Exercise set #3

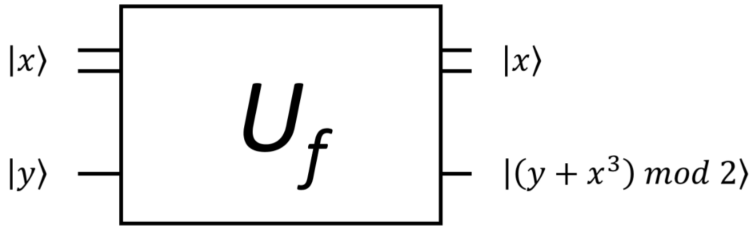
solutions

Exercise 1:

Consider the boolean function

$$f(x) = x^3 mod\, 2,$$

with $x = x_1 x_0$ a 2-bit number (x_0 is the LSB, x_1 is the MSB). We would like to encode this function in a quantum unitary, using a top register with 2 qubits and a bottom register with 1 qubit:



a) Suppose we prepare the top register in the maximal superposition state

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

and the bottom register in $|0\rangle$. What will be the state of all the qubits after the application of U_f?

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{2}(|0\rangle|0\rangle + |1\rangle|0\rangle + |2\rangle|0\rangle + |3\rangle|0\rangle) \\
&\xrightarrow{U_f} \frac{1}{2}(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|8 \; mod \; 2\rangle + |3\rangle|27 \; mod \; 2\rangle) \\
&= \frac{1}{2}(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|0\rangle + |3\rangle|1\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) \otimes \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle) \otimes \frac{1}{\sqrt{2}}|1\rangle
\end{aligned}$$

b) Suppose we obtain the result $m = +1$ (projection onto $|0\rangle$) by measuring the lower qubit in the computational basis. What is the final state of the top two qubits?

$\frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$

c)Suppose instead that we prepare the bottom register in $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. What will be the state of all the qubits after the application of U_f?

$$
\begin{aligned}
|\Psi\rangle &= \frac{1}{2\sqrt{2}}(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle + |2\rangle|0\rangle - |2\rangle|1\rangle + |3\rangle|0\rangle - |3\rangle|1\rangle) \\
&\xrightarrow{U_f} \frac{1}{2\sqrt{2}}(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|1\rangle - |1\rangle|0\rangle + |2\rangle|0\rangle - |2\rangle|1\rangle + |3\rangle|1\rangle - |3\rangle|0\rangle) \\
&= \frac{1}{2}(|0\rangle - |1\rangle + |2\rangle - |3\rangle) \otimes \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}(-|0\rangle + |1\rangle - |2\rangle + |3\rangle) \otimes \frac{1}{\sqrt{2}}|1\rangle
\end{aligned}
$$

d)Suppose we obtain the result $m = +1$ (projection onto $|0\rangle$) by measuring the lower qubit in the computational basis. What is the final state of the top two qubits?

$\frac{1}{2}(|0\rangle - |1\rangle + |2\rangle - |3\rangle)$

e) Draw the quantum circuit that implements the unitary.

We represent the two binary digits of $|x\rangle$ *as* x_1 *and* x_0 *and the one binary digit from* $|y\rangle$ *as* y_0*.*

We know that:

$$
\begin{aligned}
A \oplus B &= A \cdot \overline{B} + \overline{A} \cdot B \\
\overline{A + B} &= \overline{A} \cdot \overline{B} \\
\overline{A \cdot B} &= \overline{A} + \overline{B}
\end{aligned}
$$

In order to find the expression for x^3 *we will first find the expression for* x^2

by multiplying x with itself:

$$\begin{array}{ccccc}
 & & x_1 & x_0 & \times \\
 & & x_1 & x_0 & \\
\cline{1-4}
 & & x_1 \cdot x_0 & x_0 \cdot x_0 & \\
 & x_1 \cdot x_1 & x_1 \cdot x_0 & & \\
\cline{1-4}
x_1 \cdot x_1 \cdot x_0 \cdot x_1 & x_1 \cdot x_1 \oplus x_1 \cdot x_0 & 0 & x_0 & \\
 & & & & \\
\cline{1-4}
\multicolumn{1}{|c}{x_1 \cdot x_0} & x_1 \cdot \overline{x_0} & 0 & \multicolumn{1}{c|}{x_0} & \\
\cline{1-4}
\end{array}$$

Please observe that a digit from the addition result will propagate a carry if both terms are 1, i.e. $A \oplus B$ will propagate a carry by adding the term $A \cdot B$ to the next significant digit.

We can now multiply with x one more time to get x^3:

$$\begin{array}{cccccc}
 & x_1 \cdot x_0 & x_1 \cdot \overline{x_0} & 0 & x_0 & \times \\
 & & & x_1 & x_0 & \\
\cline{1-5}
 & x_1 \cdot x_0 \cdot x_0 & x_1 \cdot \overline{x_0} \cdot x_0 & 0 & x_0 \cdot x_0 & \\
x_1 \cdot x_1 \cdot x_0 & x_1 \cdot x_1 \cdot \overline{x_0} & 0 & x_1 \cdot x_0 & & \\
\cline{1-5}
x_1 \cdot x_0 & x_1 \cdot x_0 \oplus x_1 \cdot \overline{x_0} & 0 & x_1 \cdot x_0 & x_0 & \\
 & & & & & \\
\cline{1-5}
\multicolumn{1}{|c}{x_1 \cdot x_0} & x_1 & 0 & x_1 \cdot x_0 & \multicolumn{1}{c|}{x_0} & \\
\cline{1-5}
\end{array}$$

Finally, we will add y_0 to obtain the final result:

$$\begin{array}{cccccc}
x_1 \cdot x_0 & x_1 & 0 & x_1 \cdot x_0 & x_0 & + \\
 & & & & y_0 & \\
\cline{1-5}
x_1 \cdot x_0 & x_1 & x_1 \cdot x_0 \cdot x_0 \cdot y_0 & x_1 \cdot x_0 \oplus x_0 \cdot y_0 & x_0 \oplus y_0 & \\
 & & & & & \\
\cline{1-5}
\multicolumn{1}{|c}{\ldots} & \ldots & \ldots & \multicolumn{1}{c|}{\ldots} & \multicolumn{1}{c|}{x_0 \oplus y_0} & \\
\cline{1-5}
\end{array}$$

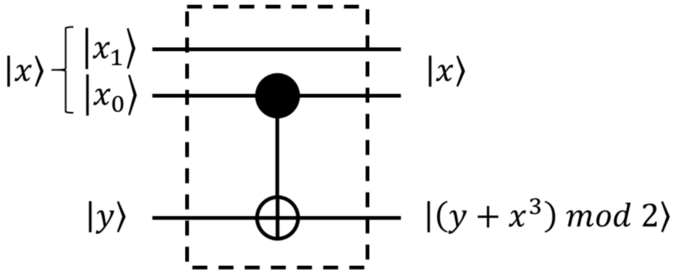
We only need to keep the least significant digit because of the mod 2 that *appears in the unitary. The answer is then:*

$$x_1 \to x_1$$

$$x_0 \to x_0$$

$$y_0 \to x_0 \oplus y_0$$

Which is represented as:



Exercise 2:

Alice wants to teleport to Bob a qubit $|\Phi\rangle = \alpha |0\rangle + \beta |1\rangle$ using an entangled qubit pair $|e\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle |0_B\rangle + |1_A\rangle |1_B\rangle)$ that they already share (Alice has qubit A and Bob has qubit B) and a classical communications channel.

a) How can Alice and Bob prepare $|e\rangle$ from $|00\rangle$?

$|e\rangle = CNOT(H \otimes I) |00\rangle$

b) Write the resulting three qubit state $|\Psi\rangle$ where Alice has the first two qubits and Bob the last one.

$$|\Psi\rangle = |\Phi\rangle \otimes |e\rangle = \tfrac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle)$$

c) Alice applies a CNOT gate on her two qubits, followed by a Hadamard gate on the first qubit. What is the resulting state $|\Psi'\rangle$?

$$\begin{aligned}|\Psi'\rangle &= (H \otimes I \otimes I)(CNOT \otimes I)\,|\Psi\rangle \\ &= \frac{1}{2}[(\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)]\end{aligned}$$

d) Alice measures her two qubits in the computational basis. What state will Bob's qubit $|\Psi_B\rangle$ be in after each one of Alice's measurement outcomes?

$$\begin{aligned}|\Psi'\rangle &= \frac{1}{2}[(\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)] \\ &= \frac{1}{2}[|00\rangle\,(\alpha\,|0\rangle + \beta\,|1\rangle) + |01\rangle\,(\alpha\,|1\rangle + \beta\,|0\rangle) + |10\rangle\,(\alpha\,|0\rangle - \beta\,|1\rangle) + |11\rangle\,(\alpha\,|1\rangle - \beta\,|0\rangle)]\end{aligned}$$

m	$\lvert\Psi_B\rangle$
00	$\alpha\,\lvert 0\rangle + \beta\,\lvert 1\rangle$
01	$\alpha\,\lvert 1\rangle + \beta\,\lvert 0\rangle$
10	$\alpha\,\lvert 0\rangle - \beta\,\lvert 1\rangle$
11	$\alpha\,\lvert 1\rangle - \beta\,\lvert 0\rangle$

e) Finally, Alice sends her measurement results to Bob. What correction does Bob need to apply to his qubit in each of the four cases so that he ends up with $|\Phi\rangle = \alpha|0\rangle + \beta|1\rangle$?

m	Applied correction
00	I
01	X
10	Z
11	ZX

f) Does this instantaneous teleportation of a qubit from Alice to Bob violate the special theory of relativity that nothing can travel faster than light?

No, because Alice's measurement results are transferred through a classical communications channel.

3.

Using CNOTs, Toffoli gates and single qubit gates implement the circuit that results in the following unitary:

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2x_1x_0$	$x_2x_1x_0$
000	110
001	111
010	100
011	101
100	010
101	011
110	001
111	000

We can analyze the truth table and of the x_0 qubit:

$x_2x_1x_0$	x_0
000	0
001	1
010	0
011	1
100	0
101	1
110	1
111	0

This corresponds to the TOFFOLI gate (the qubit changes the state only when x_2x_1=|11>), therefore the resulting circuit can start with the TOFFOLI gate. Next we analyze the truth table for x_1 taking into account that the circuit starts with TOFFOLI gate

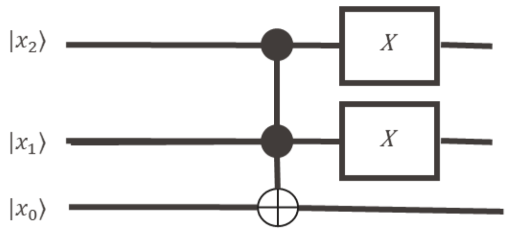
	Toffoli	
$x_2x_1x_0$	$x_2x_1x_0$	x_1
000	000	1
001	001	1
010	010	0
011	011	0
100	100	1
101	101	1

110	111	0
111	110	0

This truth table corresponds to a logical NOT, i.e. X gate, therefore the circuit starts with a TOFFOLI and continues with X gate on x_1. Finally, the truth table for the x_2

	Toffoli	$X(x_1)$	
$x_2x_1x_0$	*$x_2x_1x_0$*	*$x_2x_1x_0$*	*x_2*
000	000	010	1
001	001	011	1
010	010	000	1
011	011	001	1
100	100	110	0
101	101	111	0
110	111	101	0
111	110	100	0

Which is again equivalent to an X gate. Finally, the circuit can be identified as:



Check:

$$U = (X \otimes X \otimes I)TOFFOLI = \begin{bmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$