Homework #2

solutions

Suppose we have two qubits in the following states:

$$|\Psi_1\rangle = \alpha_1\,|0\rangle + \beta_1|1\rangle$$

$$|\Psi_2\rangle = \alpha_2\,|0\rangle + \beta_2|1\rangle$$

a) *(20 points)* Write down a two qubit unitary U such that $U(|\Psi_1\rangle \otimes |\Psi_2\rangle) = |\Psi_2\rangle \otimes |\Psi_1\rangle$ for any two qubits $|\Psi_1\rangle$ and $|\Psi_2\rangle$.

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

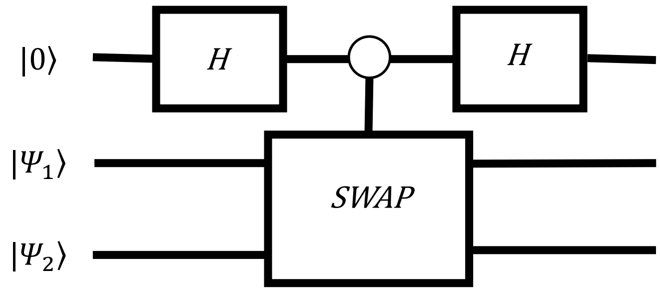
b) *(20 points)* Starting with U, write down a zero-controlled SWAP gate. Moreprecisely, write down the unitary ZCSWAP which does:

$$ZCSWAP\,(|0\rangle \otimes |\Psi_1\rangle \otimes |\Psi_2\rangle) = |0\rangle \otimes |\Psi_2\rangle \otimes |\Psi_1\rangle$$

$$ZCSWAP\,(|1\rangle \otimes |\Psi_1\rangle \otimes |\Psi_2\rangle) = |1\rangle \otimes |\Psi_1\rangle \otimes |\Psi_2\rangle$$

$$ZCSWAP = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c) *(10 points)* Let us now suppose we start with three qubits in the state $|0\rangle\,|\Psi_1\rangle\,|\Psi_2\rangle$, where we will call $|0\rangle$ the control qubit. We first apply Hadamard transform on the control qubit, then we apply ZCSWAP operation from above, and finally we again apply Hadamard transformation to the control qubit. Draw a circuit diagram for this process, where you can use a box for the ZCSWAP gate.



d) *(10 points)* What is the output $|\Phi\rangle$ of the circuit from c)?

$$
\begin{aligned}
|\Phi\rangle &= (H \otimes I \otimes I) ZCSWAP (H \otimes I \otimes I))(|0\rangle|\Psi_1\rangle|\Psi_2\rangle) \\
&= (H \otimes I \otimes I) ZCSWAP \left[\frac{1}{\sqrt{2}}(|0\rangle|\Psi_1\rangle|\Psi_2\rangle + |1\rangle|\Psi_1\rangle|\Psi_2\rangle)\right] \\
&= (H \otimes I \otimes I) \left[\frac{1}{\sqrt{2}}(|0\rangle|\Psi_2\rangle|\Psi_1\rangle + |1\rangle|\Psi_1\rangle|\Psi_2\rangle)\right] \\
&= \frac{1}{2}(|0\rangle + |1\rangle)|\Psi_2\rangle|\Psi_1\rangle + \frac{1}{2}(|0\rangle - |1\rangle)|\Psi_1\rangle|\Psi_2\rangle \\
&= \frac{1}{\sqrt{2}}|0\rangle \otimes \frac{1}{\sqrt{2}}(|\Psi_2\rangle|\Psi_1\rangle + |\Psi_1\rangle|\Psi_2\rangle) + \frac{1}{\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}(|\Psi_2\rangle|\Psi_1\rangle - |\Psi_1\rangle|\Psi_2\rangle)
\end{aligned}
$$

e) *(25 points)* A convenient way to write down the probabilities of obtaining measurement outcomes when measuring the control qubit in the computational basis is by computing

$$p_0 = \langle\Phi|\, |0\rangle\langle 0| \otimes I \otimes I\, |\Phi\rangle$$

$$p_1 = \langle\Phi|\, |1\rangle\langle 1| \otimes I \otimes I\, |\Phi\rangle$$

Apply this rule to show that:

$$p_0 = \frac{1}{2} + \frac{|\langle\Psi_1|\Psi_2\rangle|^2}{2}$$

$$p_1 = \frac{1}{2} - \frac{|\langle\Psi_1|\Psi_2\rangle|^2}{2}$$

$$
\begin{aligned}
p_0 = \langle\Phi||0\rangle\langle 0| \otimes I\otimes I|\Phi\rangle \\
&= \langle\Phi||0\rangle\langle 0| \otimes I\otimes I \\
&\cdot \left[\frac{1}{2}(|0\rangle|\Psi_2\rangle|\Psi_1\rangle + |0\rangle|\Psi_1\rangle|\Psi_2\rangle + |1\rangle|\Psi_2\rangle|\Psi_1\rangle - |1\rangle|\Psi_1\rangle|\Psi_2\rangle)\right] \\
&= \langle\Phi|\frac{1}{2}(|0\rangle\langle 0|0\rangle \otimes I|\Psi_2\rangle\otimes I|\Psi_1\rangle + |0\rangle\langle 0|0\rangle \otimes I|\Psi_1\rangle\otimes I|\Psi_2\rangle \\
&+ |0\rangle\langle 0|1\rangle \otimes I|\Psi_2\rangle\otimes I|\Psi_1\rangle - |0\rangle\langle 0|1\rangle \otimes I|\Psi_1\rangle\otimes I|\Psi_2\rangle) = \\
&= \langle\Phi|\frac{1}{2}(|0\rangle|\Psi_2\rangle|\Psi_1\rangle + |0\rangle|\Psi_1\rangle|\Psi_2\rangle) \\
&= \frac{1}{2}(\langle 0|\langle\Psi_2|\langle\Psi_1| + \langle 0|\langle\Psi_1|\langle\Psi_2| + \langle 1|\langle\Psi_2|\langle\Psi_1| - \langle 1|\langle\Psi_1|\langle\Psi_2|) \\
&\cdot\frac{1}{2}(|0\rangle|\Psi_2\rangle|\Psi_1\rangle + |0\rangle|\Psi_1\rangle|\Psi_2\rangle) \\
&= \frac{1}{4}(\langle 0|0\rangle\langle\Psi_2|\Psi_2\rangle\langle\Psi_1|\Psi_1\rangle \\
&+ \langle 0|0\rangle\langle\Psi_2|\Psi_1\rangle\langle\Psi_1|\Psi_2\rangle + \langle 0|0\rangle\langle\Psi_1|\Psi_2\rangle\langle\Psi_2|\Psi_1\rangle + \langle 0|0\rangle\langle\Psi_1|\Psi_1\rangle\langle\Psi_2|\Psi_2\rangle \\
&+ \langle 1|0\rangle\langle\Psi_2|\Psi_2\rangle\langle\Psi_1|\Psi_1\rangle + \langle 1|0\rangle\langle\Psi_2|\Psi_1\rangle\langle\Psi_1|\Psi_2\rangle - \langle 1|0\rangle\langle\Psi_1|\Psi_2\rangle\langle\Psi_2|\Psi_1\rangle \\
&- \langle 1|0\rangle\langle\Psi_1|\Psi_1\rangle\langle\Psi_2|\Psi_2\rangle) \\
&= \frac{1}{4}(1 + \langle\Psi_2|\Psi_1\rangle\,\langle\Psi_1|\Psi_2\rangle + \langle\Psi_1|\Psi_2\rangle\,\langle\Psi_2|\Psi_1\rangle + 1) \\
&= \frac{1}{4}(2 + \langle\Psi_1|\Psi_2\rangle^*\,\langle\Psi_1|\Psi_2\rangle + \langle\Psi_1|\Psi_2\rangle\,\langle\Psi_1|\Psi_2\rangle^*) \\
&= \frac{1}{4}(2 + |\langle\Psi_1|\Psi_2\rangle\,|^2 + |\langle\Psi_1|\Psi_2\rangle\,|^2) = \frac{1}{2} + \frac{|\langle\Psi_1|\Psi_2\rangle\,|^2}{2}
\end{aligned}
$$

Similarly,

$$p_1 = \frac{1}{2} - \frac{|\langle\Psi_1|\Psi_2\rangle|^2}{2}$$

f) *(15 points)* How can you use this circuit for testing whether $|\Psi_1\rangle = |\Psi_2\rangle$? Explain when your procedure works well, and when you will only gain some confidence.

If $|\Psi_1\rangle = |\Psi_2\rangle$ *then* $p_0 = 1$ *and* $p_1 = 0$ *so we have* 0% *probability of measuring a* 1. *This means that if at the end of the experiment we do measure a 1, we can conclude that* $|\Psi_1\rangle \neq |\Psi_2\rangle$ ($p_1 > 0$). *If we measure a* 0*, we can't draw a conclusion, so we would need to repeat the experiment multiple times.*