Homework #3

solutions

At work, Alice, Bob and Chuck decide to prepare a GHZ state, where Alice keeps A, Bob keeps B and Chuck keeps C. Once they arrive at home, they are restricted to just classical communication, but Bob and Chuck want to communicate an arbitrary qubit to each other.

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B |0\rangle_C + |1\rangle_A |1\rangle_B |1\rangle_C)$$

a) Alice decides to apply a Hadamard transform to her qubit. Write down the resulting state.

$$|\Psi\rangle = (H \otimes I \otimes I) |GHZ\rangle = \tfrac{1}{\sqrt{2}} |0\rangle \otimes \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \tfrac{1}{\sqrt{2}} |1\rangle \otimes \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

b) After applying the Hadamard transform, imagine now that Alice measures her qubit in the computational basis. Write down the state shared by Bob and Chuck for the case that Alice obtained measurement outcome $|0\rangle$ and for the case that she obtained outcome $|1\rangle$.

If $|0\rangle$*, then* $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

If $|1\rangle$*, then* $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

c) Describe a protocol after which Bob and Chuck share the positive even parity Bell state $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, where Alice is allowed to send one bit of classical communication to either Bob or Chuck.

They follow the steps from a) and b). Alice then sends the measurement result to Bob/Chuck. If the outcome was $|1\rangle$*, they have to apply* $I \otimes Z$ *or* $Z \otimes I$*. Otherwise, they don't do anything.*

2.

Using CNOTs, Toffoli gates and single qubit gates implement the circuit that results in the following unitary:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Truth table of this unitary:

$x_2x_1x_0$	$x_2x_1x_0$
000	000
001	010
010	011
011	100
100	101
101	110
110	111
111	001

We can analyze the truth table and of the x_2 qubit:

$x_2x_1x_0$	x_2
000	0
001	0
010	0
011	1
100	1
101	1
110	1
111	0

This corresponds to the TOFFOLI gate (the qubit changes the state only when x_1x_0=|11>), therefore the resulting circuit can start with the TOFFOLI gate. Next we analyze the truth table for x_1 taking into account that the circuit started with a TOFFOLI targeting x_2

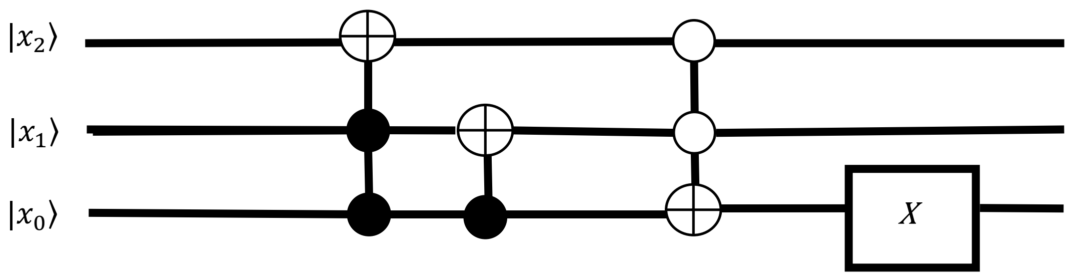
	Toffoli	
$x_2x_1x_0$	$x_2x_1x_0$	x_1
000	000	0
001	001	1
010	010	1
011	111	0
100	100	0
101	101	1

110	110	1
111	011	0

This truth table corresponds to a x_0 controlled CNOT, since x1 changes the state only when qubit x_0 is in state |1>, therefore the circuit starts with a TOFFOLI and continues with CNOT gate targeting x_1 controlled by x_0 . Finally, the truth table for the x_0

	Toffoli	CNOT(X_0-X_1)	
$x_2x_1x_0$	$x_2x_1x_0$	$x_2x_1x_0$	$x_2x_1x_0$
000	000	000	000
001	001	011	010
010	010	010	011
011	111	101	100
100	100	100	101
101	101	111	110
110	110	110	111
111	011	001	001

We can notice that x_0 does not change the state only when x_2x_1=|00>, which corresponds to a zero controlled TOFFOLI gate followed by an X gate on x_0 therefore the final circuit is:



Check:

$$U = (I\otimes I\otimes X)ZTOFFOLI(I\otimes CNOT)TOFFOLI =$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = U$$