Ex. 1 Power system of a vehicle

1. External forces acting on the vehicle

The motor of the vehicle generates a torque, which is ultimately applied to the shaft of the wheel (after some loss of efficiency). This torque is necessary to put and keep the vehicle in motion, i.e., to balance the external forces acting on it.

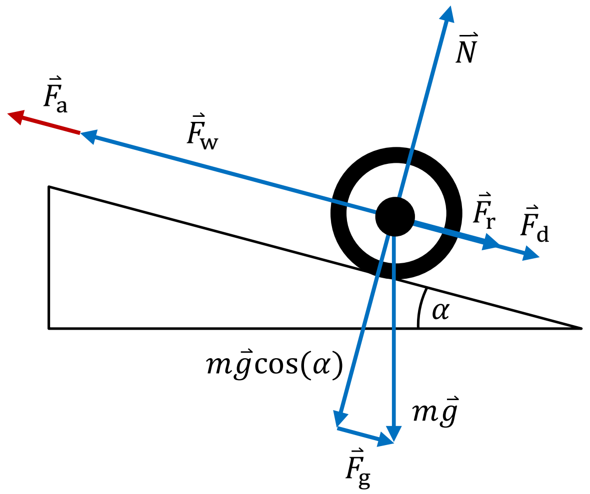

Figure 1 – Forces acting on a vehicle

Table 1 – Drag coefficient of vehicles (Source: Wikipedia)

Vehicle	C_d
Tatra 77 (1934)	0.21
Toyota Prius III	0.25
Honda Civic 2012	0.27
Peugeot 206	0.33
VW New Beetle	0.38
Citroën 2CV	0.50
Ford Model T	0.90

Figure 1 illustrates the external forces (in blue) that apply to the shaft of the wheels.[1] The magnitudes (norms) of the external forces have the following expressions:

- $F_d = \frac{1}{2}\rho v^2 A_\perp C_d$ Aerodynamic drag [2]
- $F_r = C_r N = C_r mg\cos(\alpha)$ Rolling resistance
- $F_g = mg\sin(\alpha)$ Changes in elevation (gravimetric force)
- $F_w = T_w / r_w$ Force exerted on the wheel, which corresponds to the torque that the motor must provide for.

In the direction normal to the road, the component of the weight is simply balanced by an equivalent force from the road. Note that it is assumed the aerodynamic lift (positive or negative) is negligible; it would otherwise impact the rolling resistance. In the direction parallel to the road (motion), Newton's second law of motion gives:

$$F_a = ma = \sum F_{ext} = F_w - F_d - F_r - F_g \quad \Rightarrow \quad F_w = F_a + F_d + F_r + F_g$$

Pay attention that F_a is the *net external force* resulting from the external forces. It is proportional to the resulting acceleration of the vehicle and to its mass.

[1] It is considered all forces apply to an *equivalent* single wheel, which is the sum of all four.
[2] Dynamic pressure of air times a surface, which is expressed as a reference area $A_\perp$ times a coefficient C_d.

2. General expression for the total power necessary to drive the vehicle

The total power necessary to drive the vehicle is equal to the power for the motor plus that for the auxiliary accessories. Do not forget the loss of efficiency between the motor and the wheels:

$$P_{\text{tot}} = P_{\text{aux}} + P_m = P_{\text{aux}} + \frac{P_{\text{w}}}{\varepsilon}$$

Power is energy (work) per unit time. Assuming a force F is constant along, and aligned with, a trajectory of length l, the integral to compute the work becomes simply $F \cdot l$. Hence, the power is $F \cdot v$. It also relates to torque and angular velocity as follows:

$$P_{\text{w}} = T_{\text{w}}\omega_{\text{w}} = F_{\text{w}}\omega_{\text{w}}r_{\text{w}} = F_{\text{w}}v$$

$$P_{\text{tot}} = P_{\text{aux}} + \frac{P_{\text{w}}}{\varepsilon} = P_{\text{aux}} + \frac{1}{\varepsilon}\left(mv\left(a + g\big(\sin(\alpha) + C_{\text{r}}\cos(\alpha)\big)\right) + \frac{1}{2}\rho v^3 A_{\perp} C_{\text{d}}\right)$$

Rigorously, α = atan(slope in %). However, for small angles, $\cos(\alpha) \simeq 1$ and $\sin(\alpha) \simeq \alpha$ (in radians). Besides, the percent grade of a road slope, $s_{\%}$, is usually a measure of $\sin(\alpha)$ instead of $\tan(\alpha)$. Therefore, a good estimate of the required total power is:

$$P_{\text{tot}} = P_{\text{aux}} + \frac{1}{\varepsilon}\left(mv\big(a + g(s_{\%} + C_{\text{r}})\big) + \frac{1}{2}\rho v^3 A_{\perp} C_{\text{d}}\right)$$

The parameters in this formula are assumed constant for all driving conditions of interest. In reality, *a,* C_{d}, and C_{r} typically depend on the speed and, for C_{r}, on the surfaces in contact (material, quality).

3. Minimum power needed for a conventional mid-size passenger vehicle

The following characteristics can be assumed for such a vehicle:

- Mass: $m = 1350$ kg (vehicle mass) $+ 250$ kg (mass of passengers plus cargo) $= 1600$ kg
- Coefficient of aerodynamic drag: $C_{\text{d}} = 0.3$ (see table 1 above)
- Area of the vehicle normal to flow direction:[3] $A_{\perp} = 2\ \text{m}^2$
- Coefficient of rolling resistance: $C_{\text{r}} = 0.01$
- Efficiency of motor, controller, and gearing: $\varepsilon = 0.77$
- Auxiliary power (lights, radio, wipers, air conditioner, cigarette lighter, etc.): $P_{\text{aux}} = 400$ W

The aerodynamic drag depends on atmospheric conditions. Assuming p = 1 bar, T = 15°C, and dry air (21% O_2, 79% N_2, molar) following the ideal gas law, the density is: $\rho = p\tilde{m}/\mathcal{R}T = 1.2\ \text{kg/m}^3$.

a) To sustain a speed *v* = 55 mph on a road inclined with 6.5 percent grade

v = 55 mph = 24.59 m/s. The acceleration is zero (external forces are balanced).

$$P_{\text{tot}} \cong P_{\text{aux}} + \frac{1}{\varepsilon}\left(mgv(s_{\%} + C_{\text{r}}) + \frac{1}{2}\rho v^3 A_{\perp} C_{\text{d}}\right) \approx 0.4 + 32.6 + 5.0 + 5.4 = 43.4\ \text{kW}$$

As can be observed, the contribution of the mass is large, especially to compensate for the force pulling the vehicle up the slope (75% of total).

[3] Caution: drag & lift coefficients depend on the area that is used as reference. In aeronautics, the definition is typically different (area = wingspan × chord of airfoil) than in the automotive or naval sectors.

b) To accelerate at *a* = 3 mph/s on a level road from *v* = 65 mph

On a level road, $\alpha = 0$, so that $\cos(\alpha) = 1$ and $\sin(\alpha) = 0$ (no gravimetric force due to inclination).
$v = 65$ mph = 29.06 m/s and $a = 3$ mph/s = 1.34 m/s².

$$P_{\text{tot}} = P_{\text{aux}} + \frac{1}{\varepsilon}\left(mv(a + gC_{\text{r}}) + \frac{1}{2}\rho v^3 A_{\perp} C_{\text{d}}\right) \approx 0.4 + 81 + 6 + 9 = 98.4\ \text{kW}$$

The contribution of the mass is again large, this time mostly to accelerate it (82% of total).

Thinking further:

- Estimate of the vehicle's range / dimensioning of the energy storage

The previous computations are important to dimension the power system. However, they are not representative of the consumption of energy and thereby of the range. From an *energetic* point of view, the energy related to changes in elevation may be neglected, since the potential energy for driving uphill can be retrieved while driving downhill (on average). Also, part of the energy related to acceleration can be retrieved during brakes using a dynamo (regenerative braking).
Overall, for long haul, most of the energy is dissipated into thermal energy to compensate for the rolling and aerodynamic resistances.
Assuming the power system is a fuel cell of average efficiency 45%, the quantity of hydrogen (120 MJ/kg) that must be stored to cover 200 km at an average speed[4] of 70 km/h is, at least:

$$m_{\text{H}_2} = \frac{d\left(mgC_{\text{r}} + \frac{1}{2}\rho\langle v\rangle_2{}^2 A_{\perp} C_{\text{d}}\right)}{\varepsilon \cdot \varepsilon_{\text{FC}} \cdot \Delta_{\text{r}} h^{\ominus}_{\text{H}_2,\,15°\text{C}\,(g)}} \gtrsim 1.45\ \text{kg}$$

Note how much the aerodynamic drag becomes dominant, while increasing the average speed:

- at 50 km/h (~14 m/s): $m_{\text{H}_2} = 0.8\ \text{kg} + 0.3\ \text{kg} = 1.1\ \text{kg}$ (100%)$_{\text{ref.}}$
- at 80 km/h (~22 m/s): $m_{\text{H}_2} = 0.8\ \text{kg} + 0.8\ \text{kg} = 1.6\ \text{kg}$ (150%)
- at 120 km/h (~33 m/s): $m_{\text{H}_2} = 0.8\ \text{kg} + 1.9\ \text{kg} = 2.7\ \text{kg}$ (250%)

In practice, a standardized driving profile is used, which corresponds to representative travel characteristics.

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[4] Pay attention that the average aerodynamic drag is underestimated by using the arithmetic mean of the speed. Since it is proportional to the speed squared, it is necessary to consider the weighted power mean (generalized mean) with a power of 2, i.e., the weighted quadratic mean (root mean square). For instance: $\langle v\rangle_2 = \sqrt[2]{0.6 \cdot 50^2 + 0.3 \cdot 80^2 + 0.1 \cdot 120^2} \cong 70\ \text{km/h}$, with percent of distance traveled at v_{k}.