\eta}{\cos\theta_{\mathrm{i}}}$$

$$Z_{\mathrm{t}} = \frac{\eta}{\cos\theta_{\mathrm{t}}}$$

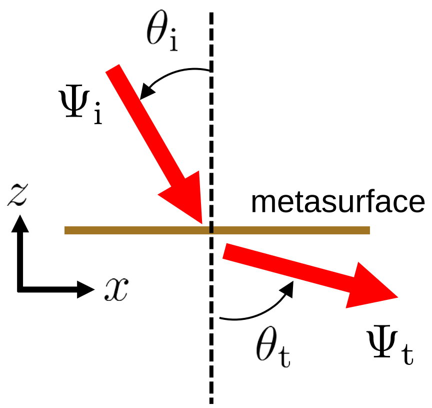
Efficiency: 1 - R

$$\zeta = 1 - \left(\frac{Z_{\mathrm{t}} - Z_{\mathrm{i}}}{Z_{\mathrm{t}} + Z_{\mathrm{i}}}\right)^2$$

Refraction efficiency



Refraction Using GSTC



$$\boxed{\begin{aligned}\chi_\mathrm{ee}^{xx} &= \frac{-\Delta H_y}{j\omega\epsilon_0 E_{x,\mathrm{av}}}\\ \chi_\mathrm{mm}^{yy} &= \frac{-\Delta E_x}{j\omega\mu_0 H_{y,\mathrm{av}}}\end{aligned}}$$

$$\mathbf{E}_\mathrm{r} = \mathbf{H}_\mathrm{r} = 0$$

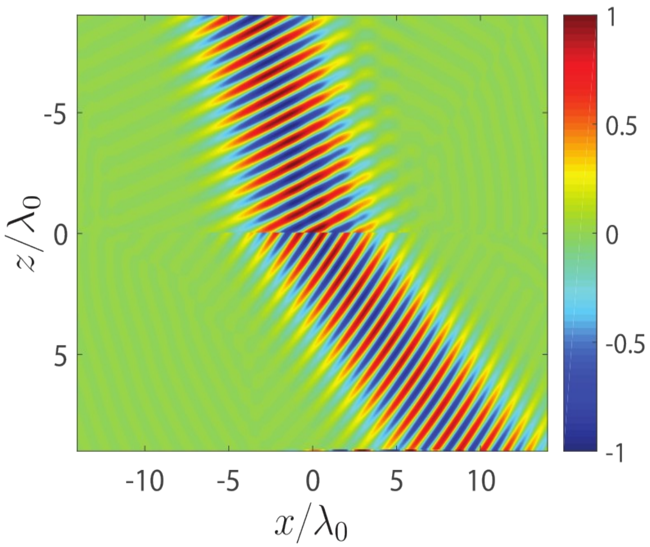
TM polarized fields

$$\begin{cases}\mathbf{E}_a = A_a\left(\hat{\mathbf{x}}\frac{k_{z,a}}{k} - \hat{\mathbf{z}}\frac{k_{x,a}}{k}\right)e^{-jk_{x,a}x}\\ \mathbf{H}_a = \frac{A_a}{\eta}\hat{\mathbf{y}}e^{-jk_{x,a}x}\end{cases}$$

for a = {i,t}

$$\chi_\mathrm{ee}^{xx} = 2j\left(\frac{A_\mathrm{t}e^{jk_{x,\mathrm{i}}x} - A_\mathrm{i}e^{jk_{x,\mathrm{t}}x}}{A_\mathrm{i}k_{z,\mathrm{i}}e^{jk_{x,\mathrm{t}}x} + A_\mathrm{t}k_{z,\mathrm{t}}e^{jk_{x,\mathrm{i}}x}}\right)$$

$$\chi_\mathrm{mm}^{yy} = \frac{2j}{k^2}\left(\frac{A_\mathrm{t}k_{z,\mathrm{t}}e^{jk_{x,\mathrm{i}}x} - A_\mathrm{i}k_{z,\mathrm{i}}e^{jk_{x,\mathrm{t}}x}}{A_\mathrm{t}e^{jk_{x,\mathrm{i}}x} + A_\mathrm{i}e^{jk_{x,\mathrm{t}}x}}\right)$$

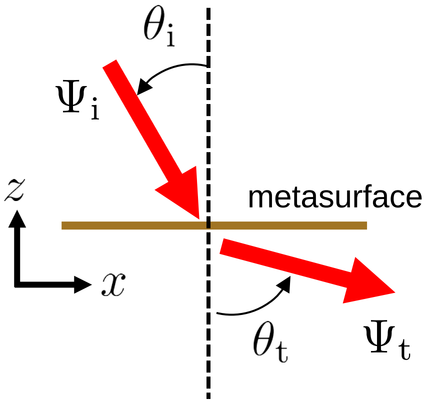


Power conservation requires

$$A_\mathrm{t} = A_\mathrm{i}\sqrt{\frac{\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{t}}}$$

Note that the susceptibilities are <u>complex</u> functions of position

Power Analysis



$$\chi_\mathrm{ee}^{xx} = 2j\left(\frac{A_\mathrm{t}e^{jk_{x,\mathrm{i}}x} - A_\mathrm{i}e^{jk_{x,\mathrm{t}}x}}{A_\mathrm{i}k_{z,\mathrm{i}}e^{jk_{x,\mathrm{t}}x} + A_\mathrm{t}k_{z,\mathrm{t}}e^{jk_{x,\mathrm{i}}x}}\right)$$

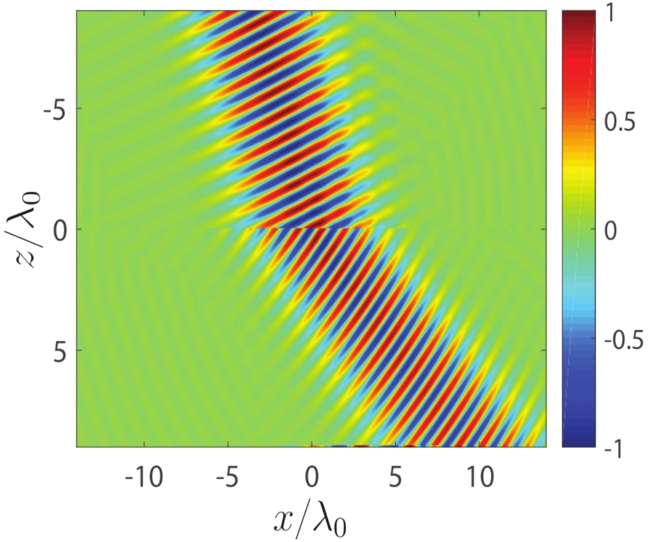
$$\chi_\mathrm{mm}^{yy} = \frac{2j}{k^2}\left(\frac{A_\mathrm{t}k_{z,\mathrm{t}}e^{jk_{x,\mathrm{i}}x} - A_\mathrm{i}k_{z,\mathrm{i}}e^{jk_{x,\mathrm{t}}x}}{A_\mathrm{t}e^{jk_{x,\mathrm{i}}x} + A_\mathrm{i}e^{jk_{x,\mathrm{t}}x}}\right)$$

Divergence of Poynting vector

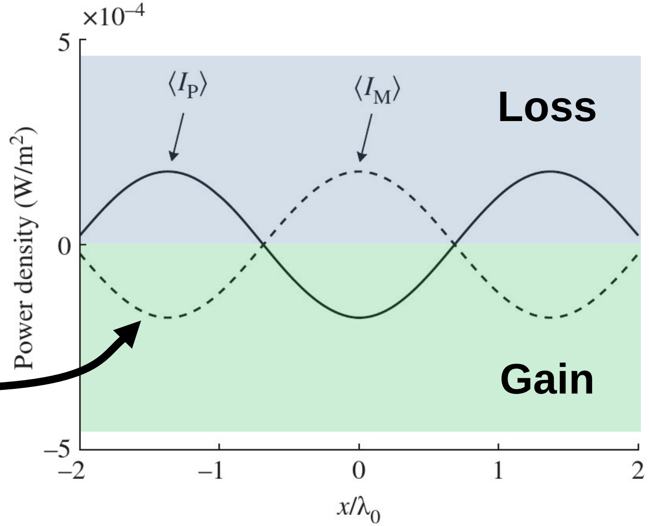
$$\boxed{\nabla\cdot\langle\mathbf{S}\rangle = -\langle I_\mathrm{P}\rangle - \langle I_\mathrm{M}\rangle}$$

$$\langle I_\mathrm{P}\rangle = \frac{\omega\epsilon_0}{4}\mathrm{Im}\left[\mathbf{E}^*\cdot\left(\overline{\overline{\chi}}_\mathrm{ee}^\dagger - \overline{\overline{\chi}}_\mathrm{ee}\right)\cdot\mathbf{E}\right] = -\frac{\omega\epsilon_0}{2}|E_{x,\mathrm{av}}|^2\mathrm{Im}\left[\chi_\mathrm{ee}^{xx}\right]$$

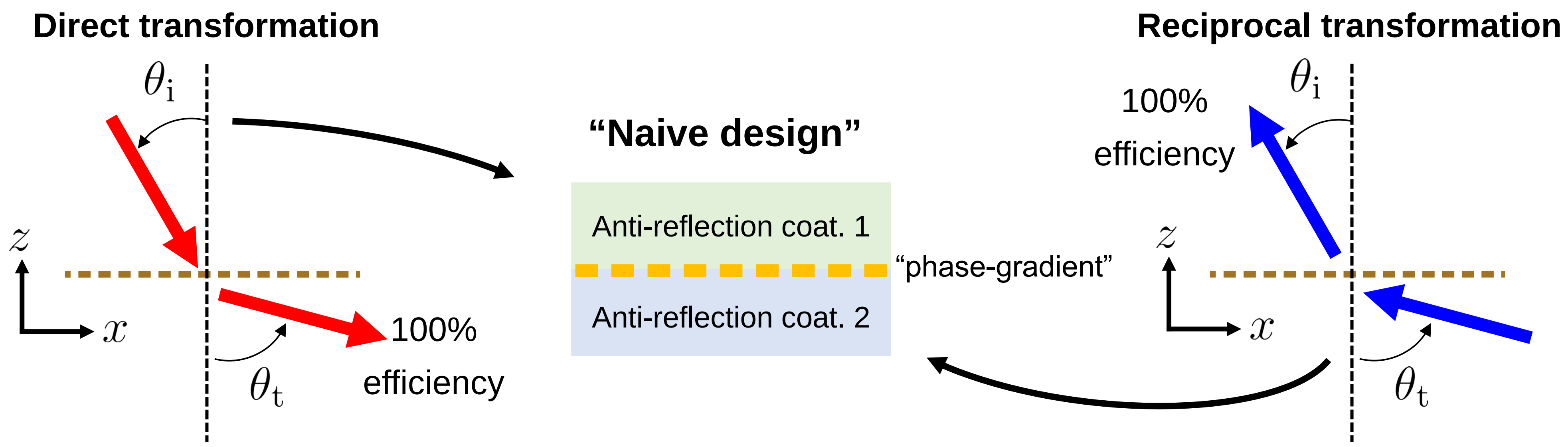
$$\langle I_\mathrm{M}\rangle = \frac{\omega\mu_0}{4}\mathrm{Im}\left[\mathbf{H}^*\cdot\left(\overline{\overline{\chi}}_\mathrm{mm}^\dagger - \overline{\overline{\chi}}_\mathrm{mm}\right)\cdot\mathbf{H}\right] = -\frac{\omega\mu_0}{2}|H_{y,\mathrm{av}}|^2\mathrm{Im}\left[\chi_\mathrm{mm}^{yy}\right]$$



The overall metasurface fully refracts the incident energy (power conservation). However, this is achieved by a periodically varying and self-canceling electric/magnetic loss/gain.



How to Achieve Lossless, Gainless and Perfect Refraction

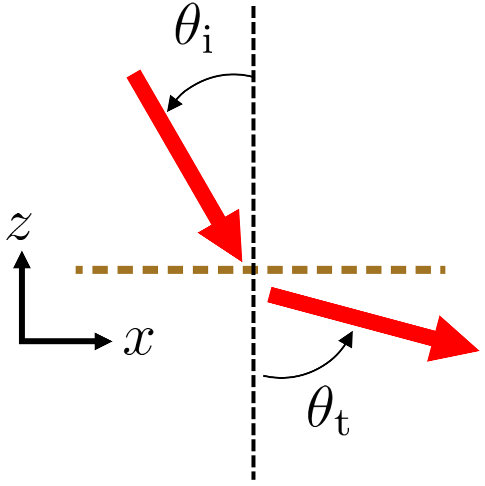


$$\begin{bmatrix} \Delta H_{y1} & \Delta H_{y2} \\ \Delta E_{x1} & \Delta E_{x2} \\ \Delta E_{y1} & \Delta E_{y2} \\ \Delta E_{x1} & \Delta E_{x2} \end{bmatrix} = \begin{bmatrix} \hat{\chi}_{\rm ee}^{xx} & 0 & 0 & \hat{\chi}_{\rm em}^{xy} \\ 0 & \hat{\chi}_{\rm ee}^{yy} & \hat{\chi}_{\rm em}^{yx} & 0 \\ 0 & \hat{\chi}_{\rm me}^{xy} & \hat{\chi}_{\rm mm}^{xx} & 0 \\ \hat{\chi}_{\rm me}^{yx} & 0 & 0 & \hat{\chi}_{\rm mm}^{yy} \end{bmatrix} \cdot \begin{bmatrix} E_{x1,\rm av} & E_{x2,\rm av} \\ E_{y1,\rm av} & E_{y2,\rm av} \\ H_{x1,\rm av} & H_{x2,\rm av} \\ H_{y1,\rm av} & H_{y2,\rm av} \end{bmatrix}$$

Terms related to broken σ_z symmetry

How to Achieve Lossless, Gainless, Perfect Refraction

Direct transformation



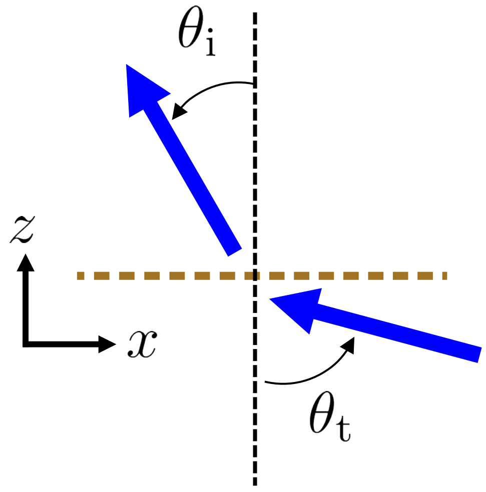
Solution for TM waves

$$\chi_\mathrm{ee}^{xx} = \frac{4\sin(\alpha x)}{\beta\cos(\alpha x) + \sqrt{\beta^2 - \gamma^2}}$$

$$\chi_\mathrm{mm}^{yy} = \frac{\beta^2 - \gamma^2}{4k^2}\frac{4\sin(\alpha x)}{\beta\cos(\alpha x) + \sqrt{\beta^2 - \gamma^2}}$$

$$\chi_\mathrm{em}^{xy} = -\chi_\mathrm{me}^{yx} = \frac{2j}{k}\frac{4\cos(\alpha x)}{\beta\cos(\alpha x) + \sqrt{\beta^2 - \gamma^2}}$$

Reciprocal transformation



where

$$\beta = k_{z_\mathrm{i}} + k_{z_\mathrm{t}}$$

$$\gamma = k_{z_\mathrm{i}} - k_{z_\mathrm{t}}$$

$$\alpha = k_{x_\mathrm{t}} - k_{x_\mathrm{i}}$$

Energy conservation relations

$$\overline{\overline{\chi}}_\mathrm{ee} = \overline{\overline{\chi}}_\mathrm{ee}^{\dagger} \qquad \overline{\overline{\chi}}_\mathrm{mm} = \overline{\overline{\chi}}_\mathrm{mm}^{\dagger} \qquad \overline{\overline{\chi}}_\mathrm{em} = \overline{\overline{\chi}}_\mathrm{me}^{\dagger}$$

The metasurface is lossless and gainless

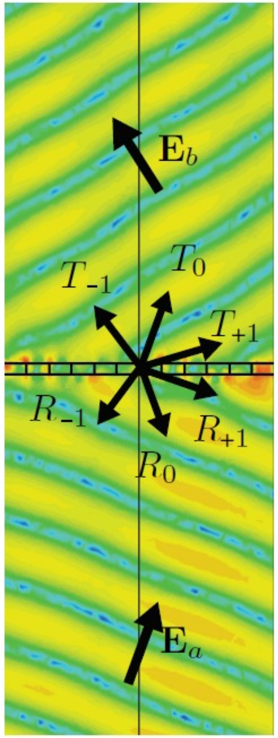
Reciprocity conditions

$$\overline{\overline{\chi}}_\mathrm{ee} = \overline{\overline{\chi}}_\mathrm{ee}^{T} \qquad \overline{\overline{\chi}}_\mathrm{mm} = \overline{\overline{\chi}}_\mathrm{mm}^{T} \qquad \overline{\overline{\chi}}_\mathrm{em} = -\overline{\overline{\chi}}_\mathrm{me}^{T}$$

It is also reciprocal

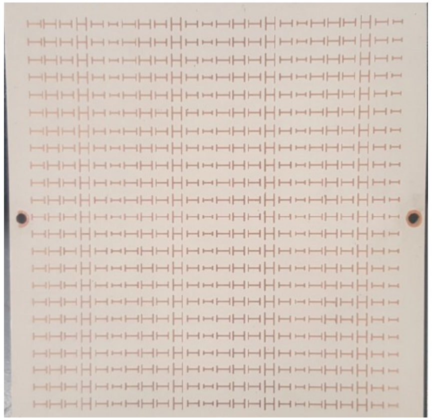
Achouri K, “Electromagnetic metasurfaces: Theory and applications”, 2021

Implementation of Lossless, Gainless, Perfect Refraction

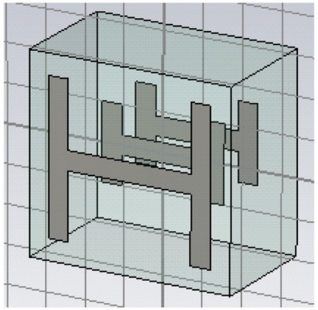


Incidence angle: 20°

Refraction angle: -28°

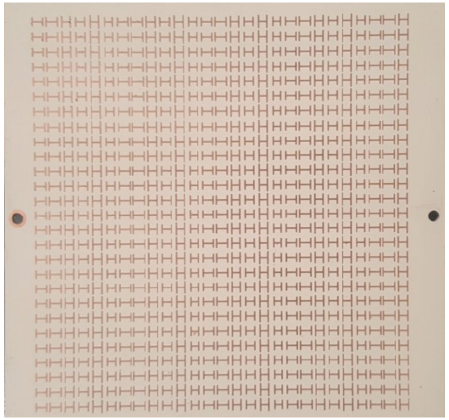


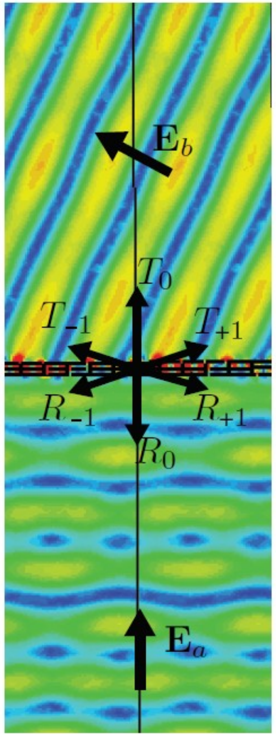
Asymmetric structure



Incidence angle: 0°

Refraction angle: -70°

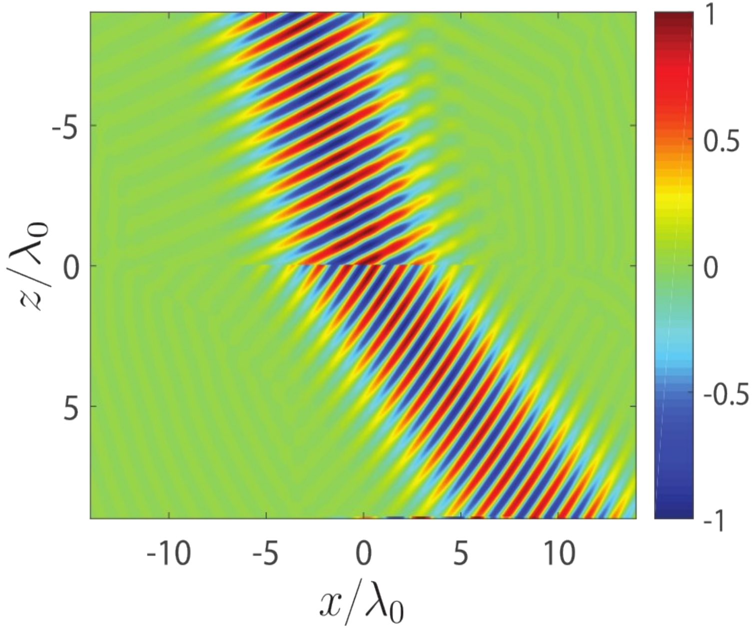




https://doi.org/10.1109/TAP.2018.2793958

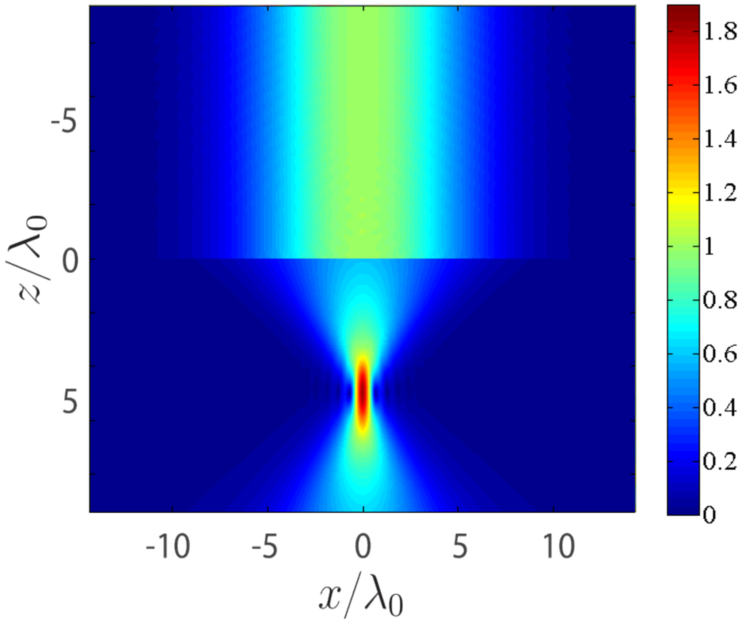
Why Does It Work ?

Refraction



The component of the Poynting vector normal to the surface of the incident and refracted waves can be equalized at <u>each</u> (x,y) positions. This is why “perfect” refraction is possible.

Collimation



The Poynting vector of the incident and transmitted waves have different spatial variations preventing them to be equal everywhere. We cannot collimate an incident plane wave with 100% efficiency.

What Have We Learned So Far….

- Phase-gradient metasurfaces (full transmission, only phase variations) can be used to change the direction of light propagation (refraction). However, they do not correspond to exact solutions of Maxwell equations and thus lead to reduced efficiency for large refraction angles.
- The GSTC can be used to synthesize a refractive metasurface. In the most simple case, the metasurface is assumed to be symmetric and only the incident and transmitted waves are defined. This leads to a design that balances electric/magnetic gain/loss to achieve full transmission, which is impractical to fabricate.
- A better design is to consider an asymmetric metasurface and use incident/transmitted fields from both sides. The asymmetry of the structure allows for properly matching both incident and transmitted waves. This design is perfectly lossless and gainless and requires bi-anisotropic susceptibilities due to the asymmetry of the structure.

Phase Shifting

Scattering with Lorentzian Responses

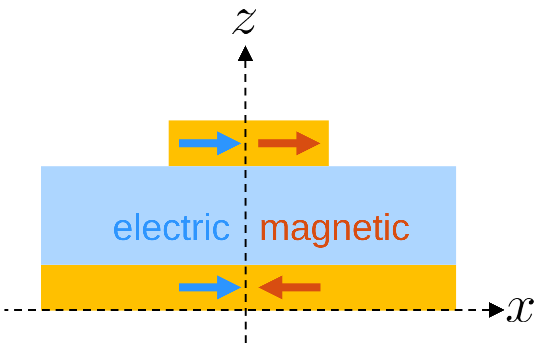
Scattering parameters

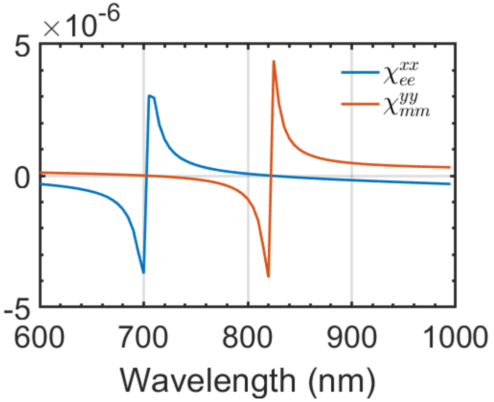
$$t = \frac{4 + \chi_{\mathrm{ee}}\chi_{\mathrm{mm}}k^2}{\left(2 + jk\chi_{\mathrm{ee}}\right)\left(2 + jk\chi_{\mathrm{mm}}\right)}$$

$$r = \frac{2jk\left(\chi_{\mathrm{mm}} - \chi_{\mathrm{ee}}\right)}{\left(2 + jk\chi_{\mathrm{ee}}\right)\left(2 + jk\chi_{\mathrm{mm}}\right)}$$

Let's consider that the susceptibilities are simple Lorentzian functions

$$\begin{cases} \chi_{\mathrm{mm}}(\omega) = \dfrac{A_{\mathrm{m}}}{\omega_{\mathrm{m}}^2 - \omega^2 - j\gamma_{\mathrm{m}}\omega} \\ \chi_{\mathrm{ee}}(\omega) = \dfrac{A_{\mathrm{e}}}{\omega_{\mathrm{e}}^2 - \omega^2 - j\gamma_{\mathrm{e}}\omega} \end{cases}$$





$$t = 1 - f_{\mathrm{m}} - f_{\mathrm{e}}$$

$$r = f_{\mathrm{m}} - f_{\mathrm{e}}$$

$$\begin{cases} f_{\mathrm{e}} = \dfrac{A_{\mathrm{e}}k}{A_{\mathrm{e}}k - 2\gamma_{\mathrm{e}}\omega + 2j(\omega - \omega_{\mathrm{e}})(\omega + \omega_{\mathrm{e}})} \\ f_{\mathrm{m}} = \dfrac{A_{\mathrm{m}}k}{A_{\mathrm{m}}k - 2\gamma_{\mathrm{m}}\omega + 2j(\omega - \omega_{\mathrm{m}})(\omega + \omega_{\mathrm{m}})} \end{cases}$$

Notice that the sign of f_{m} is not the same in the reflection and transmission coefficients. This means that if $f_{\mathrm{m}} = f_{\mathrm{e}}$, then we can completely cancel the reflection, which is equivalent to having $\chi_{\mathrm{mm}} = \chi_{\mathrm{ee}}$.

Achieving 2π Phase Shift

Scattering parameters

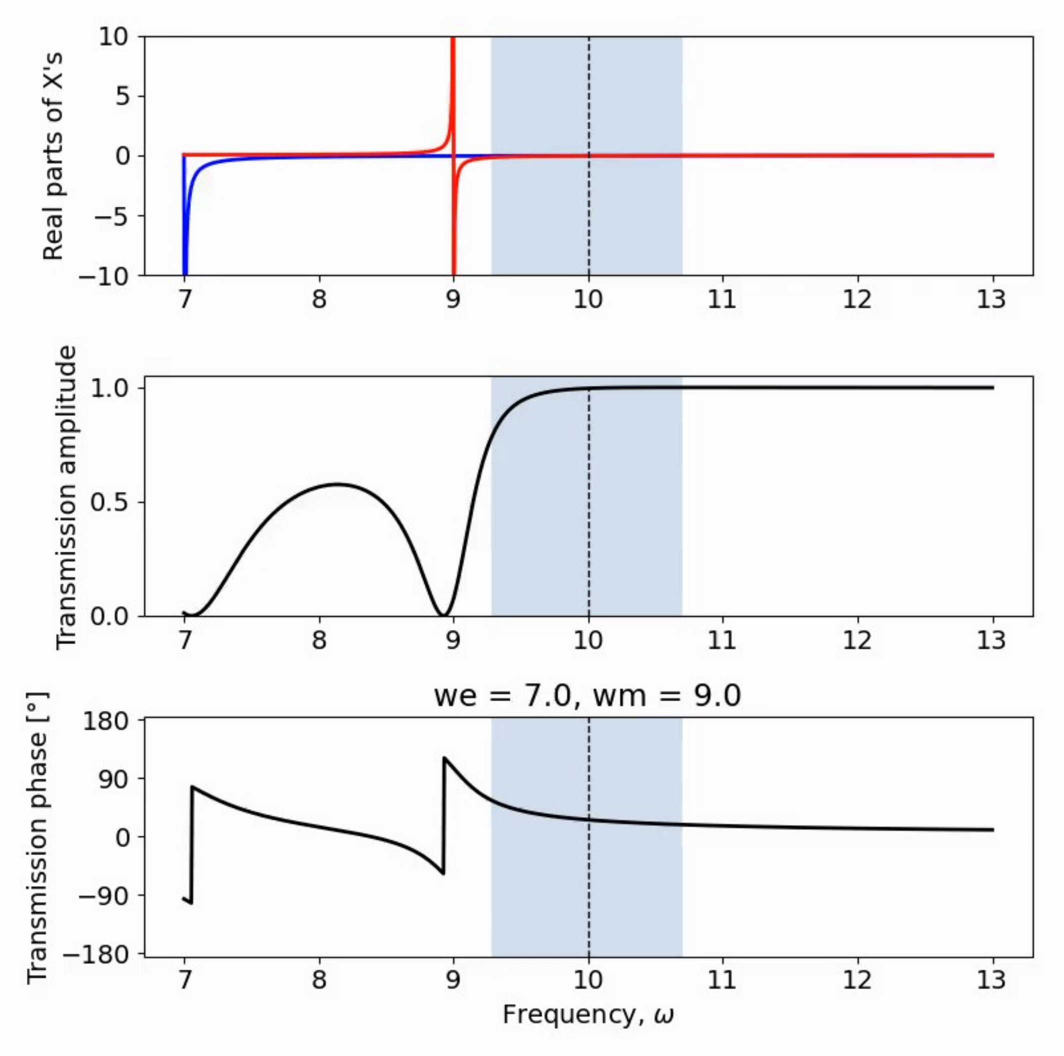
$$t = 1 - f_{\mathrm{m}} - f_{\mathrm{e}}$$

$$r = f_{\mathrm{m}} - f_{\mathrm{e}}$$

$$f_{\mathrm{e}} = \frac{A_{\mathrm{e}}k}{A_{\mathrm{e}}k - 2\gamma_{\mathrm{e}}\omega + 2j(\omega - \omega_{\mathrm{e}})(\omega + \omega_{\mathrm{e}})}$$

$$f_{\mathrm{m}} = \frac{A_{\mathrm{m}}k}{A_{\mathrm{m}}k - 2\gamma_{\mathrm{m}}\omega + 2j(\omega - \omega_{\mathrm{m}})(\omega + \omega_{\mathrm{m}})}$$

When the electric and magnetic resonances are close enough their interaction leads to a 2π phase shift



Complex Plane Analysis

Transmission coefficient

2 zeros

$$t = \frac{4 + \chi_{\mathrm{ee}}\chi_{\mathrm{mm}}k^2}{(2 + jk\chi_{\mathrm{ee}})(2 + jk\chi_{\mathrm{mm}})} \qquad \begin{cases} \chi_{\mathrm{mm}} = \dfrac{A_{\mathrm{m}}}{\omega_{\mathrm{m}}^2 - \omega^2} \\ \chi_{\mathrm{ee}} = \dfrac{A_{\mathrm{e}}}{\omega_{\mathrm{e}}^2 - \omega^2} \end{cases}$$

2 poles

Complex frequency

$$\boxed{\omega = \omega_{\mathrm{r}} + j\omega_{\mathrm{i}}} \longrightarrow k = \frac{\omega}{c}$$

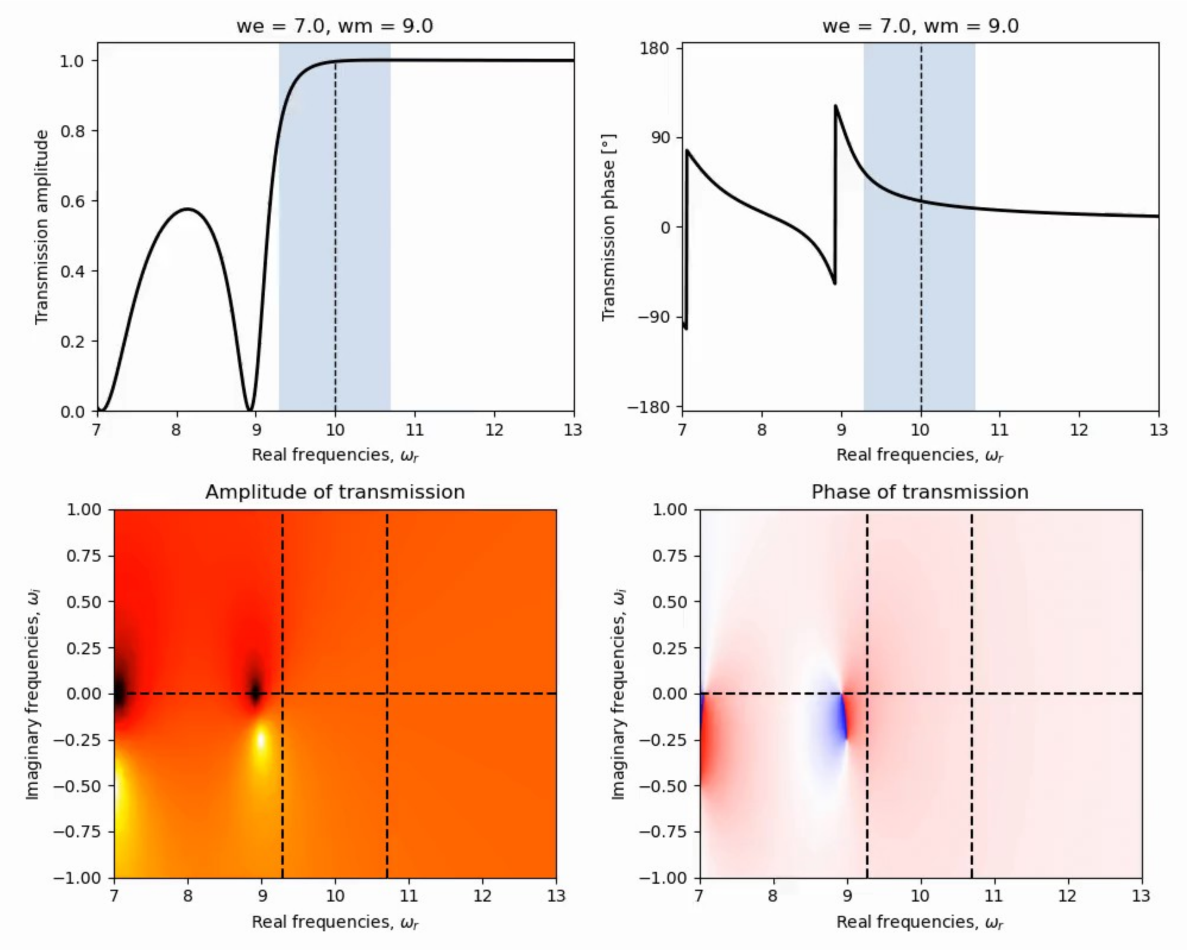
When the electric and magnetic resonances strongly interact, the transmission zeros become complex. This leads to a 2π phase-shift on the real frequency axis.

The interaction region is defined by

$$(\omega_{\mathrm{e}} - \omega_{\mathrm{m}})^2 \leq \frac{A_{\mathrm{e}}A_{\mathrm{m}}}{4c^2}$$

The edges of the interaction region are when

$$\omega = \sqrt{\omega_{\mathrm{e}}\omega_{\mathrm{m}}}$$



We assume a lossless case $\gamma_{\mathrm{e}} = \gamma_{\mathrm{m}} = 0$

What Have We Learned So Far….

- In a simple metasurface with only electric and magnetic dipolar responses modeled in terms of the susceptibilities χ_{ee} and χ_{mm}, the scattering parameters are conveniently expressed as the sums of electric and magnetic Lorentzian functions
- This allows us to understand how a metasurface can achieve a 2π phase shift. We know that a single Lorentzian function induces a π phase shift. Therefore, a metasurface whose response is expressed as the sum of an electric and a magnetic Lorentzian resonance can achieve a 2π phase shift.
- If the two electric and magnetic resonances only weakly interact, the total phase shift consists of two individual π phase shift that do not merge together. However, if the two resonances strongly interact, then their phase shifts merge into a continuous 2π phase shift.
- Within the region of high interaction, the two zeros of the transmission function become complex and split equally on both sides of the real frequency axis while the frequencies of the two poles are complex with negative imaginary parts.
- The fact that the zeros move away from the real frequency axis leads to high transmittance and the fact that one zero has positive imaginary frequency means that its interaction with one of the pole leads to 2π phase shift on the real frequency axis.