Exercise set #2

solutions

September 30, 2021

Exercise 1:

Given the following qubits:

$$|\Psi_1\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$
$$|\Psi_2\rangle = \frac{1}{5}(4\,|00\rangle + 2\,|01\rangle + 2\,|10\rangle + |11\rangle)$$
$$|\Psi_3\rangle = \frac{1}{2}(i\,|00\rangle - i\,|01\rangle - |10\rangle - i\,|11\rangle)$$
$$|\Psi_4\rangle = \frac{1}{2}(|0\rangle + i\,|1\rangle + i\,|2\rangle - |3\rangle)$$
$$|\Psi_5\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|10\rangle) - \frac{i}{\sqrt{2}}\,|11\rangle$$

a) Are the qubits entangled?

$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ - *not entangled*

$|\Psi_2\rangle = \frac{1}{\sqrt{5}}(2\,|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{5}}(2\,|0\rangle + |1\rangle)$ - *not entangled*

$|\Psi_3\rangle$ - *entangled*

$|\Psi_4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ - *not entangled*

$|\Psi_5\rangle$ - *entangled*

b) Determine the tensor product $X \otimes H$.

$$X \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

c) Determine the tensor product $Z \otimes X$.

$$Z \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

d) Determine the tensor product $X \otimes Y$.

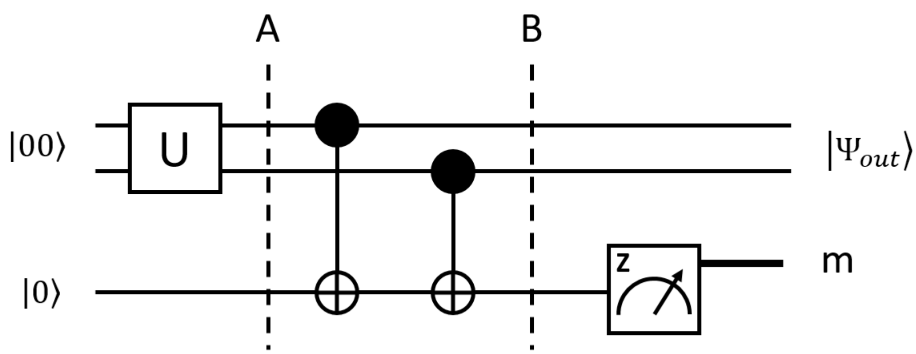
$$X \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

e) What is the result of applying $X \otimes H$ to $|\Psi_1\rangle$?

$$X \otimes H\,|\Psi_1\rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Exercise 2:

Given the following circuit:



a) Determine U so that at point A the top register will be in the maximum superposition state $|\Psi_A^{top}\rangle = \frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle)$.

$$U = H \otimes H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

b) What is the state of the three cubit system at point A?

$$
\begin{aligned}
|\Psi_A\rangle &= \frac{1}{2}(|00\rangle\,|0\rangle + |01\rangle\,|0\rangle + |10\rangle\,|0\rangle + |11\rangle\,|0\rangle) \\
&= \frac{1}{2}(|0\rangle + |2\rangle + |4\rangle + |6\rangle)
\end{aligned}
$$

c) What is the state of the three cubit system at point B?

$$
\begin{aligned}
|\Psi_A\rangle &= \frac{1}{2}(|00\rangle\,|0\rangle + |01\rangle\,|0\rangle + |10\rangle\,|0\rangle + |11\rangle\,|0\rangle) \\
&\xrightarrow{CNOT} \frac{1}{2}(|00\rangle\,|0\rangle + |01\rangle\,|0\rangle + |10\rangle\,|1\rangle + |11\rangle\,|1\rangle) \\
&\xrightarrow{CNOT} \frac{1}{2}(|00\rangle\,|0\rangle + |01\rangle\,|1\rangle + |10\rangle\,|1\rangle + |11\rangle\,|0\rangle) \\
|\Psi_B\rangle &= \frac{1}{2}(|0\rangle + |3\rangle + |5\rangle + |6\rangle)
\end{aligned}
$$

d) What is the probability we obtain $m = +1$ (outcome $|0\rangle$) when we measure the bottom qubit in the computational basis?

$$
\begin{aligned}
|\Psi_B\rangle &= \frac{1}{2}(|00\rangle\,|0\rangle + |01\rangle\,|1\rangle + |10\rangle\,|1\rangle + |11\rangle\,|0\rangle) \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes \frac{1}{\sqrt{2}}\,|1\rangle \\
&= \frac{1}{\sqrt{2}}(|\Phi_+ 0\rangle + |\Psi_+ 1\rangle) \quad -> p = \frac{1}{2}
\end{aligned}
$$

or

$p = \sum_{e \in S} \langle \Psi_B | e \rangle \langle e | \Psi_B \rangle = \frac{1}{2}$,

where S is the set of all basis vectors e such that the qubit in question takes the requested measurement value.

In this case, $S = \{00\underline{0}, 01\underline{0}, 10\underline{0}, 11\underline{0}\}$.

e) What is the final state $|\Psi_{out}\rangle$ of the top register when we obtained $m = +1$ (outcome $|0\rangle$) when we measured the bottom qubit in the computational basis?

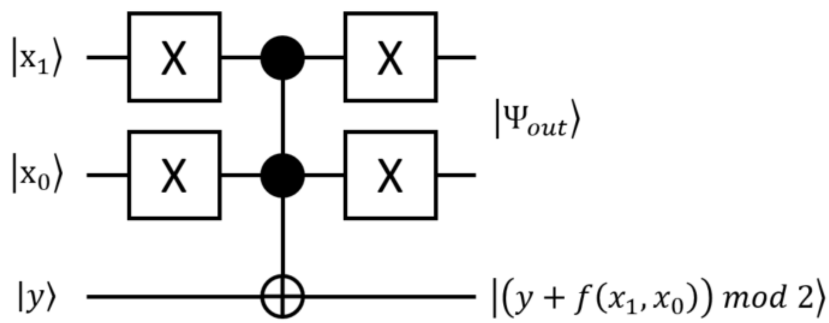
$|\Psi_{out}\rangle = |\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

f) Is the final state obtained at e) entangled?

Yes.

Exercise 3:

Given the following circuit:



a) What is the value of $|\Psi_{out}\rangle$?

$|\Psi_{out}\rangle = |x_1 x_0\rangle$

b) Suppose $|x_1 x_0\rangle = |00\rangle$ and $|y\rangle = |0\rangle$, what is the final three qubit state?

$|\Psi\rangle = (X \otimes X \otimes I) TOFFOLI (X \otimes X \otimes I) |000\rangle = |001\rangle$

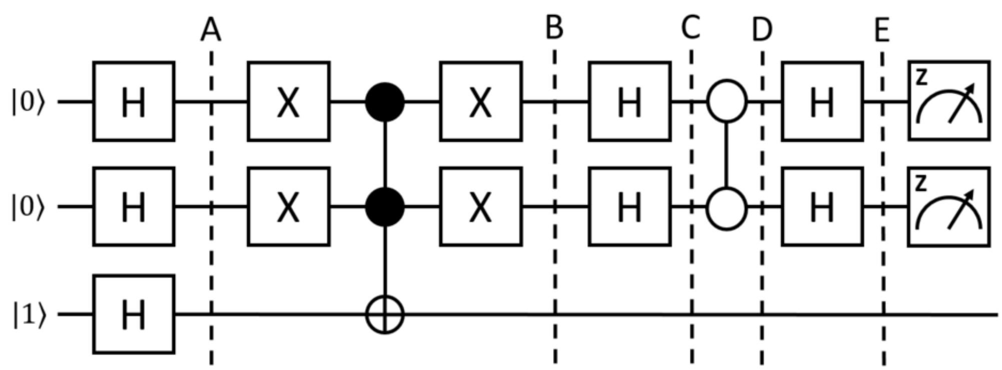
c) Repeat for all the other input states and write the final value of the bottom qubit into the following table:

x_1	x_0	y	$(y + f(x_1, x_0)) mod\, 2$
0	0	0	1
0	1	0	0
1	0	0	0
1	1	0	0
0	0	1	0
0	1	1	1
1	0	1	1
1	1	1	1

d) This function corresponds to that of a search problem with a specific solution x^*. What is x^*?

$x^* = 00$

We will implement Grover's algorithm as shown in the following figure:



e) Determine the state of the system at points A,B,C,D and E, knowing that the gate between C and D has the following unitary:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|\Psi_A\rangle = (H \otimes H \otimes H)\,|001\rangle = \frac{1}{\sqrt{8}}(|0\rangle - |1\rangle + |2\rangle - |3\rangle + |4\rangle - |5\rangle + |6\rangle - |7\rangle)$$

$$\begin{aligned} |\Psi_B\rangle &= (X \otimes X \otimes I)TOFFOLI(X \otimes X \otimes I)\,|\Psi_A\rangle \\ &= \frac{1}{\sqrt{8}}(-|0\rangle + |1\rangle + |2\rangle - |3\rangle + |4\rangle - |5\rangle + |6\rangle - |7\rangle) \end{aligned}$$

$$|\Psi_C\rangle = (H \otimes H \otimes I)\,|\Psi_B\rangle = \frac{1}{\sqrt{8}}(|0\rangle - |1\rangle - |2\rangle + |3\rangle - |4\rangle + |5\rangle - |6\rangle + |7\rangle)$$

$$|\Psi_D\rangle = (\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes I)\,|\Psi_C\rangle = \frac{1}{\sqrt{8}}(-|0\rangle + |1\rangle - |2\rangle + |3\rangle - |4\rangle + |5\rangle - |6\rangle + |7\rangle)$$

$$|\Psi_E\rangle = H \otimes H \otimes I\,|\Psi_D\rangle = \frac{1}{\sqrt{2}}(-|000\rangle + |001\rangle) = |00\rangle \otimes \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

f) Which of the five states from e) have entanglement in the top two qubits?

$$|\Psi_B\rangle = \frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|\Psi_C\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

g) Did the algorithm succeed in finding $x^* = 00$?

Yes, the measurement outcome is "00" with 100% probability.