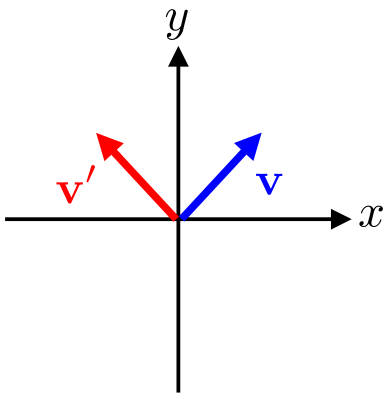
Lecture 10

Spatial Symmetries

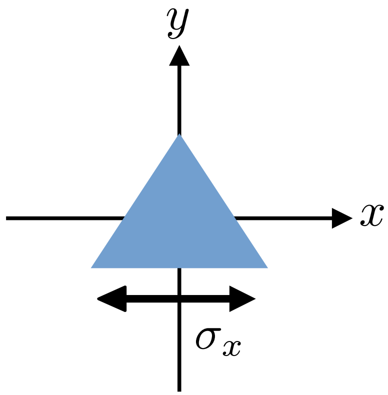
Spatial Symmetries and Material Parameters

Concept of Spatial Symmetries

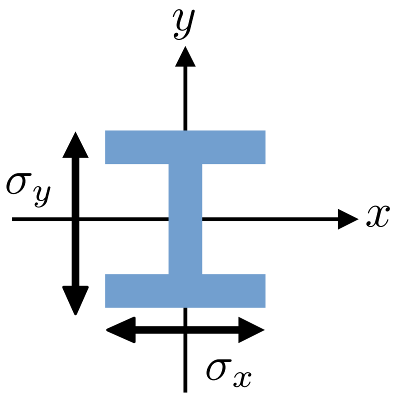


$$\mathbf{v}' = \sigma_x \cdot \mathbf{v}$$

$$\begin{bmatrix} -v_x \\ v_y \\ v_z \end{bmatrix} = \sigma_x \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



The triangle is invariant under σ_x



The dog bone is invariant under σ_x and σ_y

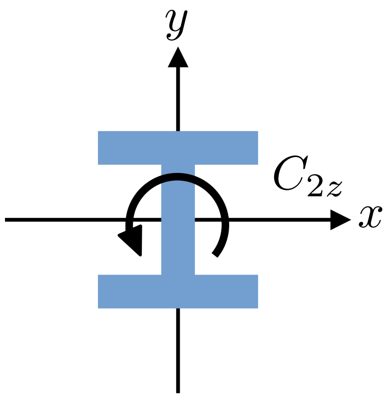
Mirror symmetries

$$\sigma_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Parity

$$P = -\bar{\bar{\mathrm{I}}}$$

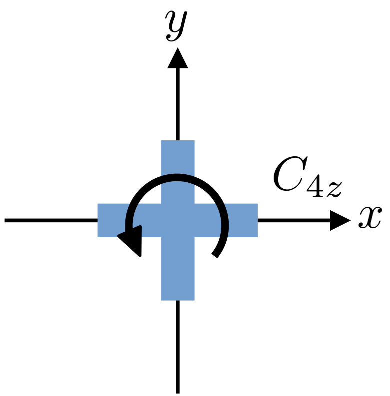
Rotation Symmetries



The dog bone is invariant under C_{2z} (180° rotation symmetry along the z-axis)

Rotations as integer fractions of 2π

$$C_{N,i} = R_i\left(\frac{2\pi}{N}\right)$$

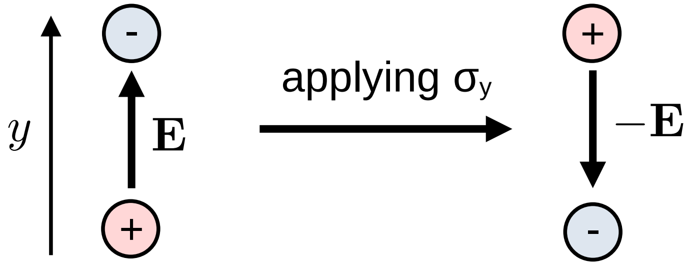


The cross is invariant under C_{4z} (90° rotation symmetry along the z-axis)

Reflection symmetries

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Space Inversion Properties



The electric field is odd under space inversion

$$P\{\mathbf{E}\} = -\mathbf{E} \quad \text{where} \quad P: \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$$

Looking a Maxwell equations, we can deduce that the magnetic field is even under space inversion

$$P\{\mathbf{H}\} = +\mathbf{H}$$

Space inversion properties of Maxwell equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}$$

$$\nabla \times \mathbf{E} = -\mathbf{K} - \frac{\partial}{\partial t}\mathbf{B}$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{e}}$$

$$\nabla \cdot \mathbf{B} = \rho_{\mathrm{m}}$$

Space inversion properties of material properties

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\xi}} \cdot \mathbf{H}$$

$$\mathbf{B} = \overline{\overline{\zeta}} \cdot \mathbf{E} + \overline{\overline{\mu}} \cdot \mathbf{H}$$

How do the individual components of the material tensors behave under spatial transformations ?

Spatial Transformations and the Fields

Let's consider an arbitrary spatial transformation (reflection or rotation) given by $\overline{\overline{\Lambda}}$

Under this spatial transformation, the electric field transforms as

$$\mathbf{E}' = \overline{\overline{\Lambda}} \cdot \mathbf{E} \longrightarrow \mathbf{E} = \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{E}' \quad \text{we also have that} \quad \nabla' = \overline{\overline{\Lambda}} \cdot \nabla \longrightarrow \nabla = \overline{\overline{\Lambda}}^{-1} \cdot \nabla'$$

To find how the magnetic field transforms, we consider $\mathbf{H} = \frac{1}{j\omega\epsilon} \nabla \times \mathbf{E}$

$$\mathbf{H} = \frac{1}{j\omega\epsilon} \left(\overline{\overline{\Lambda}}^{-1} \cdot \nabla' \right) \times \left(\overline{\overline{\Lambda}}^{-1} \cdot \mathbf{E}' \right) = \frac{1}{j\omega\epsilon} \det\left(\overline{\overline{\Lambda}}^{-1} \right) \overline{\overline{\Lambda}}^{-1} \cdot (\nabla' \times \mathbf{E}') = \det\left(\overline{\overline{\Lambda}}^{-1} \right) \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{H}'$$

$$\mathbf{E}' = \overline{\overline{\Lambda}} \cdot \mathbf{E}$$

$$\mathbf{H}' = \det\left(\overline{\overline{\Lambda}} \right) \overline{\overline{\Lambda}} \cdot \mathbf{H}$$

$$\det\left(\overline{\overline{\Lambda}} \right) = \begin{cases} +1, & \text{for rotations} \\ -1, & \text{for reflections} \end{cases}$$

Change of Basis for Material Parameters

Change of basis for the electric and magnetic fields

$$\mathbf{E}' = \overline{\overline{\Lambda}} \cdot \mathbf{E} \longrightarrow \mathbf{E} = \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{E}'$$

$$\mathbf{H}' = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \mathbf{H} \longrightarrow \mathbf{H} = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{H}'$$

$$\det\left(\overline{\overline{\Lambda}}\right) = \begin{cases} +1, & \text{for rotations} \\ -1, & \text{for reflections} \end{cases}$$

Change of basis for material parameters

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\xi}} \cdot \mathbf{H}$$

$$\overline{\overline{\Lambda}}^{-1} \cdot \mathbf{D}' = \overline{\overline{\epsilon}} \cdot \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{E}' + \overline{\overline{\xi}} \cdot \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{H}'$$

$$\mathbf{D}' = \overline{\overline{\Lambda}} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{E}' + \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{\Lambda}}^{-1} \cdot \mathbf{H}'$$

$$\mathbf{D}' = \overline{\overline{\epsilon}}' \cdot \mathbf{E}' + \overline{\overline{\xi}}' \cdot \mathbf{H}'$$

Change of Basis for Material Parameters

$$\overline{\overline{\epsilon}}' = \overline{\overline{\Lambda}} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\mu}}' = \overline{\overline{\Lambda}} \cdot \overline{\overline{\mu}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\zeta}}' = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\xi}}' = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{\Lambda}}^T$$

$$\det\left(\overline{\overline{\Lambda}}\right) = \begin{cases} +1, & \text{for rotations} \\ -1, & \text{for reflections} \end{cases}$$

where we have used the fact that an orthogonal matrix satisfies

$$\overline{\overline{\Lambda}} \cdot \overline{\overline{\Lambda}}^T = \overline{\overline{I}} \longrightarrow \overline{\overline{\Lambda}}^{-1} = \overline{\overline{\Lambda}}^T$$

Material Parameters Invariance Conditions

Neumann's principle

If a given structure is invariant under a symmetry operation, then its material parameters should also be invariant under the same operation.

Change of basis

$$\overline{\overline{\epsilon}}' = \overline{\overline{\Lambda}} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\mu}}' = \overline{\overline{\Lambda}} \cdot \overline{\overline{\mu}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\zeta}}' = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\xi}}' = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{\Lambda}}^T$$

After a change of basis, the parameters remain the same. →

Invariance conditions

$$\overline{\overline{\epsilon}} = \overline{\overline{\Lambda}} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\mu}} = \overline{\overline{\Lambda}} \cdot \overline{\overline{\mu}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\zeta}} = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{\Lambda}}^T$$

$$\overline{\overline{\xi}} = \det\left(\overline{\overline{\Lambda}}\right) \overline{\overline{\Lambda}} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{\Lambda}}^T$$

Generalization to Higher-Order Nonlocal Terms

dipolar and quadrupolar responses → higher-order susceptibilities ← local and nonlocal field excitations

$$\begin{bmatrix} P_i \\ M_i \\ Q^{(\mathrm{e})}_{il} \\ Q^{(\mathrm{m})}_{il} \end{bmatrix} \propto \begin{bmatrix} \chi^{ij}_{\mathrm{ee}} & \chi^{ij}_{\mathrm{em}} & \chi'^{ijk}_{\mathrm{ee}} & \chi'^{ijk}_{\mathrm{em}} \\ \chi^{ij}_{\mathrm{me}} & \chi^{ij}_{\mathrm{mm}} & \chi'^{ijk}_{\mathrm{me}} & \chi'^{ijk}_{\mathrm{mm}} \\ Q^{(\mathrm{e})ilj}_{\mathrm{ee}} & Q^{(\mathrm{e})ilj}_{\mathrm{em}} & Q^{(\mathrm{e})'iljk}_{\mathrm{ee}} & Q^{(\mathrm{e})'iljk}_{\mathrm{em}} \\ Q^{(\mathrm{m})ilj}_{\mathrm{me}} & Q^{(\mathrm{m})ilj}_{\mathrm{mm}} & Q^{(\mathrm{m})'iljk}_{\mathrm{me}} & Q^{(\mathrm{m})'iljk}_{\mathrm{mm}} \end{bmatrix} \cdot \begin{bmatrix} E_j \\ H_j \\ \nabla_k E_j \\ \nabla_k H_j \end{bmatrix}$$

Invariance conditions in tensor notation

1st order: $T_i = a\Lambda_{ij}T_j$

2nd order: $T_{ij} = a\Lambda_{im}\Lambda_{jk}T_{mk}$

3rd order: $T_{ijk} = a\Lambda_{il}\Lambda_{jm}\Lambda_{kn}T_{lmn}$

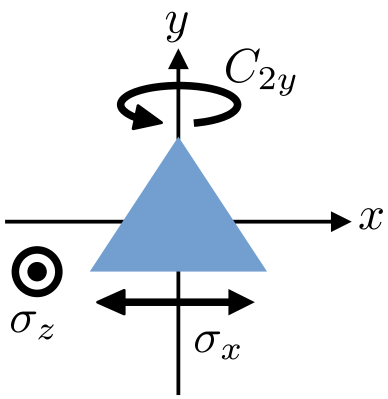
4th order: $T_{ijkl} = a\Lambda_{im}\Lambda_{jn}\Lambda_{ko}\Lambda_{lp}T_{mnop}$

where $a = \det(\overline{\overline{\Lambda}})^n$

$$n = \begin{cases} 0, & \text{for “ee” and “mm” tensors} \\ 1, & \text{for “em” and “me” tensors} \end{cases}$$

T is any of the higher-order susceptibilities

How to Apply this Formalism ?



Flat triangle

1) Start with a full tensor

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

2) Consider the invariance condition

$$\bar{\bar{\epsilon}} = \bar{\bar{\Lambda}} \cdot \bar{\bar{\epsilon}} \cdot \bar{\bar{\Lambda}}^T$$

3) Solve it to get a reduced tensor

$$\bar{\bar{\epsilon}} = ?$$

4) Repeat the process for all symmetries

$$\bar{\bar{\Lambda}} = \sigma_z \longrightarrow \bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$

$$\bar{\bar{\Lambda}} = \sigma_x \longrightarrow \bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$

$$\bar{\bar{\Lambda}} = C_{2y} \longrightarrow \bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$

Redundant symmetries

$$\sigma_i \sigma_j = C_{2k}$$

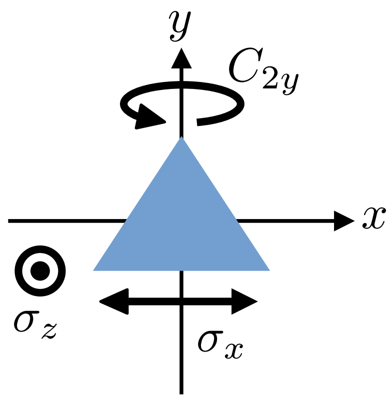
$$C_{2i} C_{2j} = C_{2k}$$

$$\sigma_x \sigma_y \sigma_z = P$$

$$\sigma_i C_{2i} = P$$

Note that C_{2y} is <u>redundant</u> because a system invariant under σ_x and σ_z is automatically invariant under C_{2y}

Full Dipolar Model



$$\bar{\bar{\epsilon}} = \bar{\bar{\Lambda}} \cdot \bar{\bar{\epsilon}} \cdot \bar{\bar{\Lambda}}^T$$

$$\bar{\bar{\mu}} = \bar{\bar{\Lambda}} \cdot \bar{\bar{\mu}} \cdot \bar{\bar{\Lambda}}^T$$

$$\bar{\bar{\zeta}} = \det\left(\bar{\bar{\Lambda}}\right) \bar{\bar{\Lambda}} \cdot \bar{\bar{\zeta}} \cdot \bar{\bar{\Lambda}}^T$$

$$\bar{\bar{\xi}} = \det\left(\bar{\bar{\Lambda}}\right) \bar{\bar{\Lambda}} \cdot \bar{\bar{\xi}} \cdot \bar{\bar{\Lambda}}^T$$

Applying the same formalism to all dipolar tensor

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E} + \bar{\bar{\xi}} \cdot \mathbf{H}$$

$$\mathbf{B} = \bar{\bar{\zeta}} \cdot \mathbf{E} + \bar{\bar{\mu}} \cdot \mathbf{H}$$

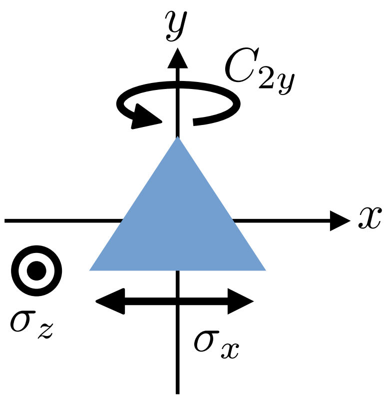
$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \qquad \bar{\bar{\xi}} = \begin{bmatrix} 0 & 0 & \xi_{xz} \\ 0 & 0 & 0 \\ \xi_{zx} & 0 & 0 \end{bmatrix}$$

$$\bar{\bar{\zeta}} = \begin{bmatrix} 0 & 0 & \zeta_{xz} \\ 0 & 0 & 0 \\ \zeta_{zx} & 0 & 0 \end{bmatrix} \qquad \bar{\bar{\mu}} = \begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix}$$

This is a bianisotropic structure !

If the system is reciprocal we also have that $\bar{\bar{\zeta}} = -\bar{\bar{\xi}}^T$

Full Quadrupolar and Dipolar Model



$$\begin{bmatrix} P_i \\ M_i \\ Q^{(\mathrm{e})}_{il} \\ Q^{(\mathrm{m})}_{il} \end{bmatrix} \propto \begin{bmatrix} \chi^{ij}_{\mathrm{ee}} & \chi^{ij}_{\mathrm{em}} & \chi'^{ijk}_{\mathrm{ee}} & \chi'^{ijk}_{\mathrm{em}} \\ \chi^{ij}_{\mathrm{me}} & \chi^{ij}_{\mathrm{mm}} & \chi'^{ijk}_{\mathrm{me}} & \chi'^{ijk}_{\mathrm{mm}} \\ Q^{(\mathrm{e})ilj}_{\mathrm{ee}} & Q^{(\mathrm{e})ilj}_{\mathrm{em}} & Q^{(\mathrm{e})'iljk}_{\mathrm{ee}} & Q^{(\mathrm{e})'iljk}_{\mathrm{em}} \\ Q^{(\mathrm{m})ilj}_{\mathrm{me}} & Q^{(\mathrm{m})ilj}_{\mathrm{mm}} & Q^{(\mathrm{m})'iljk}_{\mathrm{me}} & Q^{(\mathrm{m})'iljk}_{\mathrm{mm}} \end{bmatrix} \cdot \begin{bmatrix} E_j \\ H_j \\ \nabla_k E_j \\ \nabla_k H_j \end{bmatrix}$$

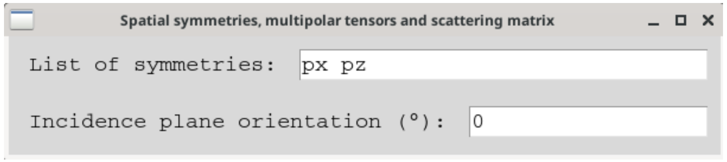
The number indicates what parameters are different or equal to each other

dipole only (p, m rows × E, H columns)

	E_x	E_y	E_z	H_x	H_y	H_z	∂^e_{xy}	∂^e_{xz}	∂^e_{yz}	∂^e_{xx}	∂^e_{yy}	∂^e_{zz}	∂^h_{xy}	∂^h_{xz}	∂^h_{yz}	∂^h_{xx}	∂^h_{yy}	∂^h_{zz}
p_x	3					4	9								28			
p_y		2								13	10	11		27				
p_z			1	5					12				26					
m_x			-5	8					15				31					
m_y					7			16								29	32	30
m_z	-4					6	14								33			
Q_{xy}	9					-14	20								34			
Q_{xz}					-16			17								37	40	41
Q_{yz}			12	-15					18				35					
Q_{xx}		13								21	19	24		39				
Q_{yy}		10								19	22	23		38				
Q_{zz}		11								24	23	25		36				
S_{xy}			-26	31					-35				50					
S_{xz}		-27								-39	-38	-36		45				
S_{yz}	-28					33	-34								46			
S_{xx}					29			-37								44	48	43
S_{yy}					32			-40								48	49	42
S_{zz}					30			-41								43	42	47

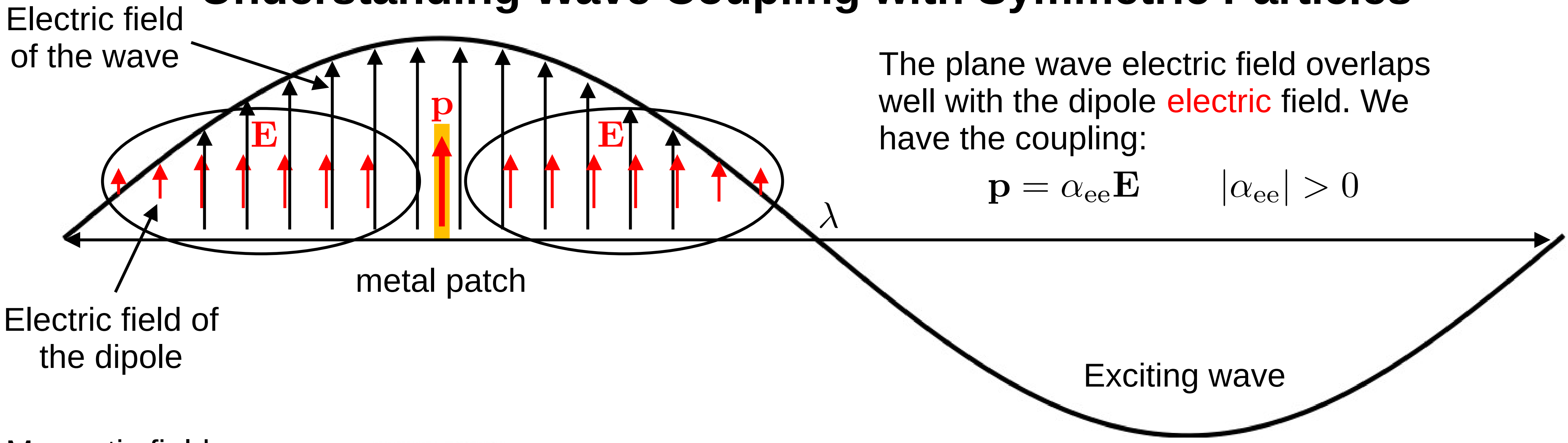
E-field derivative — H-field derivative

$\partial_x E_y + \partial_y E_x$ → ∂^e_{xy}



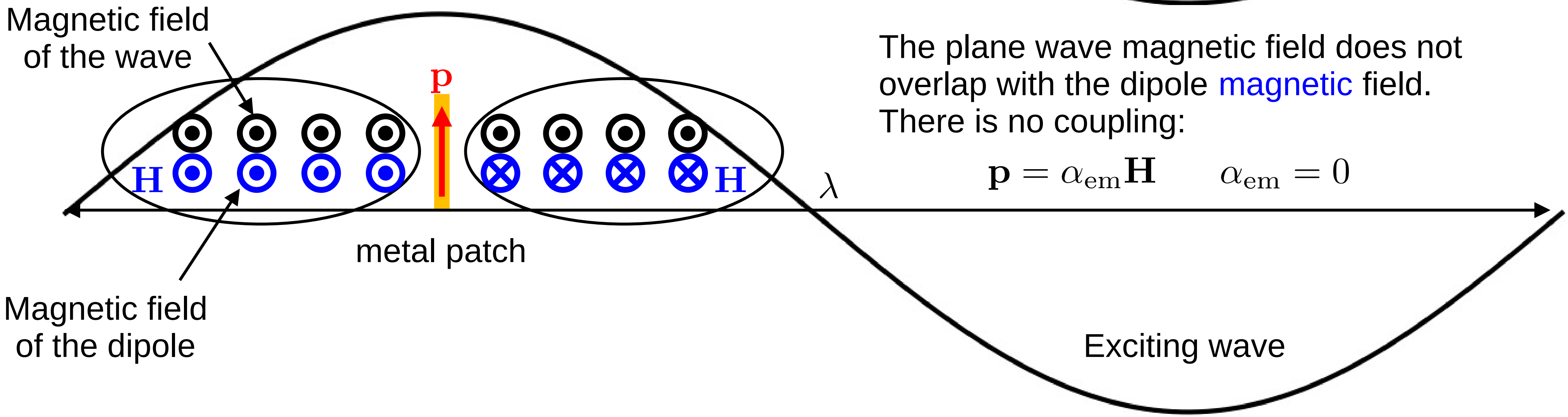
https://github.com/kachourim/sym

Understanding Wave Coupling with Symmetric Particles



The plane wave electric field overlaps well with the dipole electric field. We have the coupling:

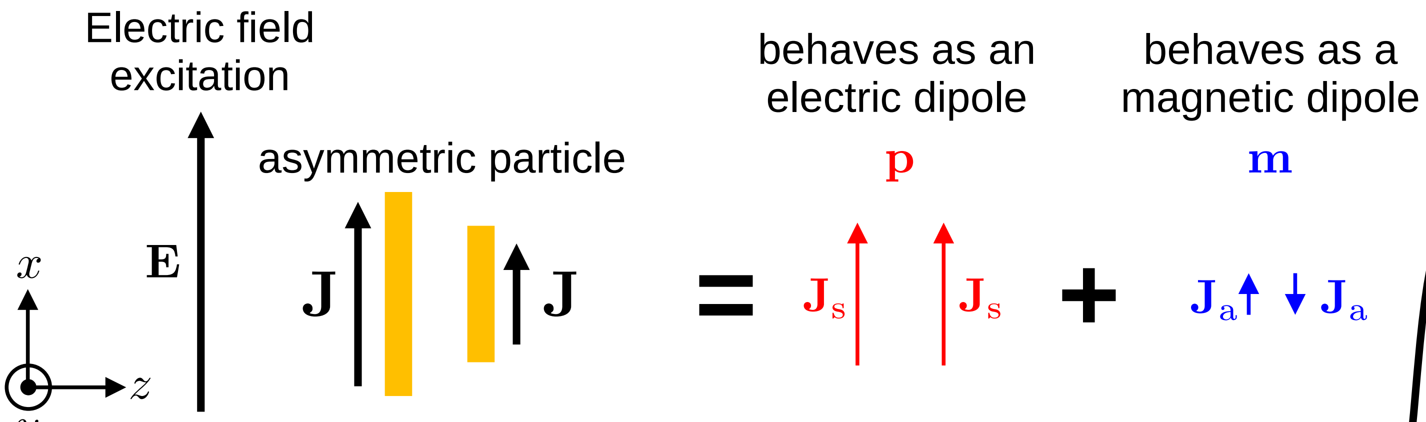
$$\mathbf{p} = \alpha_{\mathrm{ee}}\mathbf{E} \qquad |\alpha_{\mathrm{ee}}| > 0$$



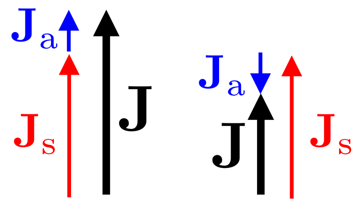
The plane wave magnetic field does not overlap with the dipole magnetic field. There is no coupling:

$$\mathbf{p} = \alpha_{\mathrm{em}}\mathbf{H} \qquad \alpha_{\mathrm{em}} = 0$$

Understanding Wave Coupling with Asymmetric Particles



Splitting into symmetric and asymmetric currents



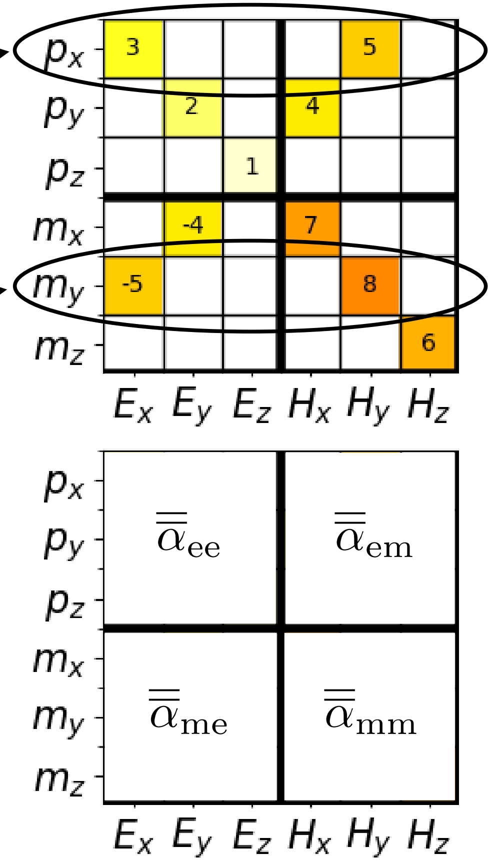
Electric field excitation

$$p_x = \alpha_{\mathrm{ee}}^{xx} E_x + \alpha_{\mathrm{em}}^{xy} H_y$$

$$m_y = \alpha_{\mathrm{me}}^{yx} E_x + \alpha_{\mathrm{mm}}^{yy} H_y$$

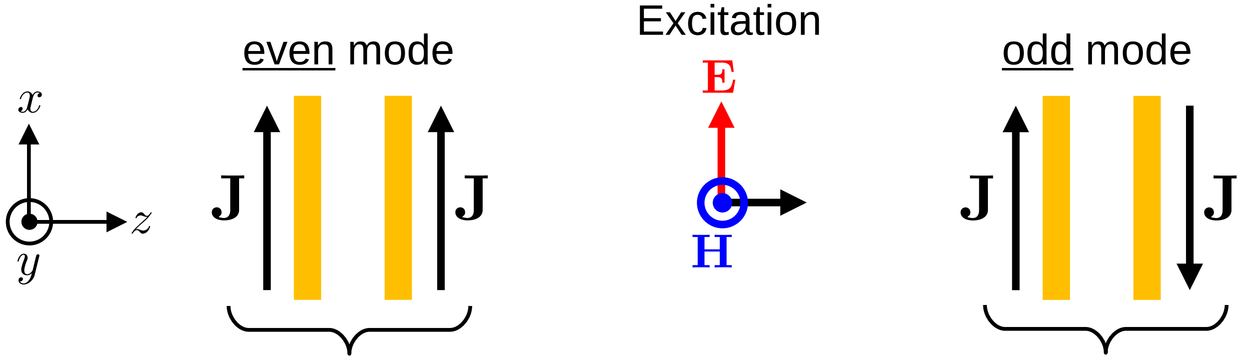
exists because of the broken symmetry along z

Dipolar structure asymmetric along *z*



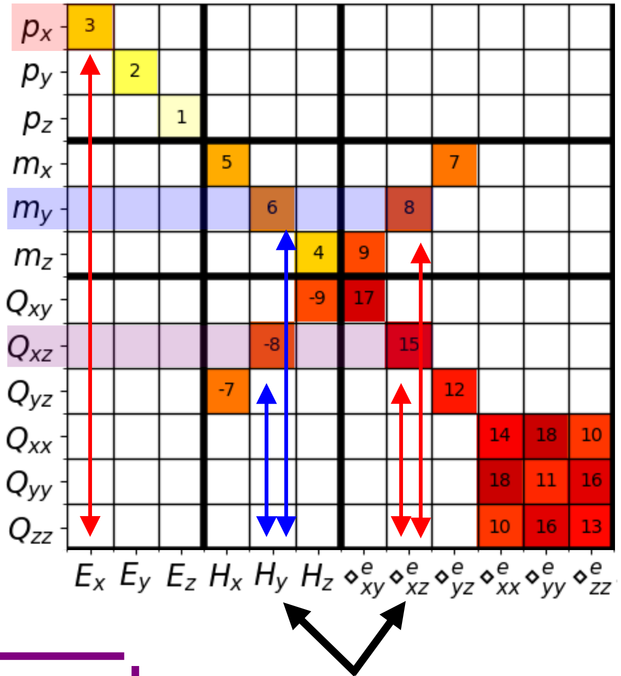
Even and Odd Modes in Quadrupolar Symmetric Particles

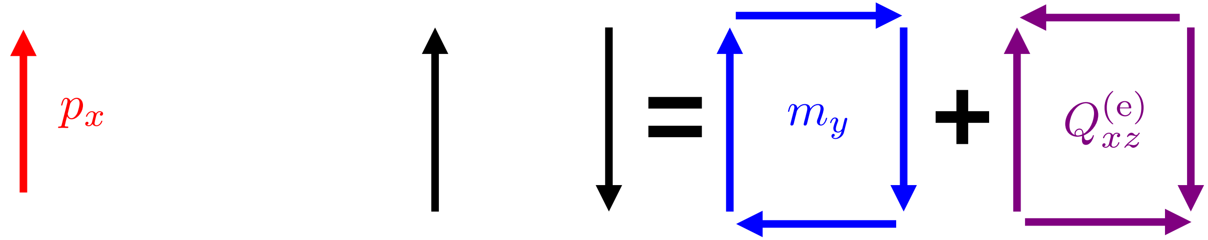
The currents on a symmetric double metallic patch structure may be decomposed into even and odd modes



modeled as an electric dipole

modeled as a combination of magnetic dipole and electric quadrupole





Could be excited either by H_y or by the derivative of E_x along z

Parity of Multipole Moments and Coupling Properties

Vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\frac{e^{-jkR}}{R}\,dV'$$

For a single particle

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\mathbf{J}\frac{e^{-jkr}}{r}$$

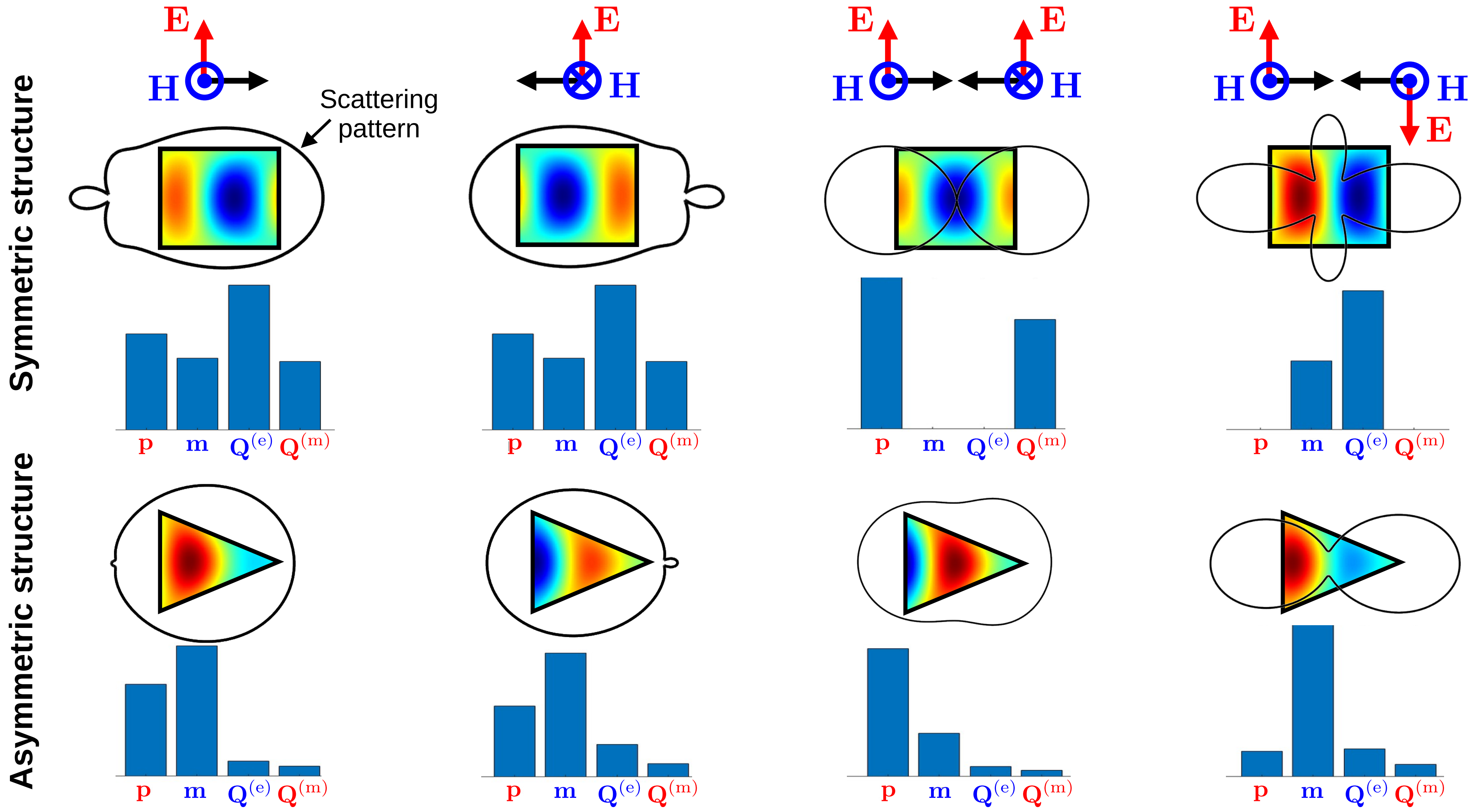
Multipolar expansion

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\underbrace{\left(j\omega\mathbf{p} - \frac{j\omega}{2}\mathbf{Q}^{(\mathrm{e})}\cdot\nabla + \nabla\times\mathbf{m} + \frac{j\omega}{6}\mathbf{O}^{(\mathrm{e})}:\nabla\nabla - \frac{1}{2}\nabla\times(\mathbf{Q}^{(\mathrm{m})}\cdot\nabla) + \dots\right)}_{\mathbf{J}}\frac{e^{-jkr}}{r}$$

In a <u>symmetric</u> particle, odd excitations (like the electric field, **E**) couple with odd multipoles. Even excitations (like the magnetic field, **H**) couple with even multipoles. In an <u>asymmetric</u> particle even/odd excitations can couple to both even and odd multipoles.

	dipole	quadrupole	octupole
Even	$\mathbf{m}$	$\mathbf{Q}^{(\mathrm{e})}$	$\mathbf{O}^{(\mathrm{m})}$
Odd	$\mathbf{p}$	$\mathbf{Q}^{(\mathrm{m})}$	$\mathbf{O}^{(\mathrm{e})}$

Parity of Multipole Moments and Coupling Properties



"Eigenvalue" Formulation

$$\mathbf{E}' = \overline{\overline{\Lambda}} \cdot \mathbf{E} \quad \longrightarrow \quad \overline{\overline{\Lambda}} \cdot \mathbf{E} = \begin{bmatrix} \lambda_x^{\pm} & 0 & 0 \\ 0 & \lambda_y^{\pm} & 0 \\ 0 & 0 & \lambda_z^{\pm} \end{bmatrix} \cdot \mathbf{E}$$

where $\lambda_i^{\pm} = \begin{cases} +1, & \text{when even} \\ -1, & \text{when odd} \end{cases}$

We want to understand how each components of the fields are odd under spatial transformations

For mirror symmetries along x, y and z

Electric field $\lambda_{\sigma,\mathrm{e}}^{-} = \begin{bmatrix} \lambda_{\sigma_x}^{-} \\ \lambda_{\sigma_y}^{-} \\ \lambda_{\sigma_z}^{-} \end{bmatrix}$ ← The x-component of the electric field is odd under σ_x

Magnetic field $\lambda_{\sigma,\mathrm{m}}^{-} = \begin{bmatrix} \lambda_{\sigma_y}^{-} \lambda_{\sigma_z}^{-} \\ \lambda_{\sigma_x}^{-} \lambda_{\sigma_z}^{-} \\ \lambda_{\sigma_z}^{-} \lambda_{\sigma_y}^{-} \end{bmatrix}$ ← The x-component of the magnetic field is odd under both σ_y and σ_z

For C_2 reflections along x, y and z

$$\lambda_{C_2,\mathrm{e}}^{-} = \lambda_{C_2,\mathrm{m}}^{-} = \begin{bmatrix} \lambda_{C_{2y}}^{-} \lambda_{C_{2z}}^{-} \\ \lambda_{C_{2x}}^{-} \lambda_{C_{2z}}^{-} \\ \lambda_{C_{2x}}^{-} \lambda_{C_{2y}}^{-} \end{bmatrix}$$

Electric and magnetic fields share the same symmetry properties under C_2

“Eigenvalue” Formulation and Material Parameters

Reflections

$$\lambda^-_{\sigma,\mathrm{e}} = \begin{bmatrix} \lambda^-_{\sigma_x} \\ \lambda^-_{\sigma_y} \\ \lambda^-_{\sigma_z} \end{bmatrix} \qquad \lambda^-_{\sigma,\mathrm{m}} = \begin{bmatrix} \lambda^-_{\sigma_y}\lambda^-_{\sigma_z} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_z} \\ \lambda^-_{\sigma_z}\lambda^-_{\sigma_y} \end{bmatrix}$$

C_2 rotations

$$\lambda^-_{C_2,\mathrm{e}} = \lambda^-_{C_2,\mathrm{m}} = \begin{bmatrix} \lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{C_{2x}}\lambda^-_{C_{2y}} \end{bmatrix}$$

From these vectors we can understand the dependence of the components of the material parameters on spatial symmetries

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\xi}} \cdot \mathbf{H}$$
$$\mathbf{B} = \overline{\overline{\zeta}} \cdot \mathbf{E} + \overline{\overline{\mu}} \cdot \mathbf{H}$$

$$\overline{\overline{\epsilon}} \propto \lambda^-_{\sigma,\mathrm{e}} \otimes \lambda^-_{\sigma,\mathrm{e}}$$
$$\overline{\overline{\mu}} \propto \lambda^-_{\sigma,\mathrm{m}} \otimes \lambda^-_{\sigma,\mathrm{m}}$$

$$\rightarrow \begin{bmatrix} 1 & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_z} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_y} & 1 & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & 1 \end{bmatrix}$$

For C_2 rotations all material parameters behave as

$$\begin{bmatrix} 1 & \lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & 1 & \lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & 1 \end{bmatrix}$$

$$\overline{\overline{\xi}} \propto \lambda^-_{\sigma,\mathrm{e}} \otimes \lambda^-_{\sigma,\mathrm{m}}$$
$$\overline{\overline{\zeta}} \propto \lambda^-_{\sigma,\mathrm{m}} \otimes \lambda^-_{\sigma,\mathrm{e}}$$

$$\rightarrow \begin{bmatrix} \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_z} & \lambda^-_{\sigma_y} \\ \lambda^-_{\sigma_z} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_x} \\ \lambda^-_{\sigma_y} & \lambda^-_{\sigma_x} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} \end{bmatrix}$$

Putting Everything Together

For this term to exist, we must break <u>all</u> of these symmetries $\sigma_x \quad \sigma_y \quad C_{2x} \quad C_{2y}$

$$\overline{\overline{\epsilon}}, \overline{\overline{\mu}} \propto \begin{bmatrix} 1 & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & 1 & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & 1 \end{bmatrix}$$

$$\overline{\overline{\zeta}}, \overline{\overline{\xi}} \propto \begin{bmatrix} \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} \end{bmatrix}$$

Chirality requires breaking all mirror symmetries

Example 1

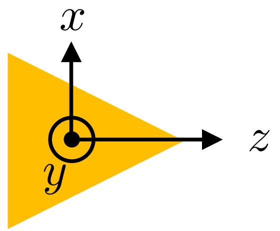
Let's assume we want to design a structure with the following material parameters

$$\overline{\overline{\epsilon}}, \overline{\overline{\mu}} \propto \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \qquad \overline{\overline{\zeta}}, \overline{\overline{\xi}} \propto \begin{bmatrix} 0 & K & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If the particle is symmetric under σ_x, we do not have (green)

If the particle is symmetric under σ_y, we do not have (blue)

flat triangle



The particle must have broken symmetries

$\sigma_z \quad C_{2x} \quad C_{2y}$

$$\overline{\overline{\epsilon}}, \overline{\overline{\mu}} \propto \begin{bmatrix} 1 & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & 1 & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & 1 \end{bmatrix}$$

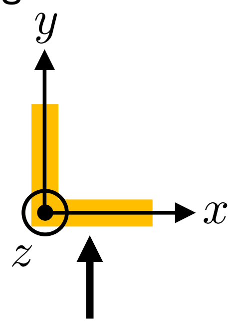
$$\overline{\overline{\zeta}}, \overline{\overline{\xi}} \propto \begin{bmatrix} \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} \end{bmatrix}$$

Example 2

Let's assume we want to design a structure with the following material parameters

$$\overline{\overline{\epsilon}}, \overline{\overline{\mu}} \propto \begin{bmatrix} A & K & 0 \\ M & B & 0 \\ 0 & 0 & C \end{bmatrix} \qquad \overline{\overline{\zeta}}, \overline{\overline{\xi}} \propto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

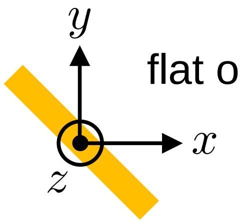
We might intuitively imagine that to get x-to-y coupling we need



asymmetric under C_{2z}

If the particle is symmetric under σ_z, we do not have [green]

If the particle is symmetric under C_{2z}, we do not have [blue]



flat oblique particle

The particle must have broken symmetries

$\sigma_x \quad \sigma_y \quad C_{2x} \quad C_{2y}$

$$\overline{\overline{\epsilon}}, \overline{\overline{\mu}} \propto \begin{bmatrix} 1 & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & 1 & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_x}\lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_y}\lambda^-_{\sigma_z}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & 1 \end{bmatrix}$$

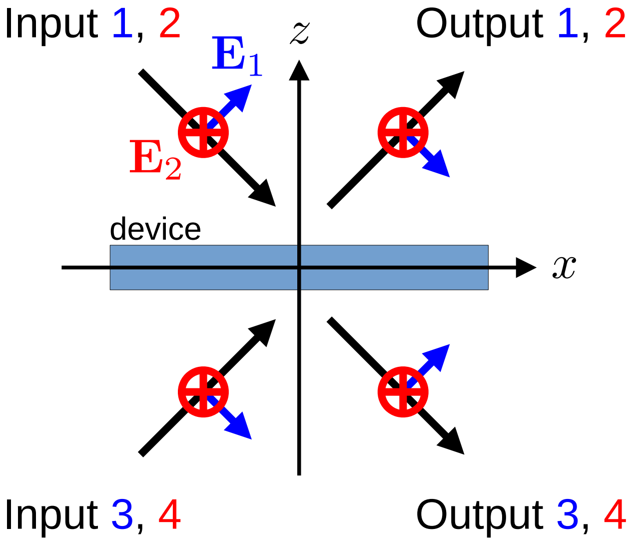
$$\overline{\overline{\zeta}}, \overline{\overline{\xi}} \propto \begin{bmatrix} \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} \end{bmatrix}$$

What Have We Learned So Far….

- Spatial symmetries are very useful to explore and understand how waves interact with structures.
- We typically consider rotation and reflection symmetries, which are operations that may be described in terms of orthogonal matrices.
- Using Maxwell equations, we can easily figure out how the **E** and **H** fields transform under a given spatial transformation. This allows us to determine how the material parameters are affected by such transformations (change of basis).
- The change of basis formulas lead to invariance conditions that are the foundation to understand the connection between spatial symmetries and material parameters (Neumann's principle).
- A simple algorithm can then be formulated to find the material parameters corresponding to a given structure described in terms of spatial symmetries.
- When investigating the coupling between a wave and a symmetric structure, we have that the even field component (**H**) couples only to even multipoles, whereas the odd field component (**E**) only couples to odd multipoles. For an asymmetric structure, even and odd field components can couple to both even and odd multipoles.
- It is possible to develop an eigenvalue formulation that allows finding which symmetries should be broken or not broken to achieve a given set of material parameter tensors. This is particularly useful to for metamaterial design.

Spatial Symmetries and Scattering Parameters

Symmetries and Scattering Matrix



2, 4 correspond to TE

1, 3 correspond to TM

$$\mathbf{E}_\mathrm{o} = \overline{\overline{\mathrm{S}}} \cdot \mathbf{E}_\mathrm{i} \quad \text{where} \quad \mathbf{E} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix}$$

Scattering matrix

reflection | transmission

$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} R_{11} & R_{12} & T_{13} & T_{14} \\ R_{21} & R_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & R_{33} & R_{34} \\ T_{41} & T_{42} & R_{43} & R_{44} \end{bmatrix}$$

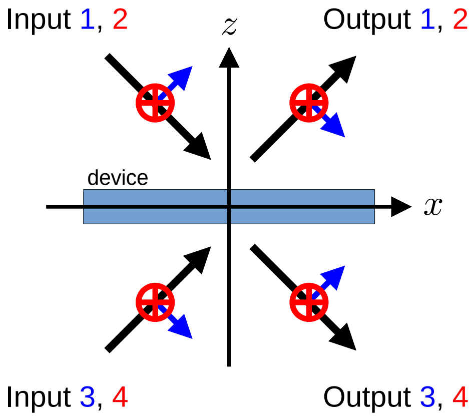
transmission | reflection

Co-polarized transmission

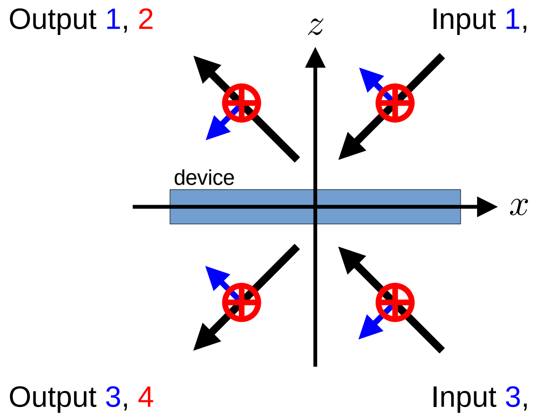
Cross-polarized transmission

https://doi.org/10.1117/1.AP.5.4.046001 https://doi.org/10.1109/TAP.2012.2220316

What Happens When Applying Symmetry Operations ?

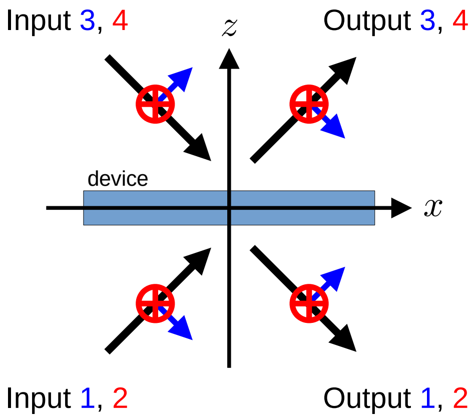


Let's apply σ_x



- TE does not flip
- TM flips
- **k**-vector flips

Let's apply σ_z



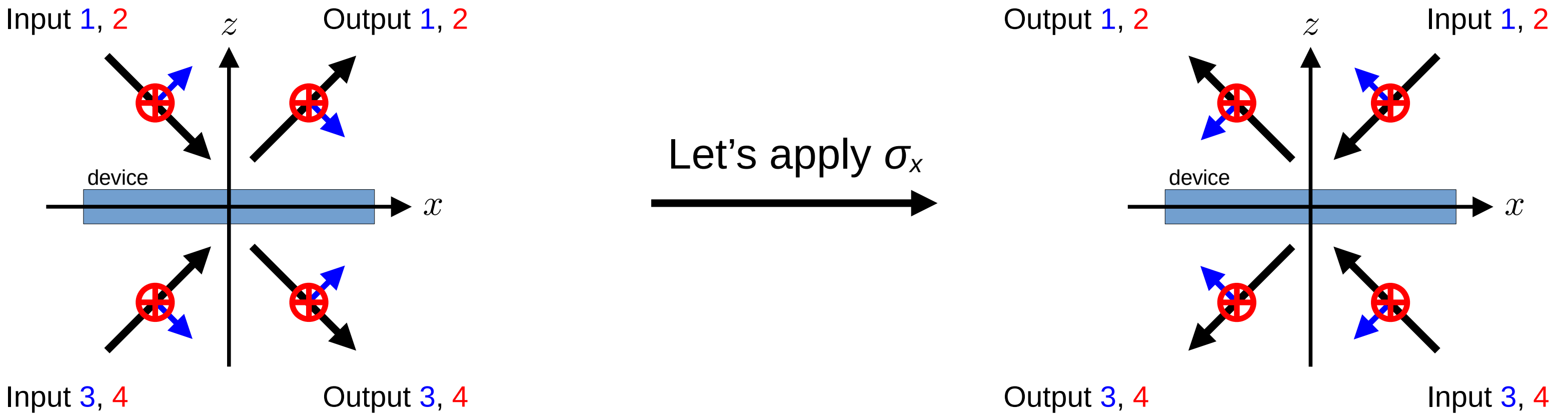
- TE does not flip
- TM does not flip
- **k**-vector does not flip

We can define a matrix M that defines how the field change under these spatial symmetries

$$\mathbf{E}' = \overline{\overline{\mathrm{M}}} \cdot \mathbf{E}$$

This transformation applies to both input and output fields

How to Find the M Matrix ?



- TE does not flip
- TM flips
- **k**-vector flips

$$\mathbf{E}' = \overline{\overline{\mathrm{M}}} \cdot \mathbf{E}$$

$$\begin{bmatrix} \mathbf{E}'_1 \\ \mathbf{E}'_2 \\ \mathbf{E}'_3 \\ \mathbf{E}'_4 \end{bmatrix} = \overline{\overline{\mathrm{M}}}_{\sigma_x} \cdot \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix}$$

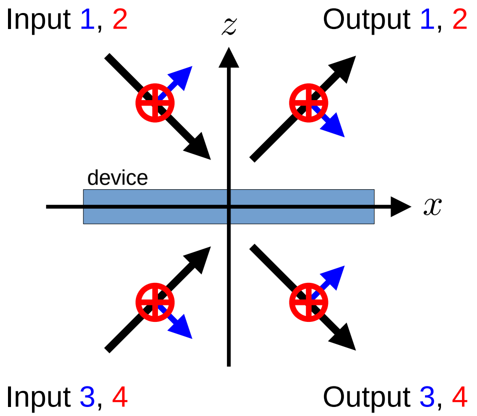
applies to both input and output fields

The matrix $M_{\sigma x}$ that defines this transformation is given by

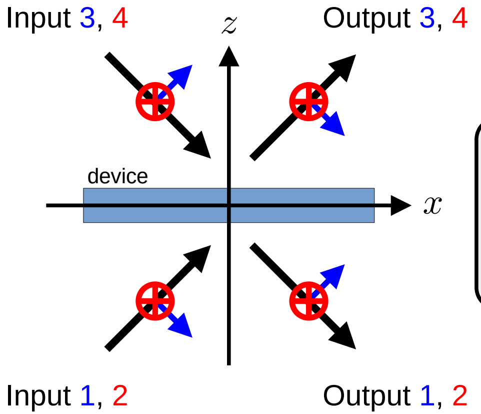
$$\overline{\overline{\mathrm{M}}}_{\sigma_x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

input/output 1 and 3 (TE) flips sign, whereas input/output 2 and 4 (TM) do not

How to Find the M Matrix ?



Let's apply σ_z



The matrix $M_{\sigma z}$ that defines this transformation is given by

$$\overline{\overline{\mathrm{M}}}_{\sigma_z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Input 1 has become input 3 and vice versa. Input 2 has become input 4 and vice versa. Ditto for the outputs

- TE does not flip
- TM does not flip
- **k**-vector does not flip

Definition of All M Matrices

$$\overline{\overline{\mathrm{M}}}_{\sigma_x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{\overline{\mathrm{M}}}_{\sigma_y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\overline{\overline{\mathrm{M}}}_{\sigma_z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\overline{\overline{\mathrm{M}}}_{C_{2x}} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\overline{\overline{\mathrm{M}}}_{C_{2y}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\overline{\overline{\mathrm{M}}}_{C_{2z}} = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{\overline{\mathrm{M}}}_{P} = -\overline{\overline{\mathrm{M}}}_{\sigma_z}$$

$$\overline{\overline{\mathrm{M}}}_{C_{4z}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

normal incidence only

(blue) = does not flip the **k**-vector

(green) = flips the **k**-vector

https://doi.org/10.1109/TAP.2012.2220316

Invariance Conditions for the Scattering Matrix

Applying the change of basis..

$$\mathbf{E}' = \overline{\overline{\mathrm{M}}} \cdot \mathbf{E} \longrightarrow \mathbf{E} = \overline{\overline{\mathrm{M}}}^{-1} \cdot \mathbf{E}'$$

..to the scattering matrix

$$\mathbf{E}_\mathrm{o} = \overline{\overline{\mathrm{S}}} \cdot \mathbf{E}_\mathrm{i}$$

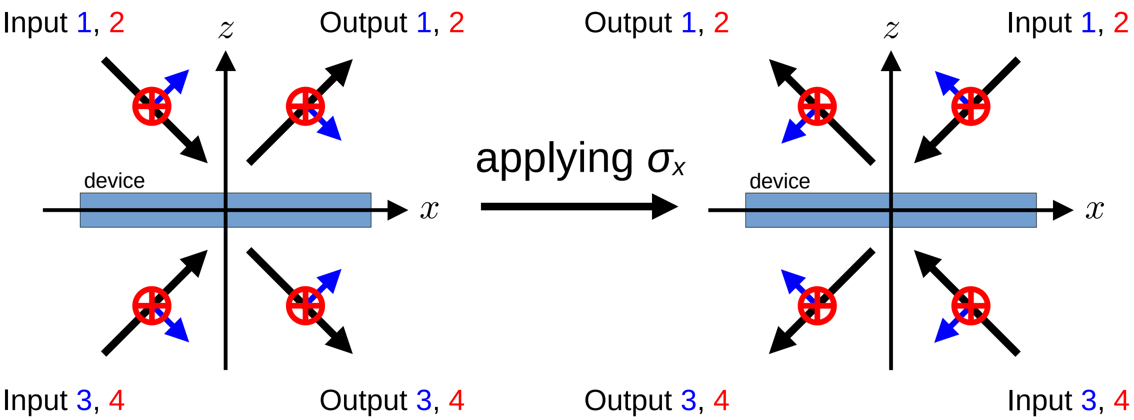
$$\downarrow$$

$$\overline{\overline{\mathrm{M}}}^{-1} \cdot \mathbf{E}'_\mathrm{o} = \overline{\overline{\mathrm{S}}} \cdot \overline{\overline{\mathrm{M}}}^{-1} \cdot \mathbf{E}'_\mathrm{i}$$

$$\downarrow$$

$$\mathbf{E}'_\mathrm{o} = \underbrace{\overline{\overline{\mathrm{M}}} \cdot \overline{\overline{\mathrm{S}}} \cdot \overline{\overline{\mathrm{M}}}^{-1}}_{\overline{\overline{\mathrm{S}}}'} \cdot \mathbf{E}'_\mathrm{i}$$

Invariance $\longrightarrow$



For symmetries that <u>do not flip</u> the **k**-vector

$$\overline{\overline{\mathrm{S}}} = \overline{\overline{\mathrm{M}}} \cdot \overline{\overline{\mathrm{S}}} \cdot \overline{\overline{\mathrm{M}}}^{-1}$$

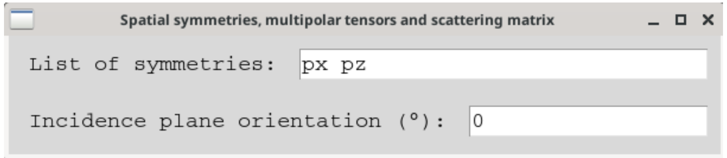
For symmetries that <u>flip</u> the **k**-vector

$$\overline{\overline{\mathrm{S}}} = \overline{\overline{\mathrm{M}}} \cdot \overline{\overline{\mathrm{S}}}^{T} \cdot \overline{\overline{\mathrm{M}}}^{-1}$$

Scattering matrix

$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} R_{11} & R_{12} & T_{13} & T_{14} \\ R_{21} & R_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & R_{33} & R_{34} \\ T_{41} & T_{42} & R_{43} & R_{44} \end{bmatrix}$$

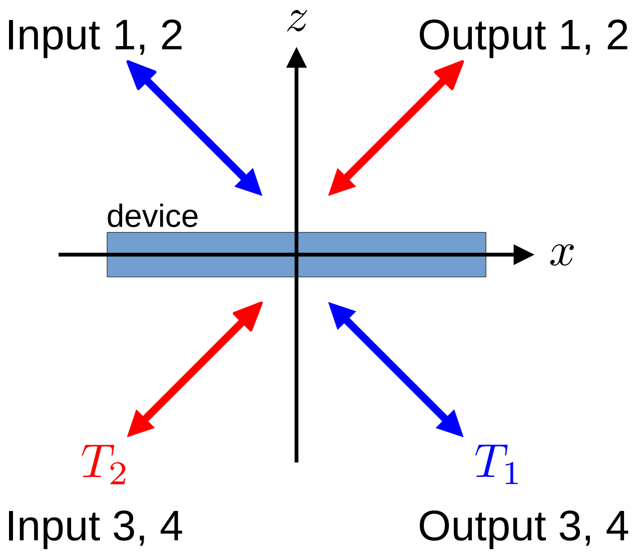
This scripts also computes the S-matrix



https://github.com/kachourim/sym

Example: How to Break Angular Transmission Symmetry ?

How to design a system that has $T_1 \neq T_2$?

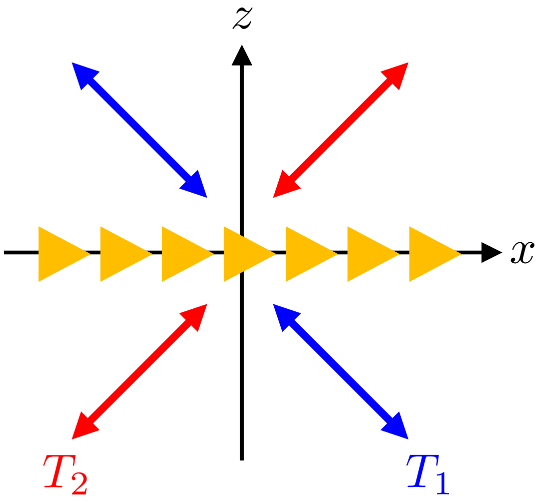


Scattering matrix

$$\bar{\bar{S}} = \begin{bmatrix} R_{11} & R_{12} & T_{13} & T_{14} \\ R_{21} & R_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & R_{33} & R_{34} \\ T_{41} & T_{42} & R_{43} & R_{44} \end{bmatrix}$$

We ignore polarization rotation

We might intuitively decide to implement a structure with broken σ_x like the following one



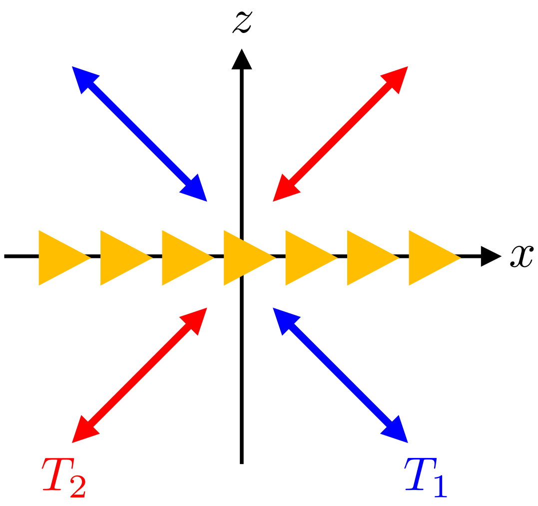
Scattering matrix

$$\bar{\bar{S}} = \begin{bmatrix} R_{11} & 0 & T_{13} & 0 \\ 0 & R_{22} & 0 & T_{24} \\ T_{13} & 0 & R_{11} & 0 \\ 0 & T_{24} & 0 & R_{22} \end{bmatrix}$$

Same values !

We have different transmissions for TE and TM waves but $T_1 = T_2$

Why Breaking σ_x Does Not Break Angular Transmission Symmetry ?

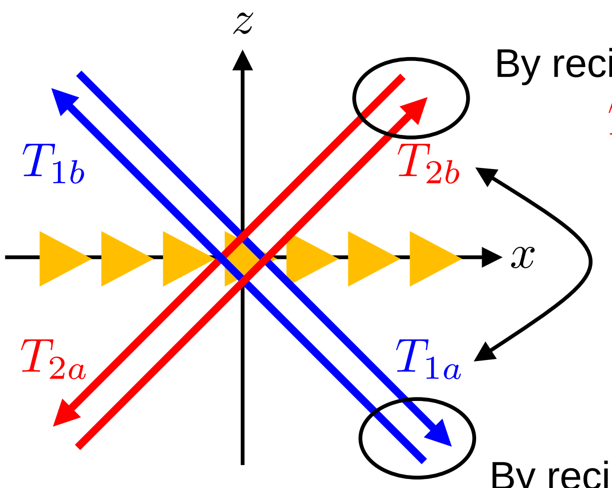


Scattering matrix

$$\bar{\bar{S}} = \begin{bmatrix} R_{11} & 0 & T_{13} & 0 \\ 0 & R_{22} & 0 & T_{24} \\ T_{13} & 0 & R_{11} & 0 \\ 0 & T_{24} & 0 & R_{22} \end{bmatrix}$$

Same values !

We have different transmissions for TE and TM waves but $T_1 = T_2$



By reciprocity, we have
$$T_{2a} = T_{2b}$$

σ_z symmetry implies
$$T_{1a} = T_{2b}$$

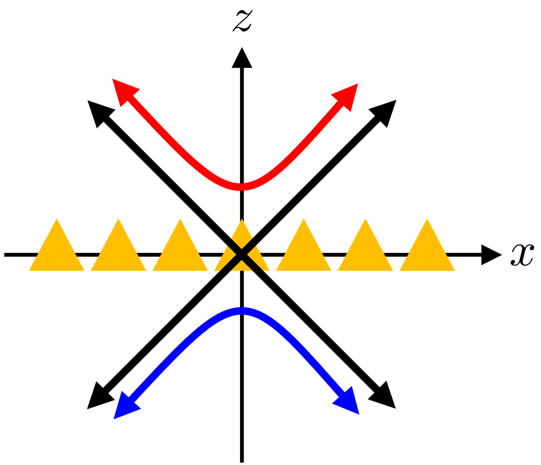
By reciprocity, we have
$$T_{1a} = T_{1b}$$

The system has a broken σ_x symmetry. However, it is reciprocal and is σ_z symmetric, which implies that

$$T_{1a} = T_{1b} = T_{2a} = T_{2b}$$

What does Breaking σ_z Do ?

structure with broken σ_z

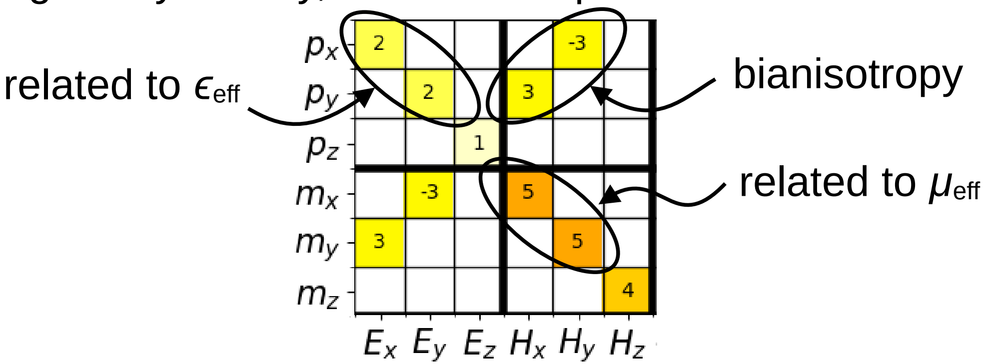


Scattering matrix

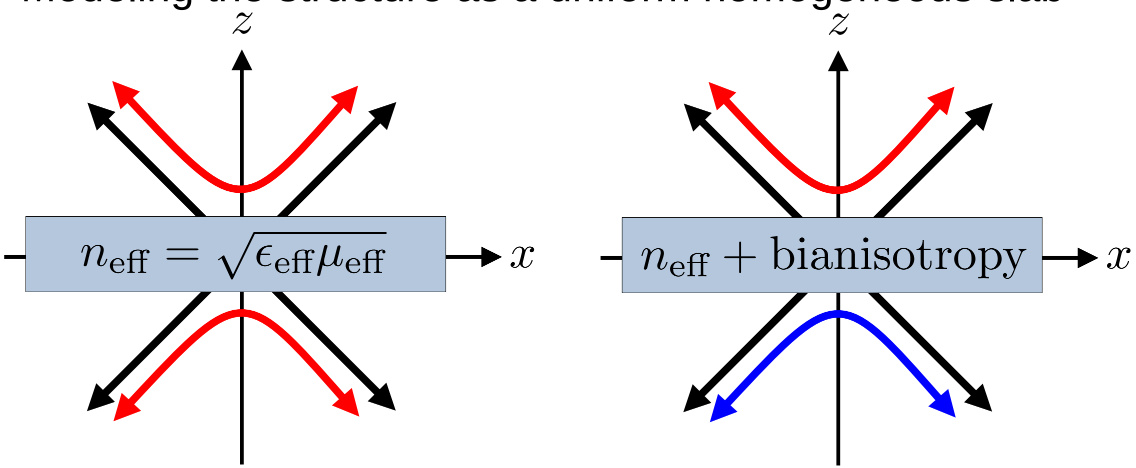
$$\overline{\overline{S}} = \begin{bmatrix} R_{11} & 0 & T_{13} & 0 \\ 0 & R_{22} & 0 & T_{24} \\ T_{13} & 0 & R_{33} & 0 \\ 0 & T_{24} & 0 & R_{44} \end{bmatrix}$$

Because of reciprocity and σ_x symmetry, all transmissions are equal. Only the reflections are different.

Assuming C_{4z} symmetry, the material parameters are bianisotropic



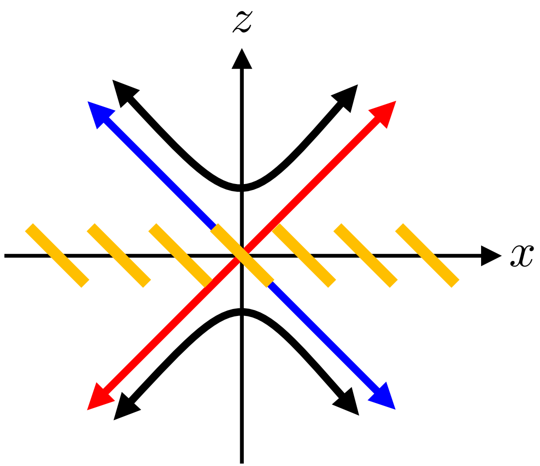
Modeling the structure as a uniform homogeneous slab



If we only model our structure with ϵ_{eff} and μ_{eff}, then its reflections must be symmetric. The only why to have reflection asymmetry is to include bianisotropy !

How do we Break Angular Transmission Symmetry ?

The structure has broken σ_x and σ_z symmetries and is σ_y and C_{2y} symmetric

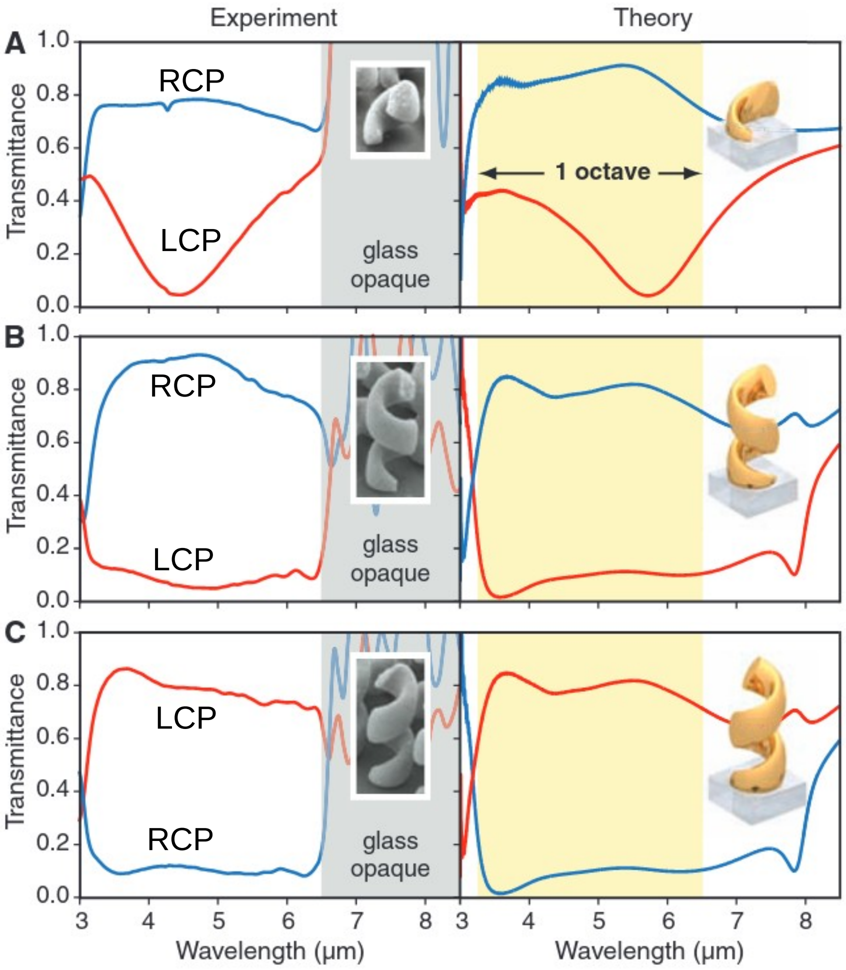


Scattering matrix

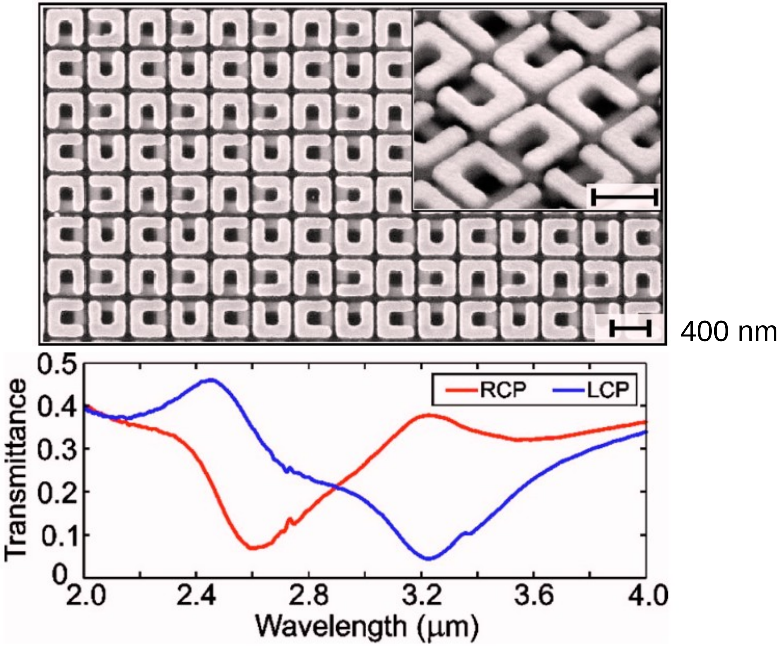
$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} R_{11} & 0 & T_{13} & 0 \\ 0 & R_{22} & 0 & T_{24} \\ T_{31} & 0 & R_{11} & 0 \\ 0 & T_{42} & 0 & R_{22} \end{bmatrix}$$

Different values !

Chiral Metasurfaces



doi.org/10.1126/science.1177031



https://doi.org/10.1364/OL.35.001593

doi.org/10.1186/s40580-015-0058-2

Chirality From Material Parameters

Chirality is obtained via the bianisotropic material paramaters

$$\overline{\overline{\zeta}}, \overline{\overline{\xi}} \propto \begin{bmatrix} \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_z}\lambda^-_{C_{2x}}\lambda^-_{C_{2y}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} \\ \lambda^-_{\sigma_y}\lambda^-_{C_{2x}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{C_{2y}}\lambda^-_{C_{2z}} & \lambda^-_{\sigma_x}\lambda^-_{\sigma_y}\lambda^-_{\sigma_z} \end{bmatrix}$$

What about the other terms ?

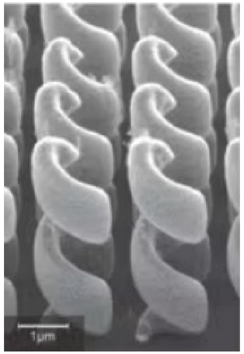
Chirality requires breaking all mirror symmetries

Extrinsic chirality

The structure is not geometrically chiral and but may lead to chiral optical responses under certain illumination conditions

Intrinsic chirality

The structure is geometrically chiral which leads to chiral optical responses



Jones Matrices and Corresponding Polarization Effects

$$\overline{\overline{\mathrm{T}}}_{\mathrm{LP}} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \longrightarrow \overline{\overline{\mathrm{T}}}_{\mathrm{CP}} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

No polarization effect

$$\overline{\overline{\mathrm{T}}}_{\mathrm{LP}} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \longrightarrow \overline{\overline{\mathrm{T}}}_{\mathrm{CP}} = \frac{1}{2}\begin{bmatrix} A+D & A-D \\ A-D & A+D \end{bmatrix}$$

Linear birefringence

$$\overline{\overline{\mathrm{T}}}_{\mathrm{LP}} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \longrightarrow \overline{\overline{\mathrm{T}}}_{\mathrm{CP}} = \begin{bmatrix} A & jB \\ -jB & A \end{bmatrix}$$

Polarization conversion

$$\overline{\overline{\mathrm{T}}}_{\mathrm{LP}} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \longrightarrow \overline{\overline{\mathrm{T}}}_{\mathrm{CP}} = \begin{bmatrix} A-jB & 0 \\ 0 & A+jB \end{bmatrix}$$

Circular birefringence

$$\overline{\overline{\mathrm{T}}}_{\mathrm{LP}} = \begin{bmatrix} A & B \\ -B & D \end{bmatrix} \longrightarrow \overline{\overline{\mathrm{T}}}_{\mathrm{CP}} = \frac{1}{2}\begin{bmatrix} A+D-j2B & A-D \\ A-D & A+D+j2B \end{bmatrix}$$

Linear/circular birefringence

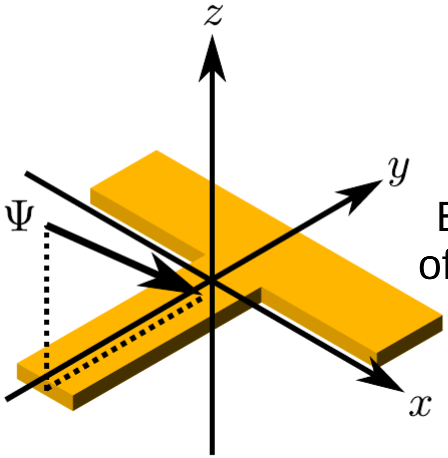
Chiral responses = circular birefringence

Linear-to-cicrular basis conversion

$$\overline{\overline{\mathrm{T}}}_{\mathrm{CP}} = \begin{bmatrix} t_{++} & t_{+-} \\ t_{-+} & t_{--} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} t_{xx}+t_{yy}-j(t_{xy}-t_{yx}) & t_{xx}-t_{yy}+j(t_{xy}+t_{yx}) \\ t_{xx}-t_{yy}-j(t_{xy}+t_{yx}) & t_{xx}+t_{yy}+j(t_{xy}-t_{yx}) \end{bmatrix}$$

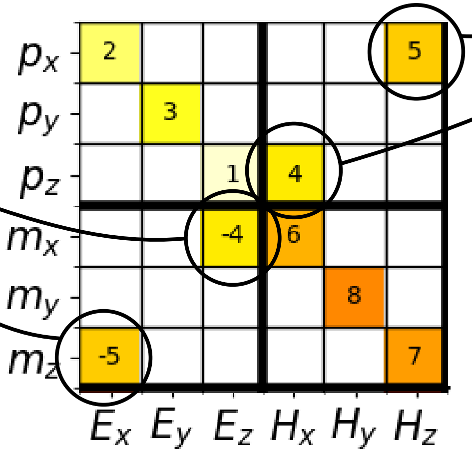
$$\overline{\overline{\mathrm{T}}}_{\mathrm{LP}} = \begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix}$$

Example of Extrinsic Chirality

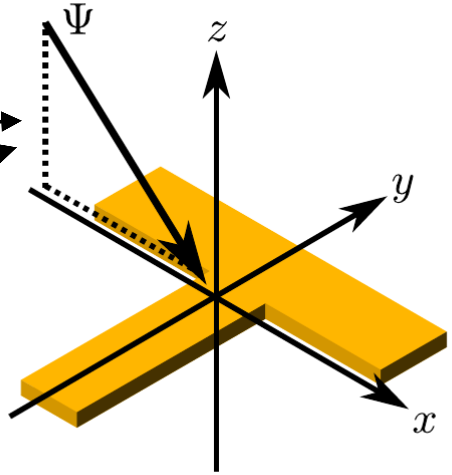


Excite either of these terms

Symmetries: σ_x, σ_z



Excite both of these terms



Scattering matrix

$$\overline{\overline{S}} = \begin{bmatrix} R_{11} & 0 & T_{13} & 0 \\ 0 & R_{22} & 0 & T_{24} \\ T_{13} & 0 & R_{11} & 0 \\ 0 & T_{24} & 0 & R_{22} \end{bmatrix}$$

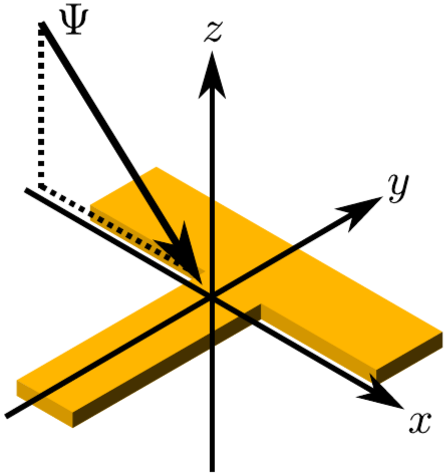
No rotation of polarization

Scattering matrix

$$\overline{\overline{S}} = \begin{bmatrix} R_{11} & R_{12} & T_{13} & -T_{14} \\ -R_{12} & R_{22} & T_{14} & T_{24} \\ T_{13} & T_{14} & R_{11} & R_{12} \\ -T_{14} & T_{24} & -R_{12} & R_{22} \end{bmatrix}$$

Extrinsic chirality: chiral response even though the structure is not chiral. The oblique wave breaks mirror symmetries σ_x and σ_z, while the structure has a broken σ_y symmetry

Example of Extrinsic Chirality



The structure has a broken σ_y symmetry, the wave break the σ_x *and* σ_z symmetries, which leads to a chiral response

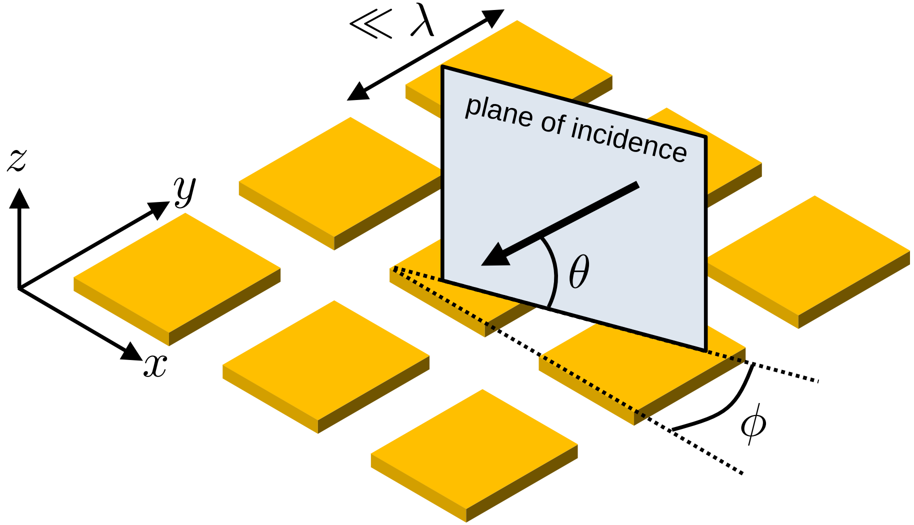
Scattering matrix

$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} R_{11} & R_{12} & T_{13} & -T_{14} \\ -R_{12} & R_{22} & T_{14} & T_{24} \\ T_{13} & -T_{14} & R_{11} & R_{12} \\ T_{14} & T_{24} & -R_{12} & R_{22} \end{bmatrix}$$

When considering scattering symmetries, we should not just consider the symmetries of the structure. Instead, we should consider the **total symmetries = structure + illumination.** The illumination therefore breaks symmetries but keep in mind that the scattering remains constrained by reciprocity.

Polarization Conversion From Isotropic Array

subwavelength array of square patches



Scattering matrix

$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} R_{11} & R_{12} & T_{13} & T_{14} \\ R_{12} & R_{22} & T_{14} & T_{24} \\ T_{13} & T_{14} & R_{11} & R_{12} \\ T_{14} & T_{24} & R_{12} & R_{22} \end{bmatrix}$$

Polarization conversion

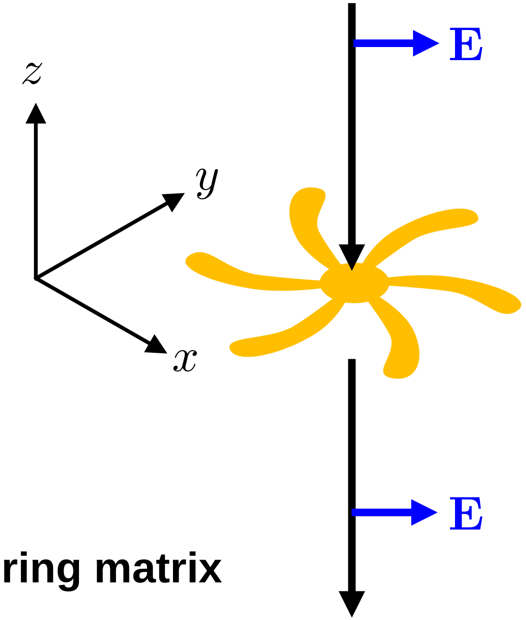
If the incident wave impinges on the array at normal incidence or oblique incidence along the xz- or yz-planes, no rotation of polarization occurs.

However, if the wave comes at an arbitrary oblique incidence, then polarization conversion is possible.

Chirality at Normal Incidence

Subwavelength array of flat spiral structures with broken σ_x and σ_y symmetries but σ_z symmetric

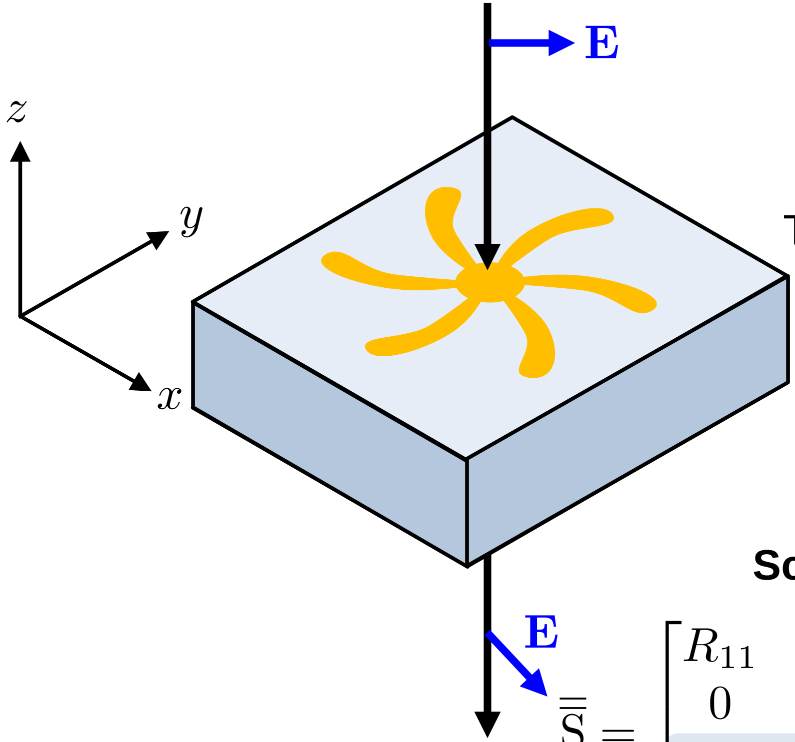
No substrate



Scattering matrix

$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} R_{11} & 0 & T_{13} & 0 \\ 0 & R_{11} & 0 & T_{13} \\ T_{13} & 0 & R_{11} & 0 \\ 0 & T_{13} & 0 & R_{11} \end{bmatrix}$$

With substrate



The substrate breaks the σ_z symmetry

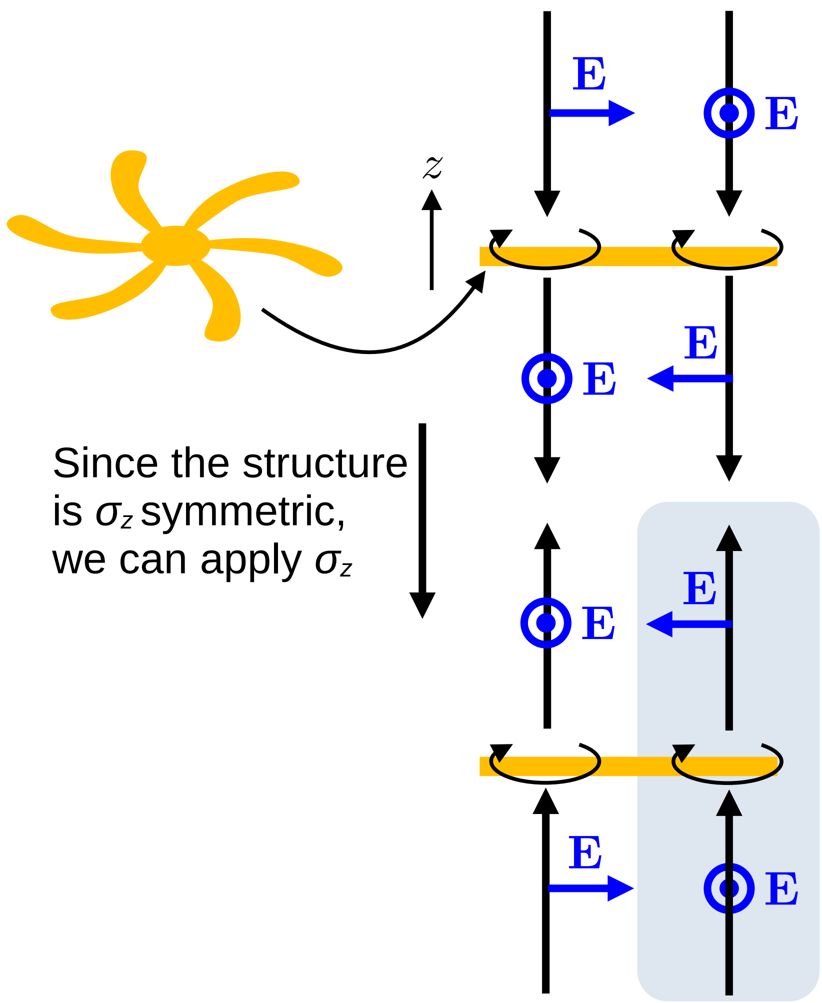
Scattering matrix

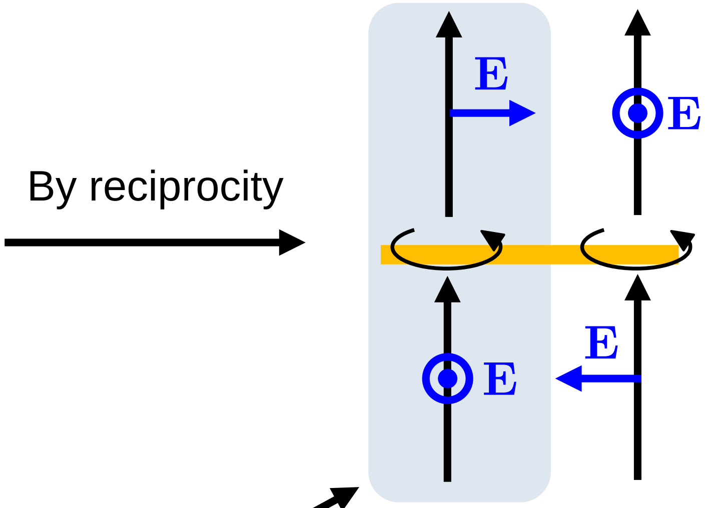
$$\overline{\overline{\mathrm{S}}} = \begin{bmatrix} R_{11} & 0 & T_{13} & T_{14} \\ 0 & R_{11} & -T_{14} & T_{13} \\ T_{13} & -T_{14} & R_{33} & 0 \\ T_{14} & T_{13} & 0 & R_{33} \end{bmatrix}$$

Circular birefringence (chirality)

Why Flat Spirals Do Not Produce Chiral Responses at Normal Incidence ?

Demonstration by absurdity: let's imagine the structure is chiral such that it rotates the polarization by 90°





Sending the wave backward must result in the same polarization state as the initial incident wave

Comparing these two situations, we see that they are not compatible. This situation is impossible! Flat spirals do not produce chiral responses at normal incidence in a reciprocal system.

If the system was geometrically chiral, then we could not apply the σ_z symmetry. Allowing the system to produce chiral responses at normal incidence.

What Have We Learned So Far….

- Spatial symmetries can also be used to assess the scattering matrix of a system.
- For some spatial symmetries, the fields of TE/TM polarizations flip and/or the **k**-vector flips direction. This allows us to formulate invariance conditions that apply on the scattering matrix of the system. Note that they are two invariance conditions, one for operations that flip the **k**-vector and another for those that do not flip the **k**-vector.
- As for the case of material parameters, we can develop an algorithm to find the scattering matrix of a given system described in terms of its spatial symmetries.
- For a subwavelength array, we see that breaking σ_x is not sufficient to break angular transmission symmetry because of reciprocity and σ_z symmetry.
- Similarly, breaking σ_z makes the reflection from both sides of the metasurface different but the angular transmission coefficients remain identical.
- To break angular transmission symmetry, we need to break both σ_x and σ_z.
- Chirality is generally a geometrical feature corresponding to broken mirror symmetries (σ_x, σ_y and σ_z). This is what we refer to as intrinsic chirality.
- Chirality leads to circular birefringence/dichroism (different phase shift/absorption for LCP and RCP waves).
- Even if the particle is not geometrically chiral, it may still exhibit chiral responses. This is known as extrinsic chirality. It works by exploiting the fact that what really matters is not just the symmetries of the structure but that of the total system (structure + illumination). So the direction of propagation of the exciting wave may further break the symmetries of the system leading to chiral responses.