Exercise set #1

solutions

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Exercise 1:

Given the following qubits:

$$|\Psi_1\rangle = |0\rangle$$

$$|\Psi_2\rangle = |0\rangle - i\,|1\rangle$$

$$|\Psi_3\rangle = \sqrt{2}\,|0\rangle + i\sqrt{2}\,|1\rangle$$

a) Verify if the three cubits are normalized and if not, normalize them.

$\langle\Psi_1|\Psi_1\rangle = 1$ - *normalized*

$\langle\Psi_2|\Psi_2\rangle = 2$ - *not normalized;* $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$

$\langle\Psi_3|\Psi_3\rangle = 4$ - *not normalized;* $|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$

b) Show that $|\Psi_2\rangle$ and $|\Psi_3\rangle$ are orthogonal.

after normalization, $\langle\Psi_2|\Psi_3\rangle = 0$ - *orthogonal*

c) Show that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are not orthogonal.

after normalization, $\langle\Psi_1|\Psi_2\rangle = \frac{1}{\sqrt{2}}$ *- not orthogonal*

d) What is the probability that if we measured $|\Psi_3\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$ we would get $|1\rangle$?

after normalization, $|\Psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle;\quad p_{|1\rangle} = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$

e) What is the probability that if we measured $|\Psi_1\rangle$ in the $\{|\Psi_2\rangle, |\Psi_3\rangle\}$ basis we would get $|\Psi_2\rangle$?

after normalization, $|\Psi_1\rangle = |0\rangle = \frac{1}{\sqrt{2}}|\Psi_2\rangle + \frac{1}{\sqrt{2}}|\Psi_3\rangle;\quad p_{|\Psi_2\rangle} = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

or

after normalization, $p_{|\Psi_2\rangle} = |\langle\Psi_2|\Psi_1\rangle|^2 = \frac{1}{2}$

Exercise 2:

Given the single cubit quantum gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the qubits:

$$|\Psi_1\rangle = |1\rangle$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

a) Show that X, Y, Z and H are unitary and hermitian.

$XX^\dagger = YY^\dagger = ZZ^\dagger = HH^\dagger = I$ - *unitary*

$X = X^\dagger;\ Y = Y^\dagger;\ Z = Z^\dagger;\ H = H^\dagger$ - *hermitian*

b) Show that $XYZ = iI$.

c) Show that $HXH = Z$.

d) Show that $HZH = X$ without explicitly multiplying the matrices.

$HXH = Z$ *and* H *is unitary and hermitian, so*

$HXH = Z <=> HHXHH = HZH <=> IXI = HZH <=> X = HZH$

e) Show that $HXHZHXH = Z$ without explicitly multiplying the matrices.

$HXH = Z$ *and* Z *is unitary and hermitian, so*

$HXHZHXH = ZZZ = IZ = Z$

f) What is the result of applying Y to $|\Psi_2\rangle$?

$$Y|\Psi_2\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(-i|0\rangle + i|1\rangle)$$

g) What is the result of applying H to $|\Psi_1\rangle$?

$$H|\Psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

h) How can we prepare $|\Psi_2\rangle$ from $|\Psi_1\rangle$?

$$|\Psi_2\rangle = ZH|\Psi_1\rangle$$

Exercise 3:

If $\hat{n} = (n_X, n_Y, n_Z)$ is a real unit vector in three dimensional space, we can define a rotation by θ around the $\hat{n}$ axis as:

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)(n_X X + n_Y Y + n_Z Z)$$

a) Show that $R_{\hat{x}}(\pi) = -iX$, where $\hat{x} = (1, 0, 0)$.

b) What is the result of applying $R_{\hat{z}}(\pi)$, where $\hat{z} = (0, 0, 1)$, to $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$?

$$R_{\hat{z}}(\pi)|\Psi\rangle = \left[\cos\left(\frac{\pi}{2}\right) I - i\sin\left(\frac{\pi}{2}\right) Z\right]|\Psi\rangle = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}(-i|0\rangle - |1\rangle)$$

c) Find $\hat{n}$ so that $H = iR_{\hat{n}}(\pi)$.

$$\hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$$

d) Show that $H = iR_{\hat{x}}(\pi)R_{\hat{y}}(\frac{\pi}{2})$, where $\hat{x} = (1, 0, 0)$ and $\hat{y} = (0, 1, 0)$.

$$iR_{\hat{x}}(\pi)R_{\hat{y}}(\tfrac{\pi}{2}) = i\,(-iX)\left(\tfrac{1}{\sqrt{2}}I - \tfrac{i}{\sqrt{2}}Y\right) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = H$$

e) Show that $R_{\hat{z}}(\theta)YR_{\hat{z}}(\theta)^{\dagger} = \cos\theta Y - \sin\theta X$.

$$\begin{aligned}
R_{\hat{z}}(\theta)YR_{\hat{z}}(\theta)^{\dagger} &= \left(\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)Z\right)Y\left(\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)Z\right)^{\dagger} \\
&= \left(\cos\left(\frac{\theta}{2}\right)Y - i\sin\left(\frac{\theta}{2}\right)ZY\right)\left(\cos\left(\frac{\theta}{2}\right)I + i\sin\left(\frac{\theta}{2}\right)Z\right) \\
&= \cos^2\left(\frac{\theta}{2}\right)Y + \sin^2\left(\frac{\theta}{2}\right)ZYZ + \\
&+ i\left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)YZ - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)ZY\right) \\
&= \left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]Y \\
&+ i\left[\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)iX - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)(-iX)\right] \\
&= \cos(\theta)Y - 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)X \\
&= \cos(\theta)Y - \sin(\theta)X
\end{aligned}$$

knowing that:

$$cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

$$sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$YZ = iX$$

$$ZYZ = -Y$$

$$YZ = -ZY$$