Exercise set #1

September 22, 2021

Exercise 1:

Given the following qubits:

$$|\Psi_1\rangle = |0\rangle$$

$$|\Psi_2\rangle = |0\rangle - i\,|1\rangle$$

$$|\Psi_3\rangle = \sqrt{2}\,|0\rangle + i\sqrt{2}\,|1\rangle$$

a) Verify if the three cubits are normalized and if not, normalize them.

b) Show that $|\Psi_2\rangle$ and $|\Psi_3\rangle$ are orthogonal.

c) Show that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are not orthogonal.

d) What is the probability that if we measured $|\Psi_3\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$ we would get $|1\rangle$?

e) What is the probability that if we measured $|\Psi_1\rangle$ in the $\{|\Psi_2\rangle, |\Psi_3\rangle\}$ basis we would get $|\Psi_2\rangle$?

Exercise 2:

Given the single cubit quantum gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the qubits:

$$|\Psi_1\rangle = |1\rangle$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)$$

a) Show that X, Y, Z and H are unitary and hermitian.

b) Show that $XYZ = iI$.

c) Show that $HXH = Z$.

d) Show that $HZH = X$ without explicitly multiplying the matrices.

e) Show that $HXHZHXH = Z$ without explicitly multiplying the matrices.

f) What is the result of applying Y to $|\Psi_2\rangle$?

g) What is the result of applying H to $|\Psi_1\rangle$?

h) How can we prepare $|\Psi_2\rangle$ from $|\Psi_1\rangle$?

Exercise 3:

If $\hat{n} = (n_X, n_Y, n_Z)$ is a real unit vector in three dimensional space, we can define a rotation by θ around the $\hat{n}$ axis as:

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right)(n_X X + n_Y Y + n_Z Z)$$

a) Show that $R_{\hat{x}}(\pi) = -iX$, where $\hat{x} = (1,0,0)$.

b) What is the result of applying $R_{\hat{z}}(\pi)$, where $\hat{z} = (0,0,1)$,

to $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$?

c) Find $\hat{n}$ so that $H = iR_{\hat{n}}(\pi)$.

d) Show that $H = iR_{\hat{x}}(\pi)R_{\hat{y}}(\frac{\pi}{2})$, where $\hat{x} = (1,0,0)$ and $\hat{y} = (0,1,0)$.

e) Show that $R_{\hat{z}}(\theta) Y R_{\hat{z}}(\theta)^\dagger = \cos\theta Y - \sin\theta X$.